GLASNIK MATEMATIČKI
Vol. 54(74)(2019), 77 – 132

FURTHER RESEARCH PROBLEMS AND THEOREMS ON PRIME POWER GROUPS

Yakov Berkovich and Zvonimir Janko

University of Haifa, Israel and University of Heidelberg, Germany

Below we state a great number of research problems concerning finite p-groups. This list is a continuation of the six lists in [B1, BJ2, BJ3, BJ4, BJ5, BJ6]. Below we also stated some new theorems with proofs. For explanation of notation see the beginning of the above volumes.

4101. Describe the p-groups all of whose subgroups of index p^k, $k \in \{2, 3, 4\}$, are normal (three problems). Consider in detail the groups of exponent p.

4102. Study the nonabelian p-groups G all of whose maximal abelian subgroups are normal (any two elements of G generate a subgroup of class ≤ 2 so our group is regular if $p > 2$, by Theorem 7.1(b) in [B1]). Consider in detail the case $p = 2$.

4103. Find the maximal possible order of the automorphism groups of the groups of maximal class of order p^p.

4104. Study the non-Dedekindian p-groups covered by nonnormal subgroups.

4105. Study the p-groups G in which the intersection of any two non-incident subgroups, say A and B, of equal order (of different orders) is normal (i) either in A or in B, (ii) in $\langle A, B\rangle$.

4106. Study the p-groups G all of whose nonabelian subgroups of equal order are isomorphic (permutable). Consider in detail the case when $\exp(G) = p$.

2010 *Mathematics Subject Classification.* 20D15.

Key words and phrases. p-groups of exponent p, metacyclic p-groups, metahamiltonian p-groups, self-centralizing subgroups, H_p-subgroups, maximal abelian subgroups, minimal nonabelian subgroups, regular p-groups.

4107. Let G be a p-group. Describe the p-groups H such that $\mathcal{L}_N(H) \cong \mathcal{L}_N(G)$, where $\mathcal{L}_N(G)$ is the lattice of normal subgroups of G. Consider in detail the case when $\exp(G) = p$.

4108. Classify the p-groups containing only one noncyclic maximal abelian subgroup (example: M_{p^n}).

4109. Study the p-groups all of whose nonabelian two-generator subgroups are either of exponent p, or minimal nonabelian, or metacyclic, or of maximal class (any 3-group of maximal class satisfies the above condition).

4110. Study the nonabelian p-groups G in which $\mathrm{cl}(S^G) = 2$ for all (i) minimal nonabelian subgroups $S < G$, (ii) maximal nonnormal abelian subgroups $S < G$.

4111. Classify the nonabelian p-groups G all of whose nonnormal maximal abelian subgroups (minimal nonabelian subgroups) are conjugate (have equal order).

4112. Classify the nonabelian p-groups G such that, whenever $A < G$ is maximal abelian and $x \in A - \mathrm{Z}(G)$, then $\mathrm{C}_G(x) = A$.

4113. Does there exist a nonabelian p-group all of whose maximal abelian subgroups have pairwise different orders?

4114. Study the p-groups containing an $\mathcal{A}_1$-subgroup S such that $\mathrm{C}_G(x)$ is an $\mathcal{A}_1$-subgroup for all $x \in S - \mathrm{Z}(S)$.

4115. Study the nonabelian p-groups G such that, whenever $A \triangleleft G$ is maximal abelian and $x \in G - A$, then $\mathrm{cl}(\langle x, A\rangle) = 2$.

4116. Study the p-groups all of whose outer p-automorphisms have order p.

4117. Study the p-groups of exponent $> p$ all of whose maximal abelian subgroups of exponent $> p$ are normal.

4118. Study the structure of a p-group G provided all its minimal nonabelian subgroups of exponent $> p$ are normal.

4119. Suppose that a p-group G of maximal class possesses an abelian subgroup of index p^4. Describe the structure of G if it has no G-invariant abelian subgroup of index p^4 (obviously, in the case under consideration, $p > 3$; see 9.2 in [B1])).

4120. Suppose that a nonabelian p-group G contains an abelian subgroup of index p. Describe the p-groups that are lattice isomorphic with such G.

4121. Study the p-groups G all of whose subgroups of index p^2 are isomorphic. Consider in detail the case $\exp(G) = p$.

4122. Study a p-group which is a product of pairwise permutable subgroups of order p^2.

4123. Estimate the derived length of a p-group which (i) is a product of n pairwise permutable cyclic (abelian, minimal nonabelian) subgroups, (ii) contains only metacyclic (minimal nonabelian) subgroups of index $\leq p^n$ for a given n.

4124. Study the p-groups G such that $\Phi(H)$ is abelian for all $H \in \Gamma_1$ but $\Phi(G)$ is nonabelian. Do the same for G' instead of $\Phi(G)$.

4125. Study the subgroup structure of the p-groups $G = ST$, where $S, T < G$ are minimal nonabelian.

4126. Study the nonabelian groups G of exponent p such that $S \cap T > \{1\}$ for any two minimal nonabelian $S, T < G$. Estimate $|G|$. Consider also the case $\exp(G) > p$.

4127. (Old problem) Study the p-groups G satisfying $|\mathrm{Aut}(G)|_p \leq |G|$. Consider the case $\exp(G) > p$ in detail.

4128. Study the p-groups G such that, whenever $H \in \Gamma_1$ is nonabelian, then all minimal nonabelian subgroups of H have equal orders (are conjugate in G).

4129. Let G be a group of exponent p such that, whenever $S < G$ is minimal nonabelian, there is in G a minimal nonabelian subgroup T such that $S \cap T = \{1\}$. Find the minimal order of G. Consider this problem for the p-groups of exponent $> p$. Estimate $|G|$.

4130. Study the p-groups containing an $\mathcal{A}_2$-subgroup S such that $\mathrm{C}_G(x)$ is abelian for all $x \in S - \mathrm{Z}(S)$.

4131. Study the p-groups $G = \Omega_1(G)$ containing a proper normal subgroup R of order p^p and exponent p such that $\langle R, x \rangle$ is of maximal class for any $x \in G - R$ of order p.

4132. Classify the non-metacyclic (non-absolutely regular) p-groups G such that $\mathrm{N}_G(M)$ is metacyclic (absolutely regular) for any nonnormal metacyclic (absolutely regular) $M < G$.

4133. Classify the p-groups G such that $\mathrm{N}_G(C)$ is abelian (metacyclic) for any nonnormal cyclic $C < G$.

4134. Study the irregular p-groups G such that $\mathrm{N}_G(R)$ is regular for any nonnormal regular $R < G$.

4135. Study the p-groups G such that (i) $\mathrm{d}(\mathrm{N}_G(H)) = 2$ for any nonnormal $H < G$, (ii) $\mathrm{N}_G(A)/A$ is cyclic for any nonnormal noncyclic $A < G$.

4136. Study the primary $\mathcal{A}_n$-groups G, $n > 1$, (i) in which the intersection of all minimal nonabelian subgroups is not contained in $\mathrm{Z}(G)$, (ii) such that $S_G = \{1\}$ for any minimal nonabelian $S < G$.

4137. Given n, estimate $\mathrm{c}_n(\mathrm{Aut}(M)_p))$, where M is a p-group of maximal class and exponent p^n.

4138. Study the p-groups G such that every their nonnormal abelian subgroup is contained in minimal nonabelian subgroup of G (any 2-group of maximal class satisfies the above condition).

4139. Study the p-groups such that all factors of their upper central series, except of the last one, are cyclic.

4140. Study the p-groups G such that the centralizer (normalizer) of any nonnormal cyclic (metacyclic) $C < G$ is metacyclic.

4141. Study the p-groups G such that $|S : S_G| = p$ for all nonnormal (minimal nonabelian) $S < G$.

4142. Study the p-groups G such that $G/\mathrm{Z}(G)$ is minimal nonabelian. What can we say about the members of the set Γ_1? Consider in detail the case $\mathrm{d}(G) = 2$.

4143. Classify the nonabelian p-groups G that are not generated by $\alpha_1(G) - 1$ minimal nonabelian subgroups (it is easy to see that, in that case, $p = 2$; indeed, if $H < G$ is generated by $\alpha_1(G) - 1$ minimal nonabelian subgroups, then $p - 1 \leq \alpha_1(G) - \alpha_1(H) = 1$).

4144. Study the nonabelian p-groups G such that whenever $S, T \leq G$ are minimal nonabelian, then $S = T^\phi$ for some $\phi \in \mathrm{Aut}(G)$. Consider also the case when that is true only for $S \cong T$.

4145. Study the p-groups in which any two-generator nonabelian subgroup is either minimal nonabelian, or metacyclic, or of maximal class.

4146. Study the p-groups G such that $|G : \mathrm{C}_G(x)| \leq p^2$ for any $x \in G - \mathrm{Z}(G)$.

4147. Study the p-groups all of whose maximal abelian and minimal nonabelian subgroups have the same order (example: $S \times \mathrm{C}_p$, where S is a minimal nonabelian p-group).

4148. Study the nonabelian p-groups G with $\mathrm{cl}(\langle A, B\rangle) = 2$ for any two distinct maximal abelian $A, B < G$.

4149. Study the nonabelian p-groups all of whose epimorphic images of equal order are isomorphic.

4150. Study the p-groups all of whose normal subgroups of equal order are isomorphic. Consider in detail this problem for groups of exponent p.

4151. Study the p-groups G such that, whenever $|H|^2 \leq |G|$ for $H < G$, then H is either cyclic or G-invariant.

4152. Study the nonabelian p-groups G such that, whenever $A, S < G$, where A is abelian and S is minimal nonabelian, then $|A| \leq |S|$.

4153. Study the p-groups G such that, whenever $|H|^2 < |G|$ for $H < G$, then $|H^G : H| \leq p$.

4154. Study the nonabelian p-groups G such that $\mathrm{cl}(S^G) = 2$ for each minimal nonabelian $S \leq G$. Moreover, consider the same conclusion in the case $\mathrm{cl}(A^G) = \mathrm{cl}(A)$ for all $A < G$.

4155. Does there exist a p-group G such that $\exp(\mathrm{M}(G)) > \exp(G)$ (here $\mathrm{M}(G)$ is the Schur multiplier of G)?

4156. Study the nonabelian p-groups G in which isomorphic nonabelian subgroups are contained in isomorphic members of the set Γ_1.

4157. Study the nonabelian p-groups G such that $\langle \Phi(H) \mid H \in \Gamma_1 \rangle < \Phi(G)$.

4158. Study the non-Dedekindian p-groups all of whose isomorphic nonnormal subgroups have isomorphic normalizers (centralizers).

4159. Study the nonabelian p-groups G such that if M, N are isomorphic G-invariant subgroups (subgroups of equal order), then $G/M \cong G/N$.

4160. Study the p-groups G such that all their subgroups containing $\Phi(G)$ as a subgroup of index p, are isomorphic.

4161. Study the p-groups G containing exactly one normal subgroup of any order $\leq |\Phi(G)|$. Consider in detail the case $|G : \Phi(G)| = p^2$. The same problems for G' instead of $\Phi(G)$.

4162. (Old problem) Study the p-groups (groups of exponent p) all of whose maximal subgroups are isomorphic.

4163. Study the p-groups G such that $G/\mathrm{K}_p(G)$ is of order p^p and exponent p. What one can say on the members of the set Γ_1?

4164. Study the p-groups G, $p > 2$, such that $\mathrm{Z}_p(G)$ is of maximal class and order p^p (of order p^{p+1}).

4165. Study the nonabelian p-groups $G = \Omega_2(G)$ that are not generated by $\mathcal{A}_1$-subgroups of order $> p^3$?

4166. It is easy to prove that a nonabelian p-group $G = \Omega_n(G)$ contains an $H \in \Gamma_1$ satisfying $\Omega_n(H) = H$. Is it true that there exists an $M \in \Gamma_1 - \{H\}$ such that $\Gamma_n(M) = M$?

4167. How many representation groups have a given homocyclic p-group?

4168. Suppose that a nonabelian p-group G, $p > 2$, contains an elementary abelian subgroup of index p. Describe the group G if it is not generated by subgroups $\cong S(p^3)$?

4169. Let $\nu_n(G)$ be the number of normal subgroups of order p^n in a p-group G. Given n, study the p-groups G such that $\nu_n(G) = \nu_{n+1}(G) = \cdots = \nu_{n+p}(G)$.

4170. (i) Let G be a p-group and ϕ its p-automorphism. Study the structure of G in the case when there is in G only one ϕ-invariant subgroup of each order. Consider in detail the case when $o(\phi) = p$. (ii) Study the nonabelian p-groups G such that each its maximal subgroup is of the form $S * A$ (central product), where A is abelian and S is (ii1) minimal nonabelian, (ii2) metacyclic.

4171. Study the p-groups containing a normal subgroup N such that all elements of the coset xN are conjugate in G for all $x \in G - N$.

4172. Study the p-groups G of exponent $> p$ such that any two cyclic subgroups $A, B < G$ with $A \cap B > \{1\}$ generate a metacyclic subgroup.

4173. Study the p-groups G such that $|\mathrm{Aut}(G) : \mathrm{Inn}(G)|_p = p$.

4174. Study the p-groups G, $p > 2$, such that $G/\mho_1(G)$ is a group of maximal class of order p^p.

4175. Study the nonabelian p-groups all of whose maximal abelian subgroups (maximal cyclic subgroups, minimal nonabelian subgroups) are permutable (three problems).

4176. Does there exist a p-group whose Schur multiplier is not characteristic in its representation group?

4177. Estimate the derived length of the p-group $G = AS$, where $A < G$ is abelian and $S < G$ is minimal nonabelian.

4178. Study the non-metacyclic p-groups all of whose metacyclic subgroups are abelian (in that case all minimal nonabelian subgroups of G are non-metacyclic).

4179. Describe the absolutely regular p-groups whose representation groups are not absolutely regular.

4180. Study the p-groups G such that $\mathrm{d}(\mathrm{N}_G(S)) = 2$ for any $\mathcal{A}_2$-subgroup $S \leq G$.

4181. Present two non-isomorphic groups G and H of exponent p and of the same class with $G' \cong H'$ that are not lattice isomorphic.

4182. Does there exist a nonabelian group G of exponent p (in that case $p > 2$) such that $\mathrm{Aut}(G)$ is a p-group?

4183. Study the 2-groups all of whose $\mathcal{A}_2$-subgroups are minimal nonmetacyclic of order 2^5.

4184. Study the irregular p-groups all of whose maximal regular subgroups are permutable.

4185. Study the p-groups all of whose subgroups of derived length 2 are either metacyclic or minimal nonabelian.

4186. Study the p-groups all of whose maximal subgroups are either regular or of maximal class with abelian subgroup of index p.

4187. Study the 3-groups all of whose regular subgroups are of class 2 (any 3-group of maximal class satisfies the above condition).

4188. Study the p-groups G of exponent $> p$ such that $\mathrm{H}_p(G) = \mho_1(G)$ (here $\mathrm{H}_p(G)$ is the Hughes subgroup of G).

4189. Given $n > 1$, does there exist a p-group G of exponent p^n such that the subgroups $\Omega_1(G), \Omega_2^*(G), \dots, \Omega_n^*(G)$ are pairwise non-incident.

4190. Study the irregular p-groups that are lattice isomorphic to regular p-groups.

4191. Is it possible to estimate the derived length of a p-group G that is lattice isomorphic to a p-group H of a given class (derived length)?

4192. Describe the p-groups that are lattice isomorphic with (i) minimal nonabelian groups, (ii) $\mathcal{A}_2$-groups.

4193. Describe the p-groups G of maximal class and order $> p^{p+1}$ such that all p-groups that are lattice isomorphic with G are isomorphic (example: $G \cong \mathrm{SD}_{2^4}$).

4194. Let G and H be irregular p-groups. Suppose that $M_1, \dots, M_k$ and $N_1, \dots, N_k$ be the sets of maximal subgroups of G and H, respectively. Is it true that $G \cong H$ if $M_i \cong N_i$ for all i?

4195. Study the 2-groups G containing a normal subgroup $E \cong \mathrm{E}_{2^k}$, $k \in \{2, 3, 4\}$, and such that G/E is of maximal class. Study the lattice isomorphisms of such G.

4196. Study the 2-groups G such that the subgroup $\Omega_2(G)$ ($\Omega_2^*(G)$) is minimal non-metacyclic of order 2^5.

4197. Study the p-groups G with an abelian (regular) H_p-subgroup.

4198. Present a nonabelian 2-group $G = \Omega_1(G)$ all of whose maximal subgroups are generated by involutions. Consider a similar problem for $p > 2$.

4199. Study the p-groups G of exponent $p > 2$ such that $\mathrm{d}(G) = \mathrm{d}(H)$ for all $H \in \Gamma_1$.

4200. Is it true that if any two $\mathcal{A}_1$-subgroups of a p-group G are permutable and $p > 3$, then G possesses a normal $\mathcal{A}_1$-subgroup?

4201. Study the non-metacyclic 2-groups $G = AB$, where A, B are cyclic. Is it true that the set Γ_1 has a metacyclic member?

4202. Study the p-groups G with $\mathrm{C}_G(S) < S$ for any $\mathcal{A}_1$-subgroup S of G (if $|S| = p^3$, then G is of maximal class; see Theorem 10.19 in [B1]).

4203. Study the 2-groups with an even number of subgroups $\cong \mathrm{E}_8$ (one of such groups is $\mathrm{D}2^n \times \mathrm{C}_2$).

4204. Classify the p-groups covered by metacyclic (non-metacyclic) $\mathcal{A}_1$-subgroups (see problem #860 in [BJ2]).

4205. (Wielandt) Study the irregular 3-groups $G = ST$ where subgroups S and T are metacyclic.

4206. Study the groups G of exponent p such that $\mathrm{C}_G(S) \cong \mathrm{E}_{p^2}$ for any $\mathcal{A}_1$-subgroup S of G. Consider a more general case when $\mathrm{C}_G(S)$ is (elementary) abelian.

4207. Study the p-groups G such that, whenever $A, B < G$ are distinct maximal abelian subgroups in G, then $A \cap B > \mathrm{Z}(G)$.

4208. Study the p-groups G subjecting $|G| = p|G'||\mathrm{Z}(G)|$ (any nonabelian p-group containing an abelian subgroup of index p, satisfies the above property; see [B1, Lemma 1.1]; also, any p-group of maximal class satisfies that condition).

4209. Study the p-groups in which any two $\mathcal{A}_1$-subgroups (abelian subgroups) generate a subgroup of derived length (class) 2.

4210. Find the maximal possible number of minimal nonabelian subgroups in an irregular group of maximal class and order p^n.

4211. Study the p-groups G in which any member of the set Γ_1 has the trivial Schur multiplier.

4212. Study the p-groups G with $|\mathrm{M}(G)| \geq |G|$ (here $\mathrm{M}(G)$ is the Schur multiplier of G; the group E_{p^n} satisfies the above condition if $n > 3$). Consider also the p-groups G subjecting $\exp(M)(G) \geq \exp(G)$.

4213. Describe the representation groups of the homocyclic p-groups (note that Schur have described the Schur multipliers of all abelian p-groups).

4214. Study the p-groups G satisfying $\Sigma_{S \in \mathcal{A}_1(G)} |S| \leq |G|$.

4215. Study the p-groups all of whose abelian (nonabelian) subgroups of (i) equal order, (ii) different orders are permutable.

4216. Study the p-groups G with a fixed $H \in \Gamma_1$ such that all abelian subgroups of G not contained in H are cyclic.

4217. Study the p-groups G such that whenever $H \in \Gamma_1$ and $A < H$ is maximal abelian, then either $\mathrm{N}_G(A) \leq H$ or $A \triangleleft G$.

4218. Classify the p-groups all of whose maximal abelian subgroups have index p^2.

4219. Classify the minimal nonabelian (metacyclic) p-groups with trivial Schur multiplier (for example, the metacyclic 2-groups Q_{2^n} and SD_{2^n} have trivial Schur multipliers).

4220. Given n, does there exist an irregular 5-group $G = AM$ of order 5^n, where A is an $\mathcal{A}_1$-subgroup and M is metacyclic?

4221. Study the groups of exponent p whose representation groups are (i) irregular, (ii) have exponent p.

4222. Study the representation groups of special p-groups. Consider in detail the extraspecial p-groups.

4223. Find $\mathrm{d}(P)$, where $P \in \mathrm{Syl}_p(\mathrm{Aut}(A))$ and A is a given abelian p-group.

4224. Describe the set of positive integers n such that if an irregular p-group G contains a regular subgroup of index p^n, then it contains a normal regular subgroup of index p^n.

4225. Study the nonabelian p-groups G such that if $x \in G - \mathrm{Z}(G)$, then $\mathrm{C}_G(x) = \langle x, \mathrm{Z}(G) \rangle$ (example: G is any minimal nonabelian p-group).

4226. Study the p-groups with abelian (Dedekindian) intersection of any two non-incident subgroups.

4227. Study the p-groups G such that $\mathrm{C}_G(S) \cong S$ for any $\mathcal{A}_1$-subgroup $S < G$. The same problem in the case where $\mathrm{C}_G(S)$ is minimal nonabelian for any $\mathcal{A}_1$-subgroup $S < G$.

4228. Study the p-groups G with $AC = CA$, where $A < G$ is maximal abelian and $C < G$ is maximal cyclic.

4229. Describe the p-groups G in which normality (quasinormality) is transitive for nonabelian subgroups.

4230. Study the p-groups G in which all subgroups of index $\geq p^4$ are normal.

4231. Study the p-groups G such that for any $H \in \Gamma_1$ and $x \in G - H$ there is $h \in H - \mathrm{Z}(G)$ such that $\langle x, h\rangle$ is an $\mathcal{A}_1$-subgroup (metacyclic subgroup).

4232. Study the $\mathcal{A}_n$-groups, $n > 2$, containing at most one $\mathcal{A}_k$-subgroup for $k \in \{2, 3\}$ (two problems).

4233. Study the metacyclic p-groups all of whose minimal nonabelian subgroups have the same order (are isomorphic).

4234. Study the p-groups G all of whose two-generator subgroups are absolutely regular.

4235. Study the p-groups G subjecting $|G : G'\mathrm{Z}(G)| \in \{p^2, p^3\}$ (two problems).

4236. Let $E \triangleleft G$ be of order p^p. Study the p-groups G such that $\langle x, E\rangle$ is of maximal class for all $x \in G$ of order p. Consider in detail the case $\exp(E) = p$.

4237. Study the p-groups G such that $|\Omega_1(G)| \geq |\Omega_1(\Gamma)|$, where Γ is a representation group of G.

4238. Study the p-groups G such that, whenever $A, B < G$ are distinct maximal abelian, then $\mathrm{dl}(\langle A, B\rangle) = 2$.

4239. Study the p-groups G such that $\mathrm{dl}(\mathrm{N}_G(S)) = 2$ for any $\mathcal{A}_1$-subgroup $S < G$.

4240. Study the p-groups G, $p > 2$, such that $|\mathrm{Aut}(G)|$ is odd.

4241. Study the nonabelian p-groups G such that $|\mathrm{Aut}(G)|_p < |G|$ (it is known that the set of such G is not empty).

4242. Study the nonabelian 2-groups G such that any $\mathcal{A}_1$-subgroup, not contained in $\Phi(G)$, is isomorphic to one of the groups $\mathrm{Q}_8, \mathrm{D}_8, \mathrm{M}_2(2, 2)$.

4243. (Wielandt) Study the irregular 3-groups $G = ST$, where S is metacyclic and T is cyclic (see #4205).

4244. Find all possible values of $\mathrm{c}_1(G) \pmod{p^p}$, where G is a p-group of maximal class of order p^n.

4245. Study the p-groups G such that, whenever $a, b \in G - \mathrm{Z}(G)$, then $b = a^\phi$ for some $\phi \in \mathrm{Aut}(G)$.

4246. Given a p-group G, study the p-groups H such that $\mathrm{s}_k(H) = \mathrm{s}_k(G)$ ($\mathrm{c}_k(H) = \mathrm{c}_k(G)$) for all k (see a partial case in #4711).

4247. Study the p-groups G such that $\mathrm{C}_G(S) = \mathrm{Z}(S)$ for any nonabelian (minimal nonabelian) $S \leq G$.

4248. Study the p-groups G such that, whenever nonnormal $A, B < G$ are isomorphic (have equal order), then $B = A^{\phi}$ for some $\phi \in \mathrm{Aut}(G)$.

4249. Study the p-groups G such that (i) $\mathrm{Aut}(G)$ is minimal nonabelian p-group, (ii) $\mathcal{A}_2$-group whose order is a power of p.

4250. Classify the metacyclic p-groups G such that a Sylow p-subgroup of $\mathrm{Aut}(G)$ is metacyclic.

4251. Find $\mathrm{d}(P)$ ($\mathrm{cl}(P)$), where $P \in \mathrm{Syl}_p(\mathrm{Aut}(A))$ for a metacyclic p-group A.

4252. Let G be a group of order p^n. Compare $|\mathrm{Aut}(G)|_p$ and $|\mathrm{GL}(n,p)|$.

4253. Does there exist an irregular p-group $G = \Omega_1(G)$ of exponent $> p$ such that all its maximal subgroups of exponent p have pairwise distinct orders (such G must be irregular, by [B1, Theorem 7.2(b)])?

4254. Classify the p-groups containing a subgroup $\cong \mathrm{M}_{p^n}$ of index p.

4255. Classify the p-groups all of whose nonnormal subgroups are abelian (nonabelian subgroups are normal).

4256. Classify the p-groups with (i) $\mathrm{f}(G) = \frac{1}{p}$, (ii) $\mathrm{f}(G) = \frac{1}{p+1}$ (here $\mathrm{f}(G) = \frac{\mathrm{T}(G)}{|G|}$, where $\mathrm{T}(G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)$).

4257. Study the irregular p-groups containing exactly $p+1$ maximal regular subgroups (any minimal irregular p-group satisfies the above property since it is two-generator).

4258. Describe $\mathrm{cd}(M \wr M)$, where M is a 2-group of maximal class (here $\mathrm{cd}(G) = \chi(1) \mid \chi \in \mathrm{Irr}(G))$. Consider also the case when M possesses an abelian subgroup of index p..

4259. Study the (irregular) p-groups all of whose maximal subgroups, except one, have exponent p.

4260. Study the non-Dedekindian p-groups G such that $|G : \mathrm{N}_G(S)| = p$ for all (i) minimal nonabelian $S < G$, (ii) nonnormal $S < G$, (iii) nonnormal abelian $S < G$.

4261. Study the p-groups all of whose two-generator subgroups contain an abelian subgroup of index p.

4262. Study the p-groups all of whose two-generator subgroups are of class ≤ 2.

4263. Study the p-groups G such that $\mathrm{cl}(\langle S, T\rangle) \leq 3$ for any minimal nonabelian (maximal abelian) $S, T < G$.

4264. Study the two-generator p-groups G such that $G/\mathrm{Z}(G)$ is (i) metacyclic, (ii) of maximal class, (iii) minimal nonabelian.

4265. Study the p-groups G in the case when the order of $|\mathrm{Z}(G/\ker(\chi))| = p$ for any $\chi \in \mathrm{Irr}_1(G)$.

4266. Study the p-groups G such that whenever $A < B < G$, where A is maximal abelian in G and $|B : A| = p$, then (i) $|B'| = p$, (ii) $|B : B'| = p^2$.

4267. Estimate $|\Omega_1(\Delta)|$ is terms of $|\Omega_1(A)|$, where Δ is a representation group of an abelian p-group A.

4268. Study the p-groups G such that $\mathrm{c}_1(G/L) = \mathrm{c}_1(G)$ for any $L \triangleleft G$ of order p.

4269. Study the p-groups G such that $\mathrm{d}(G) < \mathrm{d}(H)$ $(\mathrm{d}(G) = \mathrm{d}(H))$ for any $H \in \Gamma_1$.

4270. Study the p-groups G such that $\Phi(G) = \Phi(H)$ $(G' = H')$ for all $H \in \Gamma_1$.

4271. Study the p-groups G such that whenever $A, B < G$ with $\langle A, B\rangle = G$, then $AB = BA$.

4272. Describe the special p-groups all of whose nonabelian epimorphic images are special (example: the nonabelian derived subgroup of a minimal nonnilpotent group). Describe also the non-special p-groups satisfying the above property.

4273. Describe the p-groups in which the normalizer (centralizer) of any subgroup is normal.

4274. Describe the p-groups in which any maximal metacyclic subgroup is contained in a subgroup of maximal class.

4275. Study the p-groups G of maximal class such that $o(x) = p$ for all $x \in G - G_1$, where G_1 is the fundamental subgroup of G (see §9 in [B1]).

4276. Study the p-groups G all of whose automorphisms of order p commute.

4277. (i) Is it true that a p-group G is regular if any two its $\mathcal{A}_1$-subgroups generate a regular subgroup? (ii) Estimate the number of pairs of elements generating a given minimal irregular p-group.

4278. Study the p-groups G such that all $\mathcal{A}_1$-subgroups of any nonabelian $H \in \Gamma_1$ (i) are H-invariant, (ii) are conjugate in G.

4279. Study the p-groups G containing $H \in \Gamma_1$ such that any abelian subgroup of G which is not contained in H has order $\leq p^3$.

4280. Study the non-metacyclic p-groups G such that $\mathrm{C}_G(M) \cong M$ for any maximal metacyclic $M < G$.

4281. Study the p-groups, $p > 2$, in which any abelian (metacyclic) subgroup is contained in regular subgroup of index p (two problems).

4282. Study the p-groups G of order p^{2n+1} such that $\mathrm{cd}(G) = \{1, p, \dots, p^n\}$.

4283. Study the p-groups in which any nonnormal subgroup (minimal nonabelian subgroup) has a complement (a normal complement).

4284. Study the p-groups G all of whose nonabelian members of the set Γ_2 are isomorphic (have the same class).

4285. Study the p-groups G such that $\mathrm{cl}(A) = \mathrm{cl}(B)$ for all $A, B \in \Gamma_1$. Consider also a condition $\mathrm{dl}(A) = \mathrm{dl}(B)$ for all $A, B \in \Gamma_1$.

4286. Study the p-groups G all of whose conjugate subgroups (nonnormal subgroups of equal order) are permutable.

4287. Study the p-groups G such that $A_G = B_G$ for $A, B < G$ implies that A, B are conjugate (isomorphic, permutable).

4288. Study the group of those automorphisms that fix all $\mathcal{A}_1$-subgroups of a nonabelian p-group.

4289. Study the p-groups G in which the normal closure of any abelian subgroup is of class ≤ 2.

4290. Study the groups of order 2^n such that $\alpha_1(G) = \alpha_1(\mathrm{D}_{2^n})$.

4291. Given n, find all possible values for the number of subgroups of order p^p of maximal class in a p-group of maximal class and order p^{p+n}. Consider in details cases $n \in \{1, 2\}$.

4292. Study the p-groups G such that A is isolated in $\mathrm{N}_G(A)$ for any nonnormal $A < G$.

4293. Study the p-groups G all of whose (i) nontrivial subgroups are non-isolated, (ii) maximal metacyclic subgroups are isolated.

4294. Study the p-groups G all of whose maximal abelian subgroups, except one, are isolated.

4295. Study the p-groups all of whose nonabelian maximal subgroups are isolated.

4296. Study the p-groups all of whose $\mathcal{A}_2$-subgroups are isolated.

4297. Classify the representation groups for a given metacyclic p-group (it is known that, as a rule, such representation groups are metacyclic).

4298. Describe the representation groups of a given abelian p-group (the Schur multipliers of such groups are described by Schur). Describe the minimal nonabelian subgroups of these representation groups.

4299. Describe a Sylow p-subgroup of the holomorph of a given homocyclic p-group.

4300. Find $\mathrm{d}(P)$ and $\mathrm{cl}(P)$, where P is a Sylow p-subgroup of the holomorph of a given abelian p-group.

4301. Study the groups G of exponent $p^e > p$ satisfying (i) $\mathrm{s}_k(G) = \mathrm{s}_{e-k}(G)$ for all $k < e$ (ii) $\mathrm{c}_k(G) = \mathrm{c}_{e-k}(G)$ for all $k < e$.

4302. Given an abelian p-group P, does there exist a p-group $G > P$ such that $P < \Phi(G)$ is G-invariant and $G/P \cong \mathrm{Aut}(P)$?

4303. Find $\mathrm{s}_k(G)$ ($\mathrm{c}_k(G)$), where G is a representation group of a given homocyclic p-group.

4304. Find $\mathrm{s}_k(G)$, $k > 1$, for a given abelian p-group G.

4305. Describe the representation groups of all primary $\mathcal{A}_2$-groups.

4306. Study the p-groups G such that $\mathrm{c}_k(G) = \mathrm{c}_k(A)$ ($\mathrm{s}_k(G) = \mathrm{s}_k(A)$) for all k and a given abelian p-group A.

4307. Find $\mathrm{c}_k(H)$ and $\mathrm{s}_k(H)$, where H is a Sylow p-subgroup of the holomorph of a two-generator abelian p-group.

4308. Study the p-groups G such that $\mathrm{N}_G(H)$ ($\mathrm{C}_G(H)$) is G-invariant for all $H < G$.

4309. Does there exist a nonabelian p-group all of whose maximal abelian subgroups are pairwise non-isomorphic (have pairwise distinct orders)?

4310. Given n, does there exist a p-group containing exactly n pairwise non-isomorphic maximal abelian subgroups?

4311. Study the irregular p-groups G (i) all of whose maximal subgroups, except one H, are isomorphic, (ii) all elements of the set $G - H$ have order p.

4312. Study the p-groups G such that $F \cap H$ is abelian (metacyclic) for any non-incident $F, H < G$ of distinct orders (see #4226).

4313. Study the p-groups G such that $\mathrm{c}_1(G)$ is equal to (i) $\mathrm{c}_1(\Sigma_{p^n})$, (ii) $\mathrm{c}_1(\mathrm{UT}(n,p))$ for $n > 1$.

4314. Let $H = AB$ be a non-metacyclic 2-group, where A, B are cyclic. Study a 2-group G subjecting to (i) $\mathrm{s}_n(G) = \mathrm{s}_n(H)$, (ii) $\mathrm{c}_n(G) = \mathrm{c}_n(H)$ for all n.

4315. Study the nonabelian p-groups G such that, whenever $H < G$ is nonabelian and $A < G$ is maximal abelian, then $H \cap A$ is maximal abelian in H.

4316. Describe the p-groups $|G|$ containing a maximal abelian subgroup A of order p^3. Consider in detail the case when A is (i) cyclic, (ii) elementary abelian.

4317. Study the p-groups G of exponent p such that $\mathrm{C}_G(x) \cong S(p^3) \times \mathrm{C}_p$ for some $x \in G - \mathrm{Z}(G)$. Consider also the case when $\mathrm{C}_G(x) \cong S(p^3) \times \mathrm{E}_{p^n}$ for $n > 1$.

4318. Study the p-groups G that have no minimal nonabelian subgroups S, T with $S \cap T = \{1\}$. Consider the case $\exp(G) = p$ is detail. Is it true that then $|G|$ is bounded?

4319. Study the p-groups G all of whose nonabelian maximal subgroups have trivial Schur multipliers.

4320. Describe the maximal abelian subgroups of the representation groups of the abelian group of type $(p, p^2, \dots, p^n)$. Describe also minimal nonabelian subgroups of this representation group.

4321. Let $H \in \Gamma_1$ be of maximal class, where G is a p-group. Study the structure of H if all elements of the set $G - H$ have order p (by Burnside, we must have $p > 2$).

4322. Estimate the number of representation groups of a given p-group G of maximal class. Consider the case $|G| \geq p^p$ in detail.

4323. Study the 2-groups containing a subgroup $\cong \mathrm{E}_{2^4}$, but not containing a normal subgroup $\cong \mathrm{E}_{2^3}$.

4324. Study the p-groups all of whose maximal cyclic subgroups are nonnormal.

4325. Study the p-groups all of whose maximal regular subgroups, except one, have exponent p.

4326. Study the p-groups all of whose $\mathcal{A}_1$-subgroups, except one, have order p^3.

4327. Study the p-groups all of whose non-quasinormal subgroups are conjugate.

4328. Study the p-groups of exponent p^e covered by minimal nonabelian subgroups of exponent p^e (see the Problem #860 in [BJ2]).

4329. Study the p-groups G covered by $\mathcal{A}_2$-subgroups (non-metacyclic $\mathcal{A}_2$-subgroups).

4330. (Isaacs-Passman) Classify the p-groups G with $|\mathrm{cd}(G)| = 2$.

4331. Study the p-groups G with $|\mathrm{cd}(H)| \leq 2$ for all $H \in \Gamma_1$.

4332. Study the p-groups G such that $\mathrm{cl}(A) \neq \mathrm{cl}(B)$ for any two distinct $A, B \in \Gamma_1$.

4333. Classify the CM-groups [B1, Chapter 9] which are 2-groups.

4334. Study the irregular p-groups G such that $\mathrm{C}_G(x)$ is abelian (absolutely regular) for all $x \in G - \mathrm{Z}(G)$.

4335. Describe the p-groups G of order p^n and exponent $p^e > p$ with maximal possible $\alpha_1(G)$.

4336. (Old problem coinciding, in this partial case, with problem #860 in [BJ2]) Classify the regular p-groups covered by $\mathcal{A}_1$-subgroups.

4337. Classify the metacyclic p-groups all of whose maximal subgroups are isomorphic (pairwise non-isomorphic).

4338. Study the p-groups in which any maximal abelian subgroup is contained in only one member of the set Γ_1 (in that case, $A \cap B$ is abelian for any distinct $A, B \in \Gamma_1$).

4339. Study the p-groups G, satisfying $|\mathrm{Aut}(G)| \leq |G|$ (see [Hor1, Hor2])?

4340. Study the p-groups G such that for any maximal abelian (minimal nonabelian) $B < G$ there is $S \triangleleft G$ with $G/S \cong B$.

4341. Study the p-groups G such that for any metacyclic $M < G$ there is $B \triangleleft G$ such that $G/B \cong M$.

4342. Is it true that if G is a metacyclic p-group with $|G'| = p^n$, then $\alpha_1(G) \le (p+1)^{n-1}$?

4343. Study the p-groups G with minimal nonabelian (i) G', (ii) $\Phi(G)$, (iii) $\Omega_2(G)$.

4344. Study the group of those automorphisms of a p-group G that fix elementwise $\Phi(G)$ (G', $\mathrm{Z}(G)$).

4345. For $\Sigma_{p^n} \in \mathrm{Syl}_p(\mathrm{S}_{p^n})$ estimate the number of maximal chains of nonnormal subgroups.

4346. Describe the subgroup structure of $\Phi(\Sigma_{p^n})$.

4347. Study the automorphism group $\mathrm{Aut}(\Sigma_{p^n})$ and its Sylow p-subgroup.

4348. Estimate the derived length of the p-group which is a product of n pairwise permutable $\mathcal{A}_k$-subgroups, $k = 1, 2$.

4349. Which values of $\alpha_1(G)$ are impossible for p-groups G?

4350. Study a 2-group G such that its Frattini subgroup $\Phi(G)$ has no subgroups $\cong \mathrm{E}_8$.

4351. Study the p-groups G such that for any maximal abelian subgroup $A < G$ and $x \in G - A$ there is $a \in A$ such that $\langle a, x\rangle$ is a metacyclic minimal nonabelian subgroup.

4352. Study the p-groups G such that (i) $G/\Omega_1(G)$ is an $\mathcal{A}_2$-group, (ii) $\Omega_2(G)$ is an $\mathcal{A}_2$-subgroup.

4353. Is it true that any minimal irregular p-group is generated by two elements of equal (different) orders?

4354. Study the p-groups in which every $\mathcal{A}_1$-subgroup has cyclic subgroup of index p^2.

4355. Study the p-groups $G = ST$, where S, T are $\mathcal{A}_1$-subgroups and any maximal subgroup of S is permutable with all maximal subgroups of T.

4356. Study the p-groups covered by nonabelian subgroups with cyclic subgroups of index p.

4357. Study the p-groups all of whose cyclic subgroups are contained in $\mathcal{A}_1$-subgroups (obviously, such groups are covered by their $\mathcal{A}_1$-subgroups; see problem #860 in [BJ2]).

4358. Classify the p-groups all of whose maximal cyclic subgroups are complemented. Consider in detail such metacyclic groups.

4359. Study the p-groups G all of whose subgroups not contained in $\Phi(G)$ are complemented.

4360. Study the p-groups G all of whose nonnormal abelian subgroups $H < G$ are complemented in $\mathrm{N}_G(H)$.

4361. Classify the non-Dedekindian p-groups all of whose nonnormal nonabelian subgroups have the same order.

4362. Study the p-groups G such that, whenever $A < G$ is maximal abelian and $S < G$ is an $\mathcal{A}_1$-subgroup, then $A \cap S \le \mathrm{Z}(S)$.

4363. Classify the 2-groups all of whose maximal Dedekindian subgroups have index ≤ 4.

4364. Study the p-groups G of maximal class with abelian Frattini subgroup $\Phi(G)$. Describe $\mathrm{Aut}(G)$ for this G.

4365. Study the p-groups all of whose nonabelian maximal subgroups are two-generator.

4366. Study the p-groups $G = AB$, where $\exp(A) = \exp(B) = p$ and A, B are not G-invariant.

4367. Study the power and normal structure of a p-group $G = AB$. where A, B are abelian (as N. Ito has shown, the derived subgroup G' is abelian).

4368. Describe the p-groups G admitting a covering $G = B_1 \cup \cdots \cup B_n$ such that $|B_i \cap B_j| \le p$ for all $i \ne j$?

4369. Study the irregular p-groups, $p > 2$, admitting a non-trivial covering by nonabelian regular subgroups all of whose contain an abelian subgroup of index p.

4370. Study the p-groups G admitting a covering by proper nonabelian subgroups of equal order. Consider in detail the case $\exp(G) = p$.

4371. Study the p-groups G in which the subgroup $\Omega_1(G)$ is contained in the join of $\mathcal{A}_1$-subgroups of G.

4372. Study the p-groups G such that for any $H \in \Gamma_1$ one has $\mathrm{N}_G(K) \le H$ for any non-G-invariant $K < H$.

4373. Classify the minimal nonabelian p-groups S such that a Sylow p-subgroup of $\mathrm{Aut}(S)$ is minimal nonabelian.

4374. Does there exist p-groups P, Q of different orders (exponents) such that $\mathrm{Aut}(P) \cong \mathrm{Aut}(Q)$?

4375. Describe the nonabelian p-groups covered by abelian subgroups of type (p^2, p^3).

4376. Study the p-groups in which the centralizer (normalizer) of any maximal cyclic subgroup is abelian.

4377. Study the p-groups G all of whose subgroups of index p^4 are abelian.

4378. Find the number of abelian subgroups of order p^{m+1} in an extraspecial group of order p^{2m+1}. In the same group find the number of subgroups of any given order.

4379. Find all possible numbers of subgroups of order p^3 in a p-group of maximal class of order p^p and exponent p.

4380. Study the p-groups of exponent p without normal subgroup $\cong S(p^3)$.

4381. How many subgroups of maximal class of order $p^k \le p^p$ and exponent p may contain a p-group?

4382. Study the p-groups G with $HH^\phi = H^\phi H$ for any $H < G$ and some p-automorphism $\phi \in \mathrm{Aut}(G)$.

4383. Study the p-groups in which $F \cap H \triangleleft G$ for all distinct $\mathcal{A}_1$-subgroups $F, H < G$.

4384. Estimate in terms A and B the difference $\alpha_1(A \times B) - \alpha_1(\langle A, B\rangle)$, where A and B are p-groups. A similar problem for $|\alpha_1(A * B) - \alpha_1(A \times B)|$.

4385. Describe the nonabelian p-groups G of order p^n and exponent p with maximal $\mathrm{s}_k(G)$, $k < n$.

4386. Is it true that if a p-group G is covered by $\mathcal{A}_1$-subgroups, then the number $|\Omega_1(G)|$ is bounded? Consider the case $\exp(G) = p$ in detail.

4387. Study the nonabelian p-groups G such that $H' = \Phi(G)$ for all nonabelian $H \in \Gamma_1$.

4388. Study the p-groups G such that, whenever an $\mathcal{A}_1$-subgroup $S < G$ and $S < H < G$, then $\mathrm{cl}(H) > 2$.

4389. Study the p-groups G such that $\exp(S^G) = \exp(S)$ for any subgroup ($\mathcal{A}_1$-subgroup) $S < G$ (any regular p-group satisfies the above property; see [B1, §7]).

4390. Study the p-groups G such that C^G is minimal nonabelian (metacyclic) for any maximal cyclic $C < G$.

4391. Study the p-groups G such that $\mathrm{C}_G(A)$ is abelian (metabelian) for any maximal cyclic $A < G$.

4392. Study the p-groups G such that $\mathrm{C}_G(A) = \mathrm{N}_G(A)$ for any maximal cyclic $A < G$.

4393. Study the p-groups G such that $\mathrm{d}(G/S^G) = 2$ for any $\mathcal{A}_1$-subgroup $S < G$.

4394. Study the p-groups G such that every its maximal cyclic subgroup is contained in exactly $p + 1$ members of the set Γ_1.

4395. Study the p-groups G such that S/S_G is an $\mathcal{A}_1$-subgroup for any $\mathcal{A}_1$-subgroup $S < G$. Here $S_G = \bigcap_{x \in G} S^x$.

4396. Study the p-groups G such that $\mathrm{c}_1(G) \le \mathrm{c}_1(G/(\mathrm{Z}(G)))$.

4397. Study the p-groups G such that $|\mathrm{c}_1(G) - \mathrm{c}_1(G/S^G)| \in \{p-1, p, p+1\}$ for any $\mathcal{A}_1$-subgroup $S < G$.

4398. Study the p-groups G such that $\mathrm{C}_G(S)$ is an $\mathcal{A}_2$-subgroup for any $\mathcal{A}_1$-subgroup $S < G$.

4399. Does there exist a p-group G such that $\mathrm{C}_G(H) \cong H$ for two distinct nonabelian regular subgroups $H < G$?

4400. Study the p-groups G containing a subgroup M of maximal class such that $\mathrm{C}_G(M)$ is of maximal class.

4401. Given n, find all primes p such that there exists a p-group covered by n pairwise non-incident subgroups of pairwise different orders.

4402. Study the p-groups G such that $\mathrm{N}_G(A)/A$ is cyclic (abelian) for any nonnormal $A < G$.

4403. Study a p-group G containing a maximal abelian subgroup A such that $A \cap S$ is maximal in S for any $\mathcal{A}_2$-subgroup $S < G$.

4404. Describe the p-groups all of whose maximal abelian and minimal nonabelian subgroups are normal (permutable).

4405. Study the p-groups G such that the centralizer $\mathrm{C}_G(x)$ is irregular for any $x \in G$.

4406. Classify the p-groups G all of whose noncyclic abelian subgroups of exponent $> p$ are normal and cover G.

4407. Describe the p-groups all of whose nonabelian subgroups are normal (all such groups are metahamiltonian).

4408. Study the p-groups such that any their proper nonabelian subgroup has a cyclic subgroup of index $\le p^2$.

4409. Study the nonabelian p-groups G such that any their proper subgroup (abelian subgroup) has a cyclic Frattini subgroup.

4410. Classify the p-groups G satisfying $|\Phi(H)| \le p^2$ for all (all nonabelian) $H \in \Gamma_1$.

4411. Let $S < G$ be a fixed $\mathcal{A}_1$-subgroup of a p-group G and let $B_1, \dots, B_{p+1}$ be all maximal subgroups of S. Study the structure of G provided any maximal abelian subgroup of G c ontaining B_i, $i \le p+1$, has index $\le p^2$ in G.

4412. Given n, find a maximal $k \ge 1$ such that a p-group of order p^m containing a subgroup $\cong \mathrm{E}_{p^n}$ possesses a normal subgroup $\cong \mathrm{E}_{p^k}$.

4413. Study the p-groups G such that, whenever $S < G$ is an $\mathcal{A}_1$-subgroup, then the centralizer of any maximal subgroup of S in G is abelian.

4414. Study the p-groups G such that, whenever $R < G$ is minimal irregular, then any maximal subgroup of R contains their centralizer in G.

4415. Classify the p-groups G such that, whenever $S < G$ is an $\mathcal{A}_1$-subgroup, then $G = S\mathrm{C}_G(A)$, where $A < S$ is maximal.

4416. Study the p-groups G such that for any nonabelian $H < G$ one has $\mathrm{C}_G(x) \not\le H$ for all $x \in H$.

4417. Study the p-groups G containing $H \in \Gamma_1$ such that all maximal cyclic (abelian) subgroups of H are maximal cyclic (abelian) subgroups of G.

4418. Study the p-groups G such that maximal cyclic subgroups of all its $\mathcal{A}_2$-subgroups are maximal cyclic subgroups of G.

4419. Study the p-groups G such that whenever $S < G$ is an $\mathcal{A}_1$-subgroup and $L < S$ is non-G-invariant then $|G : \mathrm{N}_G(L)| \le p^2$.

4420. Study the p-groups all of whose subgroups of derived length 2 have class 2.

4421. Study the p-groups $G = S_1 \dots S_k$, where $S_1, \dots, S_k$ are G-invariant $\mathcal{A}_1$-subgroups. Estimate $\mathrm{dl}(G)$, the derived length of G.

4422. Estimate $\mathrm{k}(G)$, where G is a special group of order p^{z+d} with $p^z = |\mathrm{Z}(G)|$ and $d = \mathrm{d}(G)$.

4423. Describe the p-groups G such that all p-groups containing G as a subgroup of index p have the same derived length (class) (example: $G \cong \mathrm{SD}_{2^n}$, $p = 2$).

4424. Given a p-group H, estimate the minimal (maximal) value of $\mathrm{k}(G)$, where G contains H as a subgroup of index p, in terms of H. In particular, consider the case when H is an $\mathcal{A}_s$-group, $s = 1, 2$.

4425. Describe the p-groups G such that $\mathrm{Aut}(G)$ is a special p-group (example: $G = \mathrm{D}_8$).

4426. Study the p-groups all of whose normal subgroups (normal abelian subgroups) are characteristic.

4427. Study the p-groups G such that $G/\mho_1(G)$ is special.

4428. Study the noncyclic p-groups in which any maximal cyclic subgroup has metacyclic normalizer (the noncyclic p-groups all of whose maximal cyclic subgroups are self centralizing (in that case $\Omega_1(G) = \mathrm{Z}(G)$), do not exist for $p > 2$; however, if $p = 2$, then any such group is a generalized quaternion group).

4429. Study the p-groups all of whose nonnormal nonabelian subgroups are minimal nonabelian (metacyclic).

4430. Given k, describe the p-groups containing all abelian (metacyclic) groups of order p^k (of order $\leq p^k$).

4431. Study the p-groups all of whose maximal subgroups, except one, have the same derived subgroup (Frattini subgroup).

4432. Study the p-groups all of whose maximal subgroups have pairwise distinct Frattini (derived) subgroups.

4433. Study the p-groups all of whose maximal subgroups, except one, have the derived subgroup of order $\leq p^2$.

4434. Study the p-groups G such that, whenever $\chi \in \mathrm{Irr}(G)$, then $\ker(\chi)$ is incident with G'.

4435. Study the p-groups G such that all members of the set Γ_1, except one, have derived subgroup of index p^2 (if $p = 2$, then G is of maximal class; see [B1, §§1, 9]).

4436. Study the p-groups G such that all members of the set Γ_1, except one, are two-generator.

4437. Study the p-groups G such that $\mathrm{cl}(\mathrm{N}_G(H)) = \mathrm{cl}(H)$ for all nonabelian $H < G$.

4438. Study the p-groups G of exponent $> p$ all of whose maximal abelian (regular) subgroups of exponent $> p$ are G-invariant.

4439. Study the p-groups G covered by those abelian subgroups whose exponent coincides with $\exp(G)$.

4440. Construct a p-group G such that $\mho_1(G) = A * B$, where A, B are nonabelian with cyclic centers.

4441. Describe the representation groups of special p-groups.

4442. Is it true that a nonabelian normal Sylow subgroup P of a minimal nonnilpotent group is not a central product of two non-incident subgroups whenever $|P'| > p$?

4443. Does there exist $\max\{\alpha_1(G)\}$, where G runs over all p-groups containing an $\mathcal{A}_1$-subgroup of index p?

4444. Describe the special p-groups G such that every their nonabelian epimorphic image is special.

4445. Study the p-groups all of whose $\mathcal{A}_2$-subgroups are isomorphic (have the same order).

4446. Study the irregular p-groups all of whose maximal absolutely regular (metacyclic) subgroups have the same order.

4447. Study the irregular p-groups covered by normal $\mathcal{A}_1$-subgroups (metacyclic subgroups).

4448. Study the irregular p-groups covered by normal maximal regular subgroups.

4449. Suppose that a p-group G contains a minimal nonabelian subgroup S. Study the structure of G if $S \cap C > \{1\}$ for any maximal cyclic $C < G$. Consider, in particular, the case when S runs over all minimal nonabelian subgroups of G.

4450. Describe the p-groups G of exponent $> p^k$ all of whose nonnormal subgroups have exponent $\le p^k$.

4451. Study the p-groups G in which every member of the set Γ_2 contains at most $p+1$ minimal nonabelian subgroups (the p-groups X with $\alpha_1(X) \le p+1$ are known).

4452. Study the irregular p-groups all of whose maximal regular subgroups are quasinormal.

4453. Study the p-groups G in which any maximal cyclic subgroup is contained in only one maximal regular subgroup of G.

4454. Study the non-Dedekindian p-groups G such that G/L^G is cyclic (abelian, elementary abelian) for any nonnormal (nonnormal cyclic) $L < G$.

4455. Study the irregular p-groups all of whose regular subgroups are not isolated.

4456. Study the p-groups G covered by isolated (non-isolated) proper subgroups.

4457. Find all those n for which $\alpha_2(G) = n$ is impossible for all p-groups G.

4458. Find all possible values $\mu_k(G) \pmod{p^2}$, where $\mu_k(G)$ is the number of subgroups of maximal class and order p^k in a group G of exponent p (in the case under consideration $k \le p$).

4459. Let S be a fixed $\mathcal{A}_1$-subgroup of a p-group G. Suppose that $|\langle S, T\rangle : S| \le p^2$ for any $\mathcal{A}_1$-subgroup T of G. Study the structure of G. Consider a partial case where S runs over all $\mathcal{A}_1$-subgroups of G.

4460. Study the p-groups G that are not generated by $\mathcal{A}_1$-subgroups of maximal order.

4461. Study the p-groups G such that $\mathrm{C}_G(A)$ is abelian for all maximal abelian subgroups A of any $\mathcal{A}_2$-subgroup $S < G$.

4462. Study the p-groups G such that $\mathrm{C}_G(x)$ is abelian for any $x \in H - \mathrm{Z}(H)$, where H runs over all nonabelian subgroups of G. Consider the case where H runs over all minimal nonabelian subgroups of G.

4463. Study the p-groups G such that $\mathrm{C}_G(x)$ is abelian for any $x \in G - \Phi(G)$ ($x \in G - G'$).

4464. Study the p-groups G all of whose subgroups of index p^2 are either normal or abelian.

4465. Classify the metacyclic (absolutely regular) p-groups G such that $\mathrm{Aut}(G)$ is metacyclic (absolutely regular) p-group.

4466. Let A be a metabelian p-group. Describe the p-groups G such that $\mathrm{c}_n(G) = \mathrm{c}_n(A)$ ($\mathrm{s}_n(G) = \mathrm{s}_n(A)$) for all n.

4467. Let H be a p-group of maximal class. Describe the p-groups G such that $\mathcal{L}(G) \cong \mathcal{L}(H)$, where $\mathcal{L}(G)$ is the lattice of all subgroups of G. Is it true that if H has an abelian subgroup of index p so is G?

4468. Study the p-groups G such that $|G : \mathrm{N}_G(L)| \le p$ for all subgroups (proper subgroups) L of minimal nonabelian subgroups of G (see #4670 below).

4469. Study the p-groups G all of whose abelian subgroups are either elementary abelian or have a cyclic subgroup of index p.

4470. Describe the nonabelian p-groups G such that the centralizers of all $x \in G - Z(G)$ are either abelian or minimal nonabelian.

4471. Describe the nonabelian p-groups G such that the centralizers of all maximal subgroups of their minimal nonabelian subgroups are either abelian or minimal nonabelian.

4472. Describe the group $\mathrm{Aut}(G)$, where G is a group of maximal class and order 3^n, $n > 4$, with minimal nonabelian subgroup of index 3.

4473. Given $n > 1$, study the p-groups G such that any $H < G$ containing a fixed $\mathcal{A}_1$-subgroup of index p is an $\mathcal{A}_n$-subgroup.

4474. Given n and an $\mathcal{A}_1$-group S which is a p-group, find $\mathrm{c}_n(P)$, where P is a Sylow p-subgroup of $\mathrm{Aut}(S)$.

4475. Find $\alpha_1(P)$, where P is from problem #4474 in this paper.

4476. Given a metacyclic p-group M, find $\alpha_1(P)$, where P is a Sylow p-subgroup of $\mathrm{Aut}(M)$.

4477. Study the p-groups G such that $|G : \mathrm{C}_G(x)| \leq p^2$ for any $x \in G - G'$.

4478. Let $G = AB$, where A and B are abelian p-groups. Find the maximal order of its section of exponent p in terms of A and B. Do the same if A and B are metacyclic p-groups.

4479. Find the minimal order of a p-group containing all types of p-groups of order p^4 (of abelian groups of order p^4).

4480. Find the minimal order of a p-group containing all types of abelian groups of order p^5.

4481. Given n, how many there exist minimal nonabelian (metacyclic) groups of order p^n (of order $\leq p^n$).

4482. Estimate the maximal possible value of $\alpha_2(G)$ in a p-group of order p^n.

4483. Given $n > 1$, find $\alpha_1(P)$, where P is a Sylow 2-subgroup of the automorphism group of the abelian group of type $(2, 2^2, \dots, 2^n)$.

4484. Describe the group $\mathrm{Aut}(G)$, where the group G is a non-metacyclic product of two cyclic 2-groups.

4485. Does there exist a p-group all of whose maximal abelian subgroups have pairwise distinct orders?

4486. Does there exist an irregular p-group all of whose maximal regular subgroups have pairwise distinct orders?

4487. Does there exist a p-group G such that, whenever $A, B \in \Gamma_1$ are distinct, then $\alpha_1(A) \neq \alpha_1(B)$.

4488. Given a group H of exponent p and a given order, does there exist a p-group G of exponent $> p$ such that $|G'| > |H'|$ and $\alpha_1(G) = \alpha_1(H)$?

4489. Does there exist a group of exponent p and order $> p$ all of whose maximal subgroups are characteristic?

4490. Study the p-groups G such that, whenever $A, B < G$ are cyclic and $A \cap B = \{1\}$, then there exist nonabelian $U, V \leq G$ with $A < U$, $B < V$ and $U \cap V = \{1\}$.

4491. Let G be a group of maximal class of order 3^n with nonabelian fundamental subgroup G_1. Describe the 3-groups H such that $\alpha_1(H) = \alpha_1(G)$.

4492. Suppose that a p-group G possesses a minimal nonabelian subgroup S of index p^n. Estimate $\alpha_1(G)$ and $\alpha_2(G)$.

4493. Given an $\mathcal{A}_1$-group S of order p^n, estimate maximal possible value of $\alpha_2(P)$, where P is a Sylow p-subgroup of $\mathrm{Aut}(S)$.

4494. Given a p-group P, is it possible to estimate $|\Omega_1(Q)|$, where $Q \in \mathrm{Syl}_p(\mathrm{Aut}(P))$ in the terms of $|\Omega_1(P)|$?

4495. Let $\mathrm{C}_p \cong C \lhd G$ and let G/C be a minimal nonabelian p-group. Estimate $\alpha_1(G)$ in the terms G/C. Consider also the case when $C \cong \mathrm{C}_{p^n}$.

4496. Let $S \lhd G$ be a minimal nonabelian p-group and let $G/S \cong \mathrm{C}_{p^n}$. Estimate $\alpha_1(G)$ is terms of S.

4497. Does there exist a special p-group G such that $\mathrm{C}_G(x)$ is special for all $x \in G - \mathrm{Z}(G)$?

4498. Does there exist a p-group G such that $\Phi(G)$ is special and $\Phi(H)$ is extraspecial for all $H \in \Gamma_1$?

4499. Study the p-groups G such that $\Phi(\Phi(G))$ is special.

4500. Classify the p-groups containing a cyclic maximal abelian subgroup.

4501. Classify the p-groups containing exactly one abelian maximal regular subgroup (any irregular p-group of maximal class of order $> p^{p+1}$ with an abelian subgroup of index p satisfies this condition).

4502. Does there exist an irregular p-group G such that $\mathrm{C}_G(x)$ is irregular for all $x \in G - \mathrm{Z}(G)$? If the answer is 'yes', describe such G.

4503. Study the p-groups H such that there exists a p-group G satisfying (i) $G' \cong H$, (ii) $\Phi(G) \cong H$, (iii) $\mho_1(G) \cong H$.

4504. Study the p-groups G such that $G/\mho_n(G)$ is minimal nonabelian for some $n \geq 1$.

4505. Classify the p-groups all of whose minimal nonabelian subgroups are metacyclic and have cyclic centers (see §238 in [BJ5]).

4506. Study the p-groups G such that $G/\mathrm{Z}(G)$ is metacyclic minimal nonabelian.

4507. Study the nonabelian p-groups all of whose nonabelian subgroups are (i) pairwise non-isomorphic, (ii) normal, (iii) quasinormal.

4508. Does there exist a nonabelian p-group G such that $S_G = \{1\}$ for all minimal nonabelian $S < G$?

4509. Find $\alpha_1(S \times M)$, where $S \times M$ is primary, S is an $\mathcal{A}_1$-group and M is metacyclic $\mathcal{A}_2$-group.

4510. Study the two-generator p-groups G with $G' \cong \mathrm{C}_{p^n}$ (see [BJ2, Proposition 72.1], where G is metacyclic).

4511. Study the p-groups G such that $G' = \mathrm{Z}(G) \in \{\mathrm{C}_{p^n}, \mathrm{E}_{p^n}\}$ and G/G' is homocyclic.

4512. Classify the p-groups containing ≥ 2 distinct extraspecial (special) subgroups of index p.

4513. Classify the p-groups all of whose nonabelian maximal subgroups are special (see [Cos]).

4514. Describe the irregular p-groups in which the centralizer of any maximal regular subgroup is abelian.

4515. Study the p-groups G such that $G/\Phi(\Phi(G))$ is special.

4516. Study the p-groups G with a special $\Phi(G)$ (G').

4517. Study the p-groups G with a special $\Omega_1(G)$.

4518. Study a Sylow p-subgroup of the automorphism group of a special p-group.

4519. Given a special p-group H, does there exist a special p-group containing H as a subgroup of index p?

4520. Given an absolutely regular p-group A with $|\Omega_1(A)| = p^{p-1} < |A|$, does there exist an absolutely regular p-group $G > A$ subjecting $|G : A| = p$?

4521. Describe the p-groups M of maximal class such that there exists a p-group $G > M$ of maximal class such that $|G : M| = p$ (any group containing SD_{2^n} as a subgroup of index 2 is not of maximal class).

4522. Find the maximal possible value of $\alpha_1(G)$, where G runs over all (i) groups of maximal class and order p^n, (ii) metacyclic groups of order p^n.

4523. Study the p-groups G satisfying $|\mathrm{Aut}(G)|_p \leq |G|$.

4524. Find $\mathrm{cl}(P)$, where P is a Sylow p-subgroup of the automorphism group of a given abelian p-group A. Consider in detail the case when A is homocyclic.

4525. Estimate the maximal number of pairwise noncommuting elements in a group of order p^n. Consider in detail the case when a group has exponent p.

4526. Study the groups G of order p^p admitting an automorphism of order p^2. For which p and G this is possible?

4527. Let $\delta(G)$ be the minimal degree of a representation of a group G by permutations. Study the p-groups G such that $\delta(G) = \delta(H)$ for some $H \in \Gamma_1$.

4528. Study the p-groups G such that $\delta(F) = \delta(H)$ for all $F, H \in \Gamma_1$.

4529. Study the p-groups G such that $\delta(G/F) = \delta(G/H)$ for all $F, H \in \mathrm{Z}(G)$ of order p.

4530. Study the p-groups G such that $\alpha_k(F) = \alpha_k(H)$ for all $F, H \in \Gamma_1$ and all k.

4531. Study the p-groups G in which the normalizers of all their nonnormal cyclic subgroups are metabelian (metacyclic).

4532. Study the p-groups G such that, whenever $H < G$ is nonnormal, then $\mathrm{N}_G(H)/H$ is cyclic.

4533. Let H be a normal nonabelian subgroup of index p^2 of a p-group G. Estimate $\rho(H) = \alpha_1(G) - \alpha_1(H)$. It is easy to show that if $G/H \cong \mathrm{E}_{p^2}$, then $\rho(H) \geq p^2 - 1$. What will be in that case provided $\rho(H) = p^2 - 1$? Consider also the case $G/H \cong \mathrm{C}_{p^2}$.

4534. Let $H \geq G'$ be a maximal abelian subgroup of index p^n of a (metabelian) p-group G. Estimate $\alpha_1(G)$.

4535. Estimate, for known p-groups G, the number $|\alpha_1(G) - \alpha(G/L)|$, where $L \leq \mathrm{Z}(G)$ is of order p (of index p).

4536. Study the p-groups G containing a nontrivial normal subgroup H such that $\mathrm{C}_G(x)$ is abelian (metacyclic) for all $x \in G - H$.

4537. Study the p-groups G containing a nontrivial subgroup F such that $\mathrm{C}_G(H)$ is abelian (metacyclic) for all $H < G$ non-incident with F.

4538. Study the p-groups G such that $\mathrm{C}_G(x)$ is abelian for all $x \in G - G'$ ($x \in G - \mho_1(G)$).

4539. Study the p-groups $G > \Omega_1(G)$ such that, whenever $x \in G - \Omega_1(G)$, then $\mathrm{C}_G(x)$ is abelian (minimal nonabelian, metacyclic, absolutely regular).

4540. Study the p-groups $G > \Omega_2(G)$ such that, whenever $x \in G - \mho_2(G)$, then $\mathrm{C}_G(x)$ is abelian (minimal nonabelian, metacyclic, absolutely regular).

4541. Study the p-groups G satisfying $\mho_1(G) = \mathrm{Z}(G)$ ($\mho_2(G) = \mathrm{Z}_2(G)$).

4542. Given a p-group G, find n such that $\mathrm{s}_n(G) \geq \mathrm{s}_k(G)$ for all k. Consider the following cases: G is minimal nonabelian, metacyclic, of maximal class.

4543. Study the p-groups G satisfying $S \cap \mathrm{Z}(G) = S'$ for all minimal nonabelian $S < G$.

4544. Study the p-groups G that are not generated by $\alpha_1(G) - 2$ minimal nonabelian subgroups (see #4143).

4545. Describe a Sylow p-subgroup of $\mathrm{Aut}(B)$, where B is abelian group of type $(p, p^2, \dots, p^n)$. Also describe all representation groups of B.

4546. Study the minimal irregular p-groups G such that $H' = \Phi(H)$ for all (for all nonabelian) $H \in \Gamma_1$.

4547. Study the minimal irregular p-groups G such that (i) $\mathrm{d}(F) = \mathrm{d}(H)$, (ii) $\mathrm{d}(F) \neq \mathrm{d}(H)$ for all $F, H \in \Gamma_1$.

4548. Classify the metacyclic p-groups G such that $\mathrm{Aut}(G)$ is a p-group (is metacyclic).

4549. Study the p-groups G of maximal class such that $\mathrm{Aut}(G)$ is a p-group.

4550. Study the absolutely regular p-groups G such that a Sylow p-subgroup of $\mathrm{Aut}(G)$ is irregular.

4551. Classify the p-groups G such that $\mathcal{L}_N(G) \cong \mathcal{L}_N(H)$, where H is a homocyclic (abelian) p-group (here $\mathcal{L}_N(G)$ is the lattice of normal subgroups of G).

4552. Study the p-groups that are lattice isomorphic to a p-group of maximal class with an abelian subgroup of index p (an abelian Frattini subgroup).

4553. Study the irregular p-groups that are lattice isomorphic with regular p-groups (the set of such groups is nonempty).

4554. Study the p-groups that are lattice isomorphic to an $\mathcal{A}_n$-group for ≤ 3.

4555. Estimate the derived length of a product of n pairwise permutable cyclic (abelian, minimal nonabelian, metacyclic) subgroups.

4556. Describe $\mathrm{Aut}(G)$ and the representation group of G, where $G = UV$ is a product of two cyclic p-subgroups.

4557. Describe $\mathrm{Aut}(G)$, where $G = \mathrm{C}_{p^n} \wr \mathrm{C}_p$ is of order p^{pn+1}. Describe the representation group of this group.

4558. (i) Classify the p-groups G such that $\mathrm{C}_G(S)$ is of maximal class for a minimal nonabelian $S < G$. (ii) Consider in detail the case where $\mathrm{C}_G(S) = \mathrm{Z}(S)$ for a minimal nonabelian $S \leq G$.

4559. Study the p-groups G such that $|G : \mathrm{C}_G(x)| = |G'|$ for all $x \in G - G'$.

4560. Study the p-groups G in which the centralizer of any nonabelian subgroup is abelian.

4561. Study the p-groups covered by the centralizers of their (i) minimal nonabelian subgroups (maximal metacyclic subgroups), (ii) metacyclic minimal nonabelian subgroups.

4562. Classify the p-groups G such that, whenever $S \leq G$ is minimal nonabelian, then any nonnormal subgroup of S is complemented in G. Do this in the case when S runs over all minimal nonabelian subgroups of G.

4563. Study the p-groups G such that $G/\Omega_1 G)$ is special (example: $G = \mathrm{Q}_{2^4}$).

4564. Study the primary $\mathcal{A}_n$-groups G such that $G/\mathrm{Z}(G)$ is an $\mathcal{A}_n$-group.

4565. Study the p-groups in which the centralizer of any nonabelian (minimal nonabelian) subgroup is metacyclic.

4566. Study the p-groups in which (i) the centralizer of any maximal metacyclic subgroup is abelian (minimal nonabelian), (ii) all metacyclic subgroups are abelian.

4567. Study the p-groups in which any nonabelian subgroup has the noncyclic center.

4568. Study the p-groups in which any $\mathcal{A}_2$-subgroup has the cyclic center.

4569. Study the p-groups covered by proper subgroups with cyclic (noncyclic) centers.

4570. Describe the group $\langle \phi \in \mathrm{Aut}(G) \mid S^\phi = S \rangle$, where $S < G$ runs over all maximal abelian (minimal nonabelian) subgroups and $\pi(\mathrm{o}(\phi)) = \{p\}$.

4571. Classify the p-groups in which the centralizer of any nonnormal subgroup is abelian.

4572. Let a p-group $G = Z_1 \dots Z_n$, where $Z_1, \dots, Z_n$ are pairwise permutable cyclic subgroups. Estimate the derived length of G and describe its minimal nonabelian subgroups.

4573. Study the p-groups G all of whose characteristic subgroups coincide with members of the derived series of G.

4574. Estimate $\alpha_1(G)$, where $G = S_1 \dots S_k$ and all S_i are G-invariant minimal nonabelian subgroups.

4575. Study the subgroup and normal structure of a p-group $G = S * T$ (central product), where S and T are minimal nonabelian subgroups. Describe the group $\mathrm{Aut}(G)$.

4576. Study the subgroup and normal structure of a p-group $G = A * B$ (central product), where A is of maximal class and B is minimal nonabelian. Describe the group $\mathrm{Aut}(G)$.

4577. Let A be a maximal normal abelian subgroup of a nonabelian p-group G. Estimate $\alpha_1(G)$ in terms of A and $|G : A|$. Consider in detail the case $p = 2$.

4578. Study the automorphism group of the 2-group $M * C$, where M is of maximal class and C is cyclic.

4579. Study the automorphism group of the p-group $S * C$, where S is minimal nonabelian and C is cyclic.

4580. Describe the automorphism group of a p-group G such that there is a cyclic $L \triangleleft G$ satisfying $G/L \cong S * C$, where S is minimal nonabelian and C is cyclic. Consider in detail the case $p = 2$.

4581. Describe the two-generator p-groups G such that $G/\mathrm{Z}(G)$ is of maximal class with abelian subgroup of index p. Consider in detail the case $p = 2$.

4582. Study the p-groups $G = M * C$, where M is irregular group of maximal class and $C = \mathrm{Z}(G)$ is elementary abelian. Describe the structure of the automorphism group of G in the case $p = 2$.

4583. Study the two-generator p-groups G such that $G/\Omega_1(G)$ is of maximal class. Consider in detail the case $p = 2$.

4584. Let G be a metacyclic group (irregular group of maximal class with abelian subgroup of index p) of order p^n and exponent p^e. Estimate $\mathrm{s}_k(G)$ for all k.

4585. Let G be a nonabelian metacyclic group of order p^n and exponent p^e containing the maximal possible number of normal (minimal nonabelian) subgroups among the groups subjecting the above conditions. Describe the structure of G provided $\mathrm{cl}(G) = c$.

4586. Study the p-groups G in which the number of G-classes of minimal nonabelian subgroups is equal to the number of nonabelian members of the set Γ_1 (in that case, if $S < G$ is an $\mathcal{A}_1$-subgroup, then exactly one member of the set Γ_1 contains S; also, the intersection of any two distinct members of the set Γ_1 is abelian).

4587. Study the non-metacyclic (irregular) p-groups G in which any maximal metacyclic (regular) subgroup is contained in exactly one member of the set Γ_1.

4588. Describe the automorphism group of a minimal irregular p-group. Consider in detail the irregular groups of order p^{p+1}.

4589. Study the subgroup structure of a p-group G such that $\mho_1(G)$ is cyclic (metacyclic, absolutely regular, irregular).

4590. Let G be a group of order p^n and exponent p. Let $1 < k < n-1$ be such that there is in G only one normal subgroup of order p^k. Study the structure of G.

4591. Describe the p-groups G of exponent p such that a given p^k does not divide $|\mathrm{Aut}(G)|$.

4592. Let a p-group $G = A * B$ (central product), where $A, B < G$. Estimate $\alpha_1(G)$ in the terms of $\alpha_1(A)$ and $\alpha_1(B)$.

4593. Study the p-groups G of exponent $> p$ admitting an automorphism of order p fixing all elements of order $> p$ of G.

4594. Let a p-group $G = S_1 * \cdots * S_k$ (central products), where all S_i are minimal nonabelian subgroups, $S_i \cap S_j = S_i' = S_j'$ for $i \neq j$. Describe the group $\mathrm{Aut}(G)$. The same question in the case where that p-group is the direct product of $S_1, \dots, S_k$.

4595. Classify the nonabelian p-groups G such that for any minimal nonabelian $S \leq G$ one has $\mathrm{C}_G(\mathrm{C}_G(S)) = S$. In particular, consider the same question in the case when S runs over all nonabelian subgroups of G.

4596. Study the nonabelian p-groups G such that $\mathrm{N}_G(\mathrm{N}_G(S)) = G$ for any $S < G$ (any minimal nonabelian $S < G$).

4597. Study the p-groups G containing a nonabelian subgroup H such that $\mathrm{C}_G(A)$ is abelian for any maximal abelian subgroup A of H. In particular, consider a partial case when H runs over all nonabelian (minimal nonabelian) subgroups of G.

4598. For a p-group G study the subgroup $A \leq \mathrm{Aut}(G)$ that consists from all automorphisms that left out all normal subgroups of G.

4599. Study the p-groups G containing exactly $p+1$ characteristic subgroups of index p (it is possible that $\mathrm{d}(G) > 2$) (example: $G = \mathrm{SD}_{2^n}$ for $p = 2$).

4600. Study the p-groups G containing exactly p characteristic subgroups of order p. Consider in detail the case when $\mathrm{d}(\Omega_1(G)) > 2$.

4601. Describe the p-groups G satisfying $\alpha_1(G) = p + 2$ (any 2-group of maximal class and order 2^5 satisfies that condition). Find all n such that there is no p-group G satisfying $\alpha_1(G) = p + n$.

4602. Describe the structure of a p-group containing a nonabelian subgroup H such that whenever A is a maximal abelian subgroup of H, then $\mathrm{C}_G(A)$ is abelian (metabelian). Consider a partial case when the above is true for any nonabelian subgroup of G.

4603. Study the p-groups G in which any two distinct minimal nonabelian subgroups generate a subgroup of maximal class (any p-group G of maximal class of order $> p^3$ with an abelian subgroup of index p satisfies that condition).

4604. Study the pairs of nonabelian p-groups $H < G$ such that, whenever $x \in G - H$, then $\mathrm{C}_H(x)$ is metacyclic.

4605. Study the p-groups G such that, whenever $x \in \Phi(G) - \mho_1(\Phi(G))$, then $\mathrm{C}_G(x) \in \Gamma_1$. Consider also the case when $\Phi(G)$ is replaced by G'.

4606. Given a p-group G, study the group of those automorphisms of G that fix all non-G-invariant cyclic subgroups of G (of $\Phi(G)$).

4607. Describe the group $\mathrm{Aut}(M_1 \times M_2)$ (its Sylow p-subgroup), where M_1 and M_2 are metacyclic p-groups. Consider also the case where $G = M_1 * M_2$ (central product) of such factors.

4608. Study the p-groups G all of whose subgroups of $\mho_1(G)$ are G-invariant.

4609. Find a condition sufficient for a p-group G to contain a proper normal irregular subgroup of index $\leq p^2$.

4610. Study the p-groups G all of whose nonnormal subgroups of index p^4 are abelian (regular).

4611. Estimate the class (derived length) of a p-group which is a product of n pairwise permutable $\mathcal{A}_2$-subgroups.

4612. Estimate the derived length of a p-group which is a product of n pairwise permutable subgroups of given derived lengths.

4613. Describe $\mathrm{Aut}(M_1 \times M_2)$ $(\mathrm{Aut}(M_1 * M_2))$, where M_1 and M_2 are metacyclic (absolutely regular, minimal nonabelian, of maximal class) p-groups (consider all possibilities for $M_1 \cap M_2$).

4614. Let A and B be p-groups. Estimate the index $|Aut(A \times B) : (\mathrm{Aut}(A) \times \mathrm{Aut}(B))|$. Consider in detail the cases when A and B are abelian (minimal nonabelian, metacyclic).

4615. Describe the structure of the holomorph of a given abelian (minimal nonabelian, metacyclic) p-group.

4616. Given p, describe $\mathrm{Aut}(\Sigma_2)$, where $\Sigma_2 \in \mathrm{Syl}_p(\mathrm{Sym}_{p^2})$.

4617. How many subgroups of order p^p and exponent p may have a p-group of maximal class and order $p^n > p^p$? The same question for subgroups of order p^{p-1} instead of p^p.

4618. Study the p-groups G such that $\exp(G) < \exp(\mathrm{Aut}G))_p$ (any E_{p^n} satisfies the above inequality for $n > p$).

4619. Estimate the order of the automorphism group of a p-group G of maximal class and order p^n.

4620. Describe the absolutely regular p-groups G with irregular Sylow p-subgroups of their automorphism groups. Estimate $|\Omega_1(G)|$.

4621. Describe the p-groups G satisfying $|\mathrm{Aut}(G)|_p = |G|$ ($|\mathrm{Aut}(G)|_p < |G|$).

4622. Study the p-groups all of whose proper nonabelian subgroups are partitioned by proper subgroups.

4623. Study the p-groups all of whose absolutely regular subgroups are abelian. Describe the p-groups of maximal class satisfying the above condition.

4624. Study the irregular p-groups all of whose subgroups of exponent p possess an abelian subgroup of index p.

4625. Study the p-groups whose automorphism group is an absolutely regular p-group.

4626. Study the p-groups that have no automorphisms of order p^2.

4627. Study the p-groups such that a Sylow p-subgroup of its automorphism group is of maximal class.

4628. Study the p-groups all of whose regular subgroups have order $\leq p^{p+1}$ (if all regular subgroups of an irregular p-group G have order $\leq p^p$, then $|G| = p^{p+1}$).

4629. Study the p-groups all of whose metacyclic subgroups have order $\leq p^4$.

4630. Study the p-groups all of whose absolutely regular subgroups have order $\leq p^{p+1}$.

4631. Study the p-groups G satisfying $\mathrm{c}_n(G) \leq 2p$ for all $n > 1$.

4632. Let G be a group of order p^n, $n > 2$ and $1 < e < n$. Study the p-groups satisfying (i) $\mathrm{s}_e(G) = p + 1$, (ii) $\mathrm{c}_e(G) = p$.

4633. Study the p-groups G such that $G/L \cong \mathrm{M}_{p^n}$ for some G-invariant cyclic subgroup $L < \Phi(G)$. Moreover, describe the p-groups G such that G/L is metacyclic for some G-invariant cyclic $L < \Phi(G)$.

4634. Study the p-groups G, $p > 2$, such that G/L is of maximal class for some G-invariant cyclic $L < \Phi(G)$.

4635. Study the p-groups G all of whose two non-incident subgroups have a G-invariant intersection.

4636. Does there exist a p-group in which any two nonnormal nonabelian subgroups are not permutable?

4637. Study the p-groups such that the intersection of any two their subgroups is normal in one of them.

4638. Study the two-generator p-groups G such that $G/\mathrm{Z}(G)$ is of maximal class. Consider also the case when, in addition, $G/\mathrm{Z}(G)$ has an abelian subgroup of index p. Describe, in the second case, the group $\mathrm{Aut}(G)$.

4639. Describe the group $\mathrm{Aut}(G*H)$ in the terms of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$.

4640. Study the p-groups G such that $G/\mathrm{Z}(G)$ is absolutely regular.

4641. Study the p-groups G such that $|U/U_G| \leq p$ for any (any cyclic) $U < G$.

4642. Study the p-groups in which any two-generator subgroup is either of exponent p, or minimal nonabelian, or metacyclic, or of maximal class.

4643. Study the p-groups G such that $\mathrm{Aut}(\mathrm{Aut}(G))$ is a p-group. Consider in detail the case when G is metacyclic (in particular, a 2-group of maximal class), homocyclic, minimal nonabelian. Is it true that then $p = 2$?

4644. Study the p-groups $\mathrm{Aut}(P)$, where P is a Sylow p-subgroup of the automorphism group of a given homocyclic (abelian) p-group.

4645. Study the p-groups G such that any their minimal nonabelian subgroup is contained in exactly one maximal regular subgroup of G (in that case, the intersection of any two distinct maximal regular subgroups of G is abelian).

4646. Estimate the number of (i) irregular groups of order p^{p+1}, (ii) minimal irregular groups of order p^{p+2}.

4647. Study the p-groups of exponent $> p$ all of whose maximal abelian subgroups of exponent $> p$ are normal.

4648. Study the p-groups all of whose maximal subgroups except one, are regular (irregular).

4649. Study the p-groups with a self centralizing cyclic subgroup (an abelian subgroup of type (p, p^n)).

4650. Study the p-groups G such that $|\mathrm{N}_G(S)| = p|S|$ for any minimal nonabelian $S < G$.

4651. Study the p-groups G such that for all nonabelian $H \in \Gamma_2$ one has $\mathrm{d}(G) = \mathrm{d}(H)$.

4652. Given a non-Dedekindian p-group G, study the group of all automorphisms of G that left out all nonnormal subgroups of G.

4653. Study the structure of the automorphisms groups (i) $\mathrm{Aut}(\Sigma_{p^n})$ and (ii) $\mathrm{Aut}(\mathrm{UT}(n, p))$.

4654. Study the nonabelian p-groups G such that if $A < G$ is maximal abelian, then all invariants of A are pairwise distinct.

4655. Study the irregular p-groups G of exponent $p^e > p$ such that, whenever R is a maximal regular subgroup of G of exponent p^e, then $|R : \mho_1(R)| > |\mho_1(R) : \mho_2(R)| > \cdots > |\mho_{e-1}(R) : \mho_e(R)|$, i.e., R is pyramidal (see [B1, §8]).

4656. Study the p-groups G satisfying $\mathrm{cl}(G) = \mathrm{dl}(G)$.

4657. Describe the subgroup structure of the automorphism group of a p-group G of class 2. In particular, consider the case when $G \in \mathrm{Syl}_2(\mathrm{Sz}(2^{2n+1}))$.

4658. Study the nonabelian p-groups G such that if $S < G$ is minimal nonabelian and $A < S$ is maximal, then $\mathrm{C}_G(A)$ is G-invariant abelian.

4659. Study the irregular p-groups G, $p > 2$, such that if $R < G$ is minimal irregular and $A < R$ is maximal, then any maximal regular subgroup containing A (i) has index p in G, (ii) is G-invariant.

4660. Study the p-groups G such that, whenever $M \in \Gamma_1$, then M is a product of two abelian subgroups.

4661. Describe the automorphism group of a two-generator 2-group G such that it is an extension of a group $\cong \mathrm{E}_4$ by a metacyclic group.

4662. Study the 2-groups G such that $\Omega_2(G)$ is isomorphic to the minimal non-metacyclic group of order 2^5.

4663. Study the two-generator 2-groups G such that $\Omega_2(\Phi(G))$ is isomorphic to the minimal non-metacyclic group $\mathrm{Q}_8 \times \mathrm{C}_2$.

4664. Describe the two-generator 2-groups that are extensions of an elementary abelian subgroup by a group of maximal class.

4665. Describe the 2-groups that are extensions of a group of maximal class by a group of maximal class.

4666. Describe the 2-groups containing a minimal nonabelian subgroup of order 2^5 and index 2.

4667. Study the p-groups G such that a Sylow p-subgroup of $\mathrm{Aut}(G)$ is absolutely regular.

4668. Study the p-groups G such that, whenever $S \leq G$ is minimal nonabelian, then all subgroups (all maximal subgroups) of S are quasinormal in G.

4669. Study the p-groups G such that, whenever $S \leq G$ is an $\mathcal{A}_2$-subgroup, then all maximal subgroups of S are quasinormal in G.

4670. Study the p-groups G such that, whenever $S < G$ is minimal nonabelian, then $\mathrm{C}_G(x)$ is abelian for any $x \in G - S$.

4671. Study the nonabelian p-groups G such that, whenever $A < G$ is maximal abelian, then $\mathrm{C}_G(x)$ is abelian for any $x \in G - A$.

4672. Study the non-metacyclic p-groups G such that, whenever $M < G$ is maximal metacyclic, then $\mathrm{C}_G(x)$ is abelian (metacyclic) for any $x \in G - M$.

4673. Study the p-groups G containing a subgroup M of maximal class such that $\mathrm{C}_G(x)$ is abelian (metacyclic) for any $x \in G - M$.

4674. Study the irregular p-groups G, $p > 2$, containing a subgroup M of maximal class such that any regular subgroup $R < G$ not contained in M is metacyclic.

4675. Describe the group $\mathrm{Aut}(M \times \mathrm{C}_{p^n})$, where M is a metacyclic p-group.

4676. Describe the group $\mathrm{Aut}(\mathrm{C}_{p^n} \times \mathrm{E}_{p^k})$.

4677. Study the p-groups G such that any $A < G$ is permutable with all its conjugates in G. Consider also the case when that A runs over all minimal nonabelan subgroups of G.

4678. Describe the group $\mathrm{Aut}(M \times \mathrm{C}_p)$, where M is an irregular p-group of maximal class.

4679. Let G be a special p-group. Describe the group A consisting of all automorphisms of G that fix all minimal nonabelian subgroups of G.

4680. Study the p-groups P such that, whenever $P \in \mathrm{Syl}_p(G)$, then the group G is p-nilpotent.

4681. Study the p-groups covered by extraspecial (special) subgroups.

4682. Study the p-groups covered by subgroups of maximal class.

4683. Study the p-groups G covered by cyclic subgroups of exponent $\exp(G)$.

4684. Does there exist a p-group covered by centralizers of their minimal nonabelian subgroups?

4685. Study the p-groups all of whose maximal abelian subgroups are complemented in their normalizers.

4686. Study the non-metacyclic p-groups all of whose maximal metacyclic (minimal nonabelian) subgroups are complemented in their normalizers.

4687. Study the non-metacyclic p-groups all of whose maximal metacyclic subgroups are normal.

4488. Describe the p-groups without special sections.

4489. Describe the p-groups G all of whose epimorphic images of order $\frac{1}{p}|G|$ are special.

4690. Classify the p-groups G such that $\mathrm{C}_G(x) = \langle x, \mathrm{Z}(\Phi(G))\rangle$ for all $x \in G - \Phi(G)$ (any minimal nonabelian p-group satisfies the above condition).

4691. Study the p-groups G such that $\mathrm{C}_G(x) = \langle x, \mathrm{Z}(H)\rangle$ for all $x \in G - H$, where a nonabelian $H \in \Gamma_1$ (example: G is a p-group of maximal class with an abelian subgroup of index p).

4692. Study the p-groups G such that the centralizer $\mathrm{C}_G(x)$ is special for all $x \in G - \mathrm{Z}(G)$ of order p.

4693. Study the p-groups all of whose nonabelian maximal subgroups are special (see [Cos]).

4694. Study the p-groups containing a special subgroup of index p.

4695. Given n, study the p-groups all of whose subgroups of index p^n are either minimal nonabelian or special.

4696. Describe a Sylow p-subgroup of a group consisting of those automorphisms that fix all elements of order p of a p-group G.

4697. Describe the groups of exponent p without quasinormal minimal nonabelian subgroups.

4698. Describe the groups of exponent p without normal nonabelian subgroups of order p^4.

4699. Describe the groups of exponent p such that $|H^G : H| \leq p^2$ for all $H < G$.

4700. Is it true that the order of a p-group G of exponent p is bounded if $|\mathrm{Aut}(G)|_p < |G|$?

4701. Study the p-groups G such that, whenever $M \in \Gamma_1$ and $x \in M - \mathrm{Z}(G)$, then $\mathrm{C}_G(x) \leq M$.

4702. Study the p-groups G such that, whenever $H \triangleleft G$ is of index p^2, then any normal subgroup of H is G-invariant.

4703. Classify the p-groups containing a metacyclic subgroup of index p.

4704. Study the p-groups containing an $\mathcal{A}_2$-subgroup of index p.

4705. Find the maximal possible number of $\mathcal{A}_k$-subgroups, $k = 1, 2, 3$, contained in a group of order p^n and exponent p.

4706. Find the maximal possible number of metacyclic minimal nonabelian subgroups ($\mathcal{A}_2$-subgroups) contained in a group of maximal class and order p^n, $p > 2$.

4707. Given $n > 1$ and an $\mathcal{A}_1$-group H, does there exist an $\mathcal{A}_n$-group G containing a subgroup of index p isomorphic with H and such that $\mathrm{C}_G(H) < H$?

4708. Study the p-groups G such that, whenever $F, H < G$ are nonabelian of equal order, then (i) $\alpha_1(F) = \alpha_1(H)$, (ii) $\mathrm{c}_k(F) = \mathrm{c}_k(H)$, (iii) $\mathrm{s}_k(F) = \mathrm{s}_k(H)$ for all k.

4709. Below we prove Proposition 2 in [BJ1]:

THEOREM A1. *Let G be a p-group of order $\leq p^{p-1}$ and exponent p. Then p^2 does not divide* $|\mathrm{Aut}(G)|$.

PROOF. Assume that $\alpha \in \mathrm{Aut}(G)$ is a nonidentity automorphism of order p^e. Set $H = \langle \alpha \rangle G$ (semidirect product with kernel G). Let N be an H-invariant subgroup of index p in G. Since H/N is abelian of type $(p, o(\alpha)) = (p, p^e)$ and $N \leq \mathrm{Z}_{p-2}(H)$, it follows that $\mathrm{cl}(H) \leq p-1 < p$, so that the group H is regular [B1, Theorem 7.1(b)]. Therefore, by [B1, Theorem 7.2(k)], we have $[\Omega_1(H), \mho_1(H)] = \{1\}$. Note that $G \leq \Omega_1(H)$ and $\langle \alpha^p \rangle$ is contained in

$\mho_1(H)$. It follows that

$$[\alpha^p, G] \leq [\mho_1(H), \Omega_1(H)] = \{1\},$$

i.e., α as an automorphism of the group G has order p, i.e, $e = 1$, as required. $\square$

4710. Describe the nonabelian metacyclic p-groups covered by $\mathcal{A}_1$-subgroups (see problem #860 in [BJ2]).

4711. Find all p-groups G satisfying $\mathrm{s}_k(G) = \mathrm{s}_k(M)$ ($\mathrm{c}_k(G) = \mathrm{c}_k(M)$), where M is a p-group of maximal class, for all $k > 1$. Consider in detail also the cases when (i) M has an abelian subgroup of index p, (ii) M is metacyclic.

4712. Let $G = P \wr \mathrm{C}_p$ and suppose that $\mathrm{Aut}(P)$ is known. Describe $\mathrm{Aut}(G)$.

4713. Find the number $\alpha_1(P \wr \mathrm{C}_p)$ ($\alpha_1(\mathrm{C}_p \wr P)$, $\alpha_1(P \wr \mathrm{C}_{p^2})$), where P is a minimal nonabelian p-group.

4714. Find $\mathrm{cl}(S \wr T)$, $\alpha_1(S \wr T)$ and $\alpha_2(S \wr T)$, where S and T are known minimal nonabelian p-groups.

4715. Study the p-groups G containing a nontrivial normal subgroup H such that $\mathrm{C}_G(x)H = G$ for all $x \in H$.

4716. Does there exist an absolutely regular p-groups R such that a Sylow p-subgroup of the group $\mathrm{Aut}(R)$ is irregular?

4717. Describe the nonabelian p-groups G of order p^n such that the class (derived length) of a Sylow p-subgroup of $\mathrm{Aut}(G)$ is maximal possible.

4718. Classify the p-groups G such that $\alpha_1(G) = \alpha_1(H)$, where H is a proper epimorphic image of G.

4719. Study the p-groups all of whose nonabelian maximal subgroups are isomorphic.

4720. Do there exist two p-groups of exponent p of different classes with isomorphic automorphism groups?

4721. Classify the p-groups G of maximal class such that a Sylow p-subgroup of $\mathrm{Aut}(G)$ is of maximal class.

4722. Study the irregular p-groups all of whose maximal regular subgroups are nonnormal (in this case, $p > 2$).

4723. Study the p-groups of class > 2 all of whose maximal subgroups of class 2 are nonnormal.

4724. Study the p-groups all of whose maximal nonnormal subgroups have the same order.

4725. Study the nonabelian p-groups with abelian subgroup of index p all of whose minimal nonabelian subgroups have the same order (are isomorphic).

4726. Study the nonabelian p-groups of order p^{p+n}, $n > 1$, containing only one normal subgroup of order p^k for all $k \leq n$.

4727. How many there are p-groups of maximal class and order p^n containing an abelian subgroup of index p?

4728. How many there are 3-groups of maximal class and order $3^n > 3^4$ containing a minimal nonabelian subgroup of index 3?

4729. How many there are p-groups of maximal class and order p^n whose Frattini subgroup is abelian?

4730. Study the two-generator metabelian p-groups.

4731. Describe the automorphism group of a p-group of maximal class whose Frattini subgroup is abelian.

4732. Study the p-groups with nonabelian (noncyclic) Frattini subgroup (derived subgroup) all of whose proper subgroups have abelian (cyclic) Frattini subgroups (derived subgroups). Replace in the above sentence 'derived subgroups' by 'second derived subgroups'.

4733. Study the p-groups G with noncyclic subgroup $\mho_1(G)$ all of whose $H < G$ have cyclic subgroups $\mho_1(H)$. Replace in the above sentence '$\mho_1(*)$' by '$\mho_2(*)$'.

4734. Study the p-groups G with nonabelian subgroup $\Omega_k(G)$ all of whose proper $H < G$ have abelian subgroups $\Omega_k(H)$, $k \in \{1, 2\}$.

4735. Classify the p-groups all of whose nonabelian subgroups are two-generator.

4736. Classify the p-groups all of whose subgroups (nonabelian subgroups) of index p^2 are two-generator.

4737. Study the p-groups G containing exactly p subgroups of index p and exponent p.

4738. Study the p-groups G such that $G/\mho_2(G)$ is of maximal class (minimal nonabelian, metacyclic).

4739. Study the p-groups G such that G/G'' is of maximal class (minimal nonabelian, metacyclic).

4740. Describe the subgroup structure of irregular 3-groups containing an $\mathcal{A}_1$-subgroup of index 3.

4741. Describe $\mathrm{Aut}(M)$, where M is a 5-group of maximal class containing an abelian subgroup of index 5.

4742. Study the p-groups G of order p^n containing a characteristic subgroup of order p^k for all $k \leq n$ (example: $G \cong \mathrm{SD}_{2^n}$).

4743. Given $n > 3$, does there exist a p-group of order p^n containing a minimal nonabelian subgroup of order p^k for all $k \in \{3, \dots, n-1\}$?

4744. Study the p-groups G, $p > 2$, containing a nonabelian subgroup H such that $|G : \mathrm{N}_G(L)| \leq p$ for all noncyclic $L \leq H$. Consider in detail the case when H is minimal nonabelian.

4745. Let G be a group of order p^n and exponent p. Estimate the maximal possible exponent of a Sylow p-subgroup of the group $\mathrm{Aut}(G)$.

4746. Study the non-Dedekindian p-groups G such that $|G : \mathrm{N}_G(L)^G| = p$ for all nonnormal $L < G$.

4747. Classify the non-Dedekindian p-groups G such that $|\mathrm{N}_G(L) : \mathrm{N}_G(L)_G| \leq p$ for all nonnormal $L < G$.

4748. Study the non-Dedekindian p-groups G such that H/H_G is cyclic for any nonnormal $H < G$.

4749. Study the nonabelian p-groups G such that $|G : A^G| = p$ for any maximal abelian $A < G$.

4750. Describe the p-groups G of maximal class such that G possesses a subgroup isomorphic to $G/\mathrm{Z}(G)$.

4751. Describe all possible epimorphic images of order p^p of irregular p-groups of maximal class.

4752. Describe all possible types of subgroups of order p^p and exponent p in p-groups of maximal class.

4753. Study the p-groups G such that their minimal nonabelian subgroups cover $\Omega_1(G)$.

4754. Study the p-groups G of exponent $> p$ such that any their cyclic subgroup of order $> p$ is contained in only one minimal nonabelian subgroup of G (it follows that if $S, T < G$ are distinct minimal nonabelian, then $\exp(S \cap T) \leq p$).

4755. Study the p-groups G such that any two their distinct maximal regular subgroups have abelian intersection.

4756. Study the p-groups G such that any two their non-incident subgroups have a metacyclic intersection.

4757. Describe the nonabelian p-groups all of whose nonlinear irreducible characters, except one, have equal degree.

4758. Describe the group of those automorphisms of a p-group G that fix all their nonabelian subgroups.

4759. Describe the group of those automorphisms of a p-group G of exponent $> p$ that fix all their cyclic subgroups of order $> p$.

4760. Describe the group of those automorphisms of a p-group G that fix all their maximal abelian subgroups.

4761. Describe the non-Dedekindian p-groups G such that $|G : H^G| \leq p^2$ for all nonnormal subgroups H of G.

4762. Study the p-groups all of whose nonnormal subgroups are Dedekindian.

4763. Describe the groups $\mathrm{Aut}(\mathrm{Q}_{2^n} \times \mathrm{E}_{2^m})$, $\mathrm{Aut}(\mathrm{D}_{2^n} \times \mathrm{E}_{2^m})$, $\mathrm{Aut}(\mathrm{SD}_{2^n} \times \mathrm{E}_{2^m})$ and $\mathrm{Aut}(\mathrm{M}_{2^n} \times \mathrm{E}_{2^m})$.

4764. Describe the group $\mathrm{Aut}(S \times \mathrm{E}_{p^m})$, where S is a minimal nonabelian p-group.

4765. Describe all characteristic subgroups of a group $M_1 \times M_2$, where M_1, M_2 are p-groups of maximal class.

4766. Classify the nonabelian p-groups with only one nontrivial characteristic abelian subgroup.

4767. Classify the p-groups G satisfying $\mathrm{Aut}(G) \cong G$ (example: $G \cong \mathrm{D}_8$. We do not know if there are other examples).

4768. Describe the p-groups G such that $|G : \mathrm{C}_G(S)| \le p^3$ for all nonabelian (minimal nonabelian) $S < G$.

4769. Study the p-groups of exponent $> p$ all of whose cyclic subgroups of order $> p$ are normal (nonnormal).

4770. Study the p-groups of exponent $> p$ all of whose maximal abelian subgroups of exponent $> p$ are nonnormal.

4771. Study the p-groups with cyclic Schur multiplier. In particular, consider the p-groups with trivial Schur multiplier.

4772. Describe the p-groups G such that, whenever $M \in \Gamma_1$ and $H < M$ is not G-invariant, then $\mathrm{N}_G(H) \le M$.

4773. Describe the p-groups G such that $\mathrm{Aut}(G)$ is of class 2. Is it true that if G is noncyclic, then $p = 2$?

4774. Describe the p-groups G of order $> p > 2$ such that p^2 does not divide $\exp(\mathrm{Aut}(G))$.

4775. Study the p-groups of exponent $> p$ all of whose two elements of different orders commute. Is it true that G is p-central? Recall that if $p > 2$ and $\Omega_1^{\#}(G) \le \mathrm{Z}(G)$ and if $p = 2$ and $\Omega_2^{\#}(G) \le \mathrm{Z}(G)$, then a p-group G is said to be p-central.

Solution. Let $p > 2$ and $a, b \in G$ are both of order p. Assume that $o(ab) > p$. Then $[ab, a] = [ab, b] = 1$ so that $[a, b] = 1$ which implies that $o(ab) \le p$, a contradiction. Thus, $\exp(\Omega_1(G)) = p$. In that case, by condition, $[\Omega_1(G), G - \Omega_1(G)] = 1$ so that $\Omega_1(G) \le \mathrm{Z}(G)$ since $\langle G - \Omega_1(G)\rangle = G$. It follows that G is p-central.

Now let $p = 2$ and let $a, b \in G$ be of order 4 and $ab \ne ba$. Assume that $o(ab) > 4$. Then, as above, $[ab, a] = [ab, b] = 1$ so that $[a, b] = 1$, and we conclude that $o(ab) \le 4$, a contradiction. Since elements of order 2 are permutable with elements of order 4, we get $\exp(\Omega_2(G)) = 4$. In that case, $[\Omega_2^{\#}(G), G - \Omega_2^{\#}(G)] = 1$ so that $\Omega_2^{\#}(G) \le \mathrm{Z}(G)$ since $\langle G - \Omega_2^{\#}(G)\rangle = G$. It follows that G is also 2-central.

4776. Study the p-groups G satisfying $\exp(\mathrm{Aut}(G)) = p$. Consider separately the case when $\exp(G) = p$.

4777. Study the p-groups all of whose minimal nonabelian subgroups have abelian (minimal nonabelian, metacyclic) centralizers.

4778. Study the nonabelian p-groups G satisfying $|G| \le \sum_{M\in\Gamma_1} |\mathrm{Z}(M)|$ (example: G is a minimal nonabelian p-group).

4779. Describe the nonabelian p-groups G such that, whenever $S \le G$ is minimal nonabelian, then $|G : \mathrm{C}_G(x)| \le p^2$ for all $x \in S - \mathrm{Z}(G)$ (see problem #2787 in [BJ4] and Theorem 217.1 in [BJ5]).

4780. Find the maximal possible $\mathrm{dl}(G)$ ($\mathrm{cl}(G)$), where a p-group G is a product of n pairwise permutable minimal nonabelian subgroups. The same problem for the case where G is a product of n pairwise permutable subgroups of class 2.

4781. Study the irregular p-groups $G = \Omega_1(G)$ satisfying $EF = FE$ for any two maximal subgroups $E, F < G$ of exponent p.

4782. Study the group of those automorphisms of a p-group G that fix all its minimal nonabelian subgroups.

4783. Does there exist a p-group $G = A \times B$, where $A, B > \{1\}$, that is covered by minimal nonabelian subgroups (see problem #860 in [BJ2])?

4784. Study the p-groups G such that, whenever $A < S \le G$, where S is minimal nonabelian and A is cyclic, then $|G : \mathrm{C}_G(A)| \le p$.

4785. Study the non-primary nonabelian groups G such that, whenever $A < G$ is nonabelian, then A/A' is primary cyclic.

4786. Classify the p-groups all of whose minimal nonabelian subgroups have index $\le p^2$.

4787. Classify the p-groups all of whose $\mathcal{A}_2$-subgroups have index $\le p^2$.

4788. Study the p-groups all of whose $\mathcal{A}_2$-subgroups (maximal metacyclic subgroups) have normal complements.

4789. Study the p-groups all of whose maximal absolutely regular subgroups are complemented (normally complemented).

4790. Study the non-metacyclic p-groups all of whose minimal non-metacyclic subgroups are complemented (normally complemented).

4791. Study a metacyclic p-group G with an irregular Sylow p-subgroup of the automorphism group $\mathrm{Aut}(G)$.

4792. Study the nonabelian p-groups all of whose maximal abelian subgroups are complemented (quasinormally complemented).

4793. Study the nonabelian p-groups all of whose nonnormal abelian subgroups are either cyclic or have exponent p.

4794. Let a p-group $G = M_1 \times M_2$, where $p > 2$ and the subgroups M_1, M_2 are nonabelian metacyclic. Describe all cases when a Sylow p-subgroup of the group $\mathrm{Aut}(G)$ is regular.

4795. Describe a Sylow p-subgroup of the automorphism groups of the following groups: $\mathrm{Aut}(\Sigma_{p^2})$, $\mathrm{Aut}(\Sigma_{p^2} \times \mathrm{C}_p)$, $\mathrm{Aut}(\Sigma_{p^2} \times \mathrm{C}_{p^n})$ for $n > 1$, $\mathrm{Aut}(\Sigma_{p^2} \times \Sigma_{p^2})$, $\mathrm{Aut}(\Sigma_{p^2} * \Sigma_{p^2})$ (central product).

4796. Study the structure of a nonabelian p-group G, $p > 2$, such that for any minimal nonabelian $S \leq G$ one has $|G : \mathrm{N}_G(L)| = p$, where $L < S$ is nonnormal.

4797. Is it true that if G is a p-group, $p > 2$, such that $G/\mathrm{K}_p(G)$ is of order p^p, of class $p-1$ and of exponent p, then it is of maximal class?

4798. Study the p-groups G containing an $\mathcal{A}_2$-subgroup H such that any maximal abelian subgroup of H is a maximal abelian subgroup in G.

4799. Let A be the group of those automorphisms of a p-group G that fix all maximal subgroups of its minimal nonabelian subgroups. Study the structure of the group A.

4800. Let G be a p-group of exponent $> p$ and let A be the set of all automorphisms of G that fix all cyclic subgroups of order $> p$. Describe the structure of A.

4801. Study the p-groups G such that $\alpha_1(T) = p + 1$ for any their $\mathcal{A}_2$-subgroup T.

4802. Study the p-groups G such that any their maximal metacyclic subgroup is an $\mathcal{A}_2$-group.

4803. Find $\alpha_k(M \times N)$, $k = 1, 2$, where M, N are $\mathcal{A}_2$-groups.

4804. Find the maximal possible value of $\delta(G) = \alpha_1(G) - \alpha_1(H)$, where G is a group of order p^n and H runs over the set Γ_1.

4805. Given $n \geq 3$, estimate the minimal order of a p-group containing all types of minimal nonabelian p-groups of order $\leq p^n$. The same problem for a p-group containing all types of $\mathcal{A}_2$-groups of order $\leq p^n$.

4806. Describe the p-groups that are lattice isomorphic to $\mathcal{A}_k$-groups, $k = 1, 2$.

4807. Describe the p-groups that are lattice isomorphic to p-groups of maximal class with an abelian Frattini subgroup.

4808. Study the p-groups containing only one proper irregular subgroup.

4809. Study the noncyclic p-groups all of whose maximal cyclic subgroups have the same order.

4810. Study the non-metacyclic p-groups all of whose maximal metacyclic subgroups are isomorphic (have the same order).

4811. Classify the p-groups G such that $G/\mho_2(G)$ is a group of maximal class.

4812. Let G be an irregular p-group of order p^{p+n} for $n \leq 2$. Study the structure of G.

4813. Estimate the number of groups of maximal class and order p^{p+1} (of order p^{p+2}).

4814. Find the number of subgroups of given order in a minimal nonabelian p-group (metacyclic or non-metacyclic).

4815. Let us prove the following results:

THEOREM A2. *Let G be a p-group, $p > 2$, $\Omega_1(G) \le \mathrm{Z}(G)$ and $|\Omega_1(G)| = p^2$. Then G is metacyclic.*

PROOF. By [B1, §41], the group G has no minimal non-metacyclic subgroups. It follows that G is metacyclic. □

THEOREM A3. *Assume that a p'-group Q acts on a p-group P such that Q centralizes $\Phi(P)$. Then $[P, Q] \le \Omega_1(P)$.*

PROOF. Take $x \in P$ and $\mu \in Q$. As $x^p \in \Phi(P)$, we get $x^p = (x^p)^\mu = (x^\mu)^p$. By [B1, Theorem 7.2(a)], we get $(x^{-1}x^\mu)^p = 1$ so that $x^{-1}x^\mu \in \Omega_1(P)$. Since

$$x^{-1}x^\mu = x^{-1}\mu^{-1}x\mu \in [P, Q],$$

we are done. □

4816. Find the number of subgroups of a given order in a 3-group of maximal class.

4817. Estimate the number of subgroups of a given order in a group of order $\le p^6$ and exponent p.

4818. Classify the p-groups G such that $A \cap B$ is metacyclic, where $A, B < G$ are non-incident subgroups of different orders.

4819. Classify the groups of exponent p in which intersection of any two distinct subgroups of order p^5 is abelian.

4820. Find the number of cyclic subgroups in a p-group of maximal class with an abelian subgroup of index p.

4821. Find the number of maximal abelian subgroups in a p-group of maximal class with an abelian subgroup of index p. For $p = 2$ that number is $2^{n-2} + 1$. Consider in detail the case $p = 3$ even in the case when our group has no abelian subgroup of index 3.

4822. Find the number of cyclic subgroups in an extraspecial 2-group of a given order.

4823. Let $c(G)$ be the number of nonidentity cyclic subgroups in a group G. Estimate $c(A \times B) - (c(A) \times c(B))$, where A and B are given p-groups.

4824. Study the nonabelian p-groups all of whose nonabelian subgroups are complemented.

4825. Let P be a nonabelian Sylow 2-subgroup of minimal nonnilpotent group (that subgroup is special). Find the number of cyclic subgroups in P.

4826. Study the nonabelian p-groups all of whose minimal nonabelian subgroups are complemented.

4827. Study the p-groups of exponent $> p$ all of whose subgroups of index p^2 are complemented.

4828. Study the p-groups G such that the centralizer $\mathrm{C}_G(x)$ is metacyclic for all $x \in G - \mathrm{Z}(G)$.

4829. Study the groups of exponent p all of whose nonnormal minimal nonabelian subgroup have p conjugates.

4830. Study the irregular p-groups all of whose maximal regular subgroups have pairwise different orders.

4831. Study the non-metacyclic p-groups all of whose maximal metacyclic subgroups have pairwise different orders.

4832. Describe the group $\mathrm{Aut}(S \times S)$, where S is a minimal nonabelian p-group. Describe also a Sylow p-subgroup of that group. Consider the group $\mathrm{Aut}(S \times T)$, where $T \ncong S$ are minimal nonabelian p-groups.

4833. Describe the group $\mathrm{Aut}(M \times S)$, where M is a 3-group of maximal class and S is a cyclic (minimal nonabelian) 3-group.

4834. Study the p-groups all of whose nonabelian subgroups are quasinormal (any p-group all of whose minimal nonabelian subgroups are quasinormal satisfy the above condition).

4835. Describe the group $\mathrm{Aut}(\mathrm{Aut}(M))$, where $M \cong \mathrm{M}_{p^n}$.

4836. Describe the group $\mathrm{Aut}(\mathrm{Aut}(M))$, where M is a 2-group of maximal class. Moreover, consider the case when M is a metacyclic 2-group.

4837. Study the p-groups all of whose abelian subgroups of exponent $> p$ are two-generator.

4838. Classify the non-Dedekindian p-groups all of whose nonnormal abelian subgroups have exponent p (see #4210).

4839. Study the non-Dedekindian p-groups G such that all nonnormal cyclic subgroups of their minimal nonabelian subgroups are complemented in G.

4840. Classify the metacyclic p-groups G such that, whenever H is lattice isomorphic with G, then $H \cong G$.

4841. Study the nonabelian p-groups G such that $\mathrm{C}_G(S)$ is minimal nonabelian for any minimal nonabelian $S < G$.

4842. Study the nonabelian p-groups G such that $|\mathrm{Aut}(G)| = p|\mathrm{Inn}(G)|$.

4843. Let a p-group G be irregular and let H be a regular p-group. Suppose that G and H are lattice isomorphic. Is it true that $|G|$ is bounded?

4844. Classify the non-Dedekindian p-groups of exponent $> p$ all of whose nonnormal subgroups have exponent p (this was solved by the second author; see below Theorem A4).

A nonabelian p-group G is called *metahamiltonian* if all its nonabelian subgroups are normal (by [B1, Theorem 10.28], this will be if and only if all minimal nonabelian subgroups of G are normal). This is a natural generalization of Hamiltonian p-groups (i.e., nonabelian Dedekindian p-groups).

Metahamiltonian p-groups have been classified. Here we solve [B1, Problem #4210] by proving the following result.

THEOREM A4 (Jan). *Let G be a nonabelian p-group of exponent $> p$ all of whose nonnormal subgroups have exponent p. Then G is metahamiltonian.*

PROOF. Let G be a nonabelian p-group of exponent $> p$ all of whose subgroups of exponent $> p$ are normal.

If $p = 2$, then each nonabelian subgroup H in G has exponent > 2 and so $H \trianglelefteq G$ and therefore G is metahamiltonian.

Suppose that $p > 2$ and G is not metahamiltonian. Let P be a cyclic subgroup of order p^2 in G. Then $P \trianglelefteq G$ and each subgroup of G containing P is normal in G and hence G/P is Dedekindian and so abelian [B1, Theorem 1.20]. Therefore $G' \leq P$. If $G' \cong \mathrm{C}_p$, then any $\mathcal{A}_1$-subgroup has derived subgroup coinciding with G' [B1, Exercise 1.8a] so it is normal; in that case G is metahamiltonian, contrary to the assumption. Hence we have $G' = P \cong \mathrm{C}_{p^2}$. If $\Omega_1(P)$ is the only subgroup of order p in G, then G is cyclic (see [B1, §1]), a contradiction. Let Q be a subgroup of order p in G which is distinct from $\Omega_1(P)$. Then we have either $PQ = P \times Q$ or $PQ \cong \mathrm{M}_{p^3}$ (this follows from the description of groups of order p^3). Then PQ contains a cyclic subgroup $U \neq P$ of order p^2. As above, G/U is abelian. It follows that $G/(P \cap U)$ is abelian and that $G' = P \cap U$ is of order p. As above, G is metahamiltonian, a contradiction. Our theorem is proved. □

A more general problem is the following one: Classify the p-groups of exponent $> p$ all of whose cyclic subgroups of order $> p$ are normal. Partially this problem is considered in the following theorem.

THEOREM A5. *Suppose that a nonabelian p-group G has exponent $> p$. If all cyclic subgroups of order $> p$ are G-invariant, then one of the following holds:*

(i) If $p > 2$, then $|G'| = p$ so that G is metahamiltonian.

(ii) If $p = 2$ and G is D_8-free, then it is metahamiltonian.

PROOF. Assume that G is not metahamiltonian. Then, by [B1, Theorem 10.28], G possesses a nonnormal minimal nonabelian subgroup S. It follows that S is not generated by cyclic subgroups of order $> p$. In that case either $p > 2$ and $S \cong S(p^3)$ or $p = 2$ and $S \cong \mathrm{D}_8$ (indeed, S/S' is not generated by cyclic subgroups of order $> p$ so $S/S' \cong \mathrm{E}_{p^2}$). By (ii), the second case is impossible.

Let $P \triangleleft G$ be maximal cyclic of order $> p$. Let H/P be a nonidentity cyclic subgroup of G/P.

If $|H/P| > p$, then metacyclic subgroup H is generated by two cyclic subgroups of order $> p$, and we conclude that $H \triangleleft G$. Now let $|H/P| = p$. It follows from the description of p-groups with cyclic subgroup of index p

and condition in (ii) that H is isomorphic to one of the groups $\mathrm{C}_{|P|} \times \mathrm{C}_p$, $\mathrm{M}_{p|P|}$, Q_8 so that H is generated by two cyclic subgroups of order $> p$, and we conclude that $H \triangleleft G$ again. It follows that the group G/P is Dedekindian.

If $P \cong \mathrm{C}_{p^2}$ and $p > 2$, then, by the previous paragraph, G/P is abelian [B1, Theorem 1.20]. As G is noncyclic, it contains a cyclic subgroup $Q \neq P$ of order p^2 [B1, Theorem 1.10 (b)]. By the above, G/Q is abelian so that $G/(P \cap Q)$ is isomorphic to a subgroup of $(G/P) \times (G/Q)$ so is abelian hence $G' = P \cap Q$. As $|P \cap Q| = p$, then all nonabelian subgroups of G are normal therefore G is metahamiltonian.

Now let $p = 2$ and let $S <\leq G$ be minimal nonabelian. Assume that S is nonnormal in G. Then S is not generated by cyclic subgroups of order > 2. It follows that $S \cong \mathrm{D}_8$, contrary to the assumption. Thus, all minimal nonabelian subgroups are normal in G so the group G is metahamiltonian (see [B1, Theorem 1.28]). □

4845. Classify the nonabelian p-groups G that are not generated by $\alpha_1(G) - 1$ minimal nonabelian subgroups (by [B1, Theorem 10.28], G is generated by $\alpha_1(G)$ minimal nonabelian subgroups). One may assume that G is not minimal nonabelian; then $\alpha_1(G) > 1$. It is easy to see that, in that case, $p = 2$. Indeed, if $H < G$ is generated by $\alpha_1(G) - 1$ minimal nonabelian subgroups, then $p-1 \leq \alpha_1(G)-\alpha_1(H) = 1$ which implies that $p = 2$. Let us prove this. In that case one has $|G : H| = p$ [B1, Theorem 1.28]. As G is noncyclic, we have $\mho_1(G) < H$. Let $\mho_1(G) < F < H$, where $F \triangleleft G$ and $|H : F| = p$; then $G/F \cong \mathrm{E}_{p^2}$. Let $H = H_1, \dots, H_{p+1}$ be all subgroups of index p in G that contain F. One may assume that $H_1, \dots, H_p$ are nonabelian since G is not minimal nonabelian. Let $S_i \leq H_i$ be minimal nonabelian such that $S_i \not\leq F$, $i = 1, \dots, p$ [B1, Theorem 10.28] again. Then the subgroups S_i for $i > 1$ are not contained in $H_1 = H$, and we conclude that $\alpha_1(G) - \alpha_1(H) \geq p - 1$, as claimed.

Theorem A6. *Suppose that G is a nonabelian p-group with $\alpha_1(G) > 1$. If G is not generated by $\alpha_1(G) - 1$ distinct $\mathcal{A}_1$-subgroups, then $p = 2$ and $\alpha_1(G) = 2$ (in that case G is an $\mathcal{A}_2$-group).*

Proof. By the paragraph preceding the theorem, we have $p = 2$. Let $H \in \Gamma_1$ be nonabelian. If $S < G$ is minimal nonabelian with $S \not\leq H$, then $\langle H, S \rangle = G$. It follows that $\alpha_1(H) = \alpha_1(G) - 1$, by [B1, Theorem 10.28] (otherwise, G is generated by $\alpha_1(H) + 1 \leq \alpha_1(G) - 1$ minimal nonabelian subgroups, contrary to the hypothesis). Let $M \in \Gamma_1 - \{H\}$ be nonabelian (as we know, such M exists since the number of nonabelian members of the set Γ_1, in the case under onsideration, is > 1 [B1, Exercise 1.6]. By the above, $\alpha_1(M) = \alpha_1(G) - 1$. Set $D = H \cap M$. Then $|G : D| = p^2$, by the product formula. It follows that $G = \langle D, S, T \rangle$, where $T < G$ is minimal nonabelian, and $\langle D, S\rangle$, $\langle D, T\rangle \in \Gamma_1$. In that case, by the above, we have

$$\alpha_1(H) = \alpha_1(\langle D, S\rangle) = \alpha_1(G) - 1 = \alpha_1(\langle D, T\rangle).$$

Therefore, by hypothesis, $\alpha_1(D) = \alpha_1(G) - 2 = \alpha_1(H) - 1$ (otherwise, G is generated by $< \alpha_1(G)$ minimal nonabelian subgroups). Thus, H is not generated by $\alpha_1(H) - 1$ minimal nonabelian subgroups (otherwise, G is generated by $\alpha_1(H) \le \alpha_1(G) - 1$ minimal nonabelian subgroups, a contradiction). Let $H = H_1, H_2, H_3$ be three distinct members of the set Γ_1 containing D.

Assume that D is nonabelian. Then, by the above, we have $\alpha(H_i) = \alpha_1(G) - 1$ for $i = 1, 2, 3$. As

$$\alpha_1(D) + 2 = \alpha_1(G) = \sum_1^3 \alpha_1(H_i) - 2\alpha_1(D) = \alpha_1(D) + 3,$$

we get a contradiction. If D is abelian, then $\alpha_1(G) = \alpha_1(D) + 2 = 2$ so that G is an $\mathcal{A}_2$-group. $\square$

4846. Next we prove the following

Theorem A7. *Suppose that all maximal cyclic subgroups of composite order coincide with their centralizers in a noncyclic p-group G of exponent $> p$ and G is not a 2-group of maximal class. Then there is in G a normal subgroup $R \cong \mathrm{E}_{p^2}$. Set $M = \mathrm{C}_G(R)$; then $M \in \Gamma_1$. Suppose that there is in M a maximal cyclic subgroup X of order $> p$. Then $\mathrm{H}_p(M) \le \Phi(G)$.*

Proof. Our group G has no normal abelian subgroup of type (p, p) if and only if it is a 2-group of maximal class [B1, Lemma 1.4], and any such group satisfies the condition. Now assume that there is in G a normal subgroup $R \cong \mathrm{E}_{p^2}$. As $\mathrm{Z}(G)$ is cyclic, it follows that $M = \mathrm{C}_G(R) \in \Gamma_1$. If $X < M$ is a maximal cyclic subgroup of order $> p$, then $\mathrm{C}_M(X) > R$ is noncyclic, i.e., $\mathrm{C}_G(X) > X$. It follows that, by hypothesis, X is not a maximal cyclic subgroup of G hence $X < Y < G$, where Y is a cyclic subgroup with $|Y : X| = p$. In that case, $X = \Phi(Y) \le \Phi(G)$, and we conclude that $\mathrm{H}_p(M) \le \Phi(G)$. $\square$

4847. We solve here Problem #4671 by proving the following result.

Theorem A8 (Jan). *Let G be a nonabelian p-group such that whenever $S < G$ is maximal abelian, then $\mathrm{C}_G(x)$ is abelian for any $x \in G - S$. Then either G has an abelian subgroup of index p or $\mho_1(G) \le \mathrm{Z}(G)$.*

Here we shall use the following two known results from the previous volumes of our book.

Theorem 91.2 ([BJ2]). *Let G be a nonabelian p-group. Then $A \cap B = Z(G)$ for any two distinct maximal abelian subgroups A and B of G if and only if $\mathrm{C}_G(x)$ is abelian for each $x \in G - \mathrm{Z}(G)$.*

THEOREM 255.1 ([BJ5]). *Let G be a nonabelian p-group. If $A \cap B = Z(G)$ for any two distinct maximal abelian subgroups A and B of G, then either G has an abelian subgroup of index p or $\mho_1(G) \leq \mathrm{Z}(G)$.*

PROOF OF THEOREM A8. Let G be a nonabelian p-group such that whenever $S < G$ is maximal abelian, then $\mathrm{C}_G(x)$ is abelian for any $x \in G - S$. Let A be a maximal abelian subgroup in G. Then $\mathrm{Z}(G) < A$ since $\mathrm{C}_G(A) = A$. Let $a \in A - \mathrm{Z}(G)$ and suppose that $\mathrm{C}_G(a) > A$ so that $\mathrm{C}_G(a)$ is nonabelian. Take $b \in \mathrm{C}_G(a) - A$; then $\langle a, b\rangle$ is abelian. Let $\langle a, b\rangle \leq B$, where B is a maximal abelian subgroup of G; then $B \neq A$ since $b \not\leq A$. Assume that $y \in (A \cap B) - \mathrm{Z}(G)$. Then $\mathrm{C}_G(y) \geq \langle A, B\rangle$ is nonabelian. It follows that y is contained in all maximal abelian subgroups of G (indeed, $y \in A$, by the previous sentence), and we conclude that $y \in \mathrm{Z}(G)$, contrary to the choice of y. Hence, if A is a maximal abelian subgroup of G, then we have for each $a \in A - \mathrm{Z}(G)$ that $\mathrm{C}_G(a) = A$ is abelian.

It follows that for each $x \in G - \mathrm{Z}(G)$, the subgroup $\mathrm{C}_G(x)$ is abelian (indeed, if $x \in L$, where L is maximal abelian subgroup of G, then $\mathrm{C}_G(x) = L$, by the last sentence in the previous paragraph).[1] By Theorems 91.2 in [BJ2] and 255.1 in [BJ5], we get that either G has an abelian subgroup of index p or $\mho_1(G) \leq \mathrm{Z}(G)$. Theorem A8 is proved. □

4848. It is interesting to describe the nonabelian p-groups G all of whose minimal nonabelian subgroups are normal. If $p \leq 3$, then G may be irregular (example: $G \in \mathrm{Syl}_3(\mathrm{Sym}_{3^2})$; for $p = 2$ one can take as G the group D_{2^4}). Below we consider the case $p > 3$.

THEOREM A9. *If $p > 3$ and all minimal nonabelian subgroups of a nonabelian p-group G are normal, then G is regular.*

PROOF. Indeed, let $S \leq G$ be minimal nonabelian; then S is normal in G. By [B1, Theorem 10.28], all subgroups of the quotient group G/S are normal so the group G/S is abelian [B1, Theorem 1.20]. It follows that $G' \leq S$ so that $|\Omega_1(G')| \leq \Omega_1(S) \leq p^3 < p^{p-1}$ [B1, Exercise 1.8a]. Therefore, by [B1, Theorem 9.8(c)], the group G is regular. □

What will be if we assume in Theorem A9 that $p = 3$ and G is not a group of maximal class and order 3^4?

Is the following assertion true: 'If all minimal nonabelian subgroups of a 3-group G are metacyclic and G-invariant (quasinormal), then G is regular'?

4849. Classify the p-groups that are lattice isomorphic with $\mathcal{A}_n$-groups, $n = 1, 2, 3$.

4850. Classify the p-groups that are lattice isomorphic with regular p-groups of maximal class and exponent p^2 (that groups have order $\leq p^p$; see [B1, §9]).

[1] It follows that then every maximal abelian subgroup of $G/\mathrm{Z}(G)$ is a **TI**-subgroup.

4851. Describe the p-groups all of whose minimal nonabelian subgroups are characteristic (see Theorem A9).

4852. Classify the nonabelian p-groups all of whose nonabelian subgroups are normal (these p-groups are metahamiltonian, for $p > 3$; see Theorem A9 and [B1, Theorem 10.28]).

4853. Study the group of those automorphisms of a nonabelian p-group G that fix all mnimal nonabelian subgroups of G (these automorphisms fix all nonabelian subgroups of G, by [B1, Theorem 10.28]).

4854. Describe the non-Dedekindian p-groups all of whose two subgroups of different orders (of equal order) are permutable.

4855. Study the p-groups G such that, whenever $A \cap B > \{1\}$ for some $A, B < G$, then $AB = BA$.

4856. Study the p-groups all of whose conjugate subgroups are permutable (example: D_8, E_{p^3}).

4857. Study the nonabelian p-groups all of whose minimal subgroups of class 2 have the same order.

4858. Study the non-metacyclic p-groups all of whose minimal non-metacyclic subgroups are isomorphic (have the same order).

4859. Study the p-groups all of whose two-generator subgroups are of class ≤ 2 (see #4102).

4860. Classify the nonabelian p-groups all of whose nonabelian two-generator subgroups are minimal nonabelian (any p-group G with $|G'| = p$ and any special p-group have this property). Given $n > 2$, does there exist a p-group of class n with that property?

4861. Classify the non-metacyclic p-groups all of whose two-generator subgroups are metacyclic (the minimal non-metacyclic group of order 2^5 and also the p-group $M \times C$, where M is metacyclic minimal nonabelian and $|C| = p$ satisfy the above condition).

4862. Describe those absolutely regular p-groups which are isomorphic to fundamental subgroups of p-groups of maximal class.

4863. Describe the p-groups M of maximal class such that any p-group G that contains M as a subgroup of index p is not of maximal class (for example, any 2-group of maximal class has no semidihedral subgroup of index 2; moreover, any 2-group of maximal class has no proper semidihedral subgroup; is it possible to state an analogous result for $p > 2$?).

4864. Estimate $|\mathrm{Aut}(S \times T)| - |Aut(S)||Aut(T)|$, where S and T are minimal nonabelian p-groups.

4865. Describe $\mathrm{Aut}(S * T)$, where $S * T$ is the central product of two minimal nonabelian p-groups S and T with $S \cap T = \mathrm{Z}(S) \leq \mathrm{Z}(T)$.

4866. Study the p-groups all of whose nonnormal subgroups are complemented.

4867. Study the p-groups G all of whose normal subgroups are not complemented (in that case, $\Omega_1(G) \leq \Phi(G)$).

4868. Study the p-groups that have no automorphisms of order p^2.

4869. Let S, T be minimal nonabelian p-groups. Describe the groups $\mathrm{Aut}(S \times T)$ and $\mathrm{Aut}(S * T)$, where $S \cap T \neq S', T'$ has order p (consider all possible cases for $S \cap T$). Consider the partial case $T \cong S$ in detail (see #4864).

4870. Describe the group $\mathrm{Aut}(M_1 * M_2)$, where $M_i \cong \mathrm{M}_p(p,p)$, $i = 1, 2$ and $|M_1 \cap M_2| = p$ (consider all possible cases for $M_1 \cap M_2$). Consider also the case when $M_1 \ncong M_2$.

4871. Study the p-group containing a self centralizing subgroup of maximal class and order p^{p+1}.

4872. Study the p-groups containing a self centralizing subgroup isomorphic to $\mathrm{M}_p(2,2)$.

4873. Study the p-groups containing a self centralizing subgroup isomorphic to $\mathrm{M}_p(1.1.2)$.

4874. Study a group of exponent p containing a self centralizing subgroup isomorphic to $S(p^3) \times \mathrm{C}_p$.

4875. Study a group of order $> p^3$ containing a self centralizing subgroup isomorphic to $\mathrm{C}_{p^2} \times \mathrm{C}_p$.

4876. Describe the group $\mathrm{Aut}(M \times \mathrm{C}_{p^m})$, where $M \cong \mathrm{M}_p(n,n)$.

4877. Find the Schur multiplier of the 2-group $G = M_1 * M_2$, where M_1, M_2 are groups of maximal class and $|G| = \frac{1}{2}|M_1||M_2|$.

4878. Find the Schur multiplier of the p-group $G = M_1 * M_2$, where M_1, M_2 are minimal nonabelian groups and $M_1 \cap M_2 = M_1' = M_2'$. Consider also $\mathrm{Aut}(G)$ is the case where $M_1 \cong M_2$ and $M_1 \cap M_2 = \Phi(M_i)$, $i = 1, 2$.

4879. Find the Schur multiplier of the p-group $G = M_1 * M_2$, where M_1, M_2 are nonabelian metacyclic groups and $|M_1 \cap M_2| = \Omega_1(M_1') = \Omega_1(M_2')$.

4880. Describe the group $\mathrm{Aut}(G)$, where G is one of the groups in three previous problems.

4881. Study the automorphism group of the p-group $G = L_1 L_2$, where L_1, L_2 are cyclic.

4882. Find all possible numbers of G-invariant subgroups N of a p-group G such that G/N is of maximal class and order p^{p+1}.

4883. Find (mod 4) all possible numbers of subgroups D_{2^n} in a 2-group. Do this for subgroups $\cong \mathrm{Q}_{2^n}$ and SD_{2^n}.

4884. For the p-groups G find all possible values for $\epsilon_3(G) \pmod{p^2}$, $\epsilon_4(G) \pmod{p}$. Here $\epsilon_k(G)$ is the number of elementary abelian subgroups of order p^k in a p-group G.

4885. Study the $\mathcal{A}_n$-groups G, $n > 1$, which are p-groups such that $|G : \mathrm{N}_G(S)| = p$ for all minimal nonabelian $S < G$.

4886. Given n, find $|\mathrm{Aut}(\Sigma_{p^n})|$, where Σ_{p^n} is a Sylow p-subgroup of the symmetric group of degree p^n.

4887. Describe the representation groups of the groups Σ_{p^n} and $\mathrm{UT}(n, p)$. Describe also the automorphism groups of those p-groups.

4888. Describe the Schur multiplier and the representation groups of the group $G = M_1 * M_2$, where M_1, M_2 are 2-groups of maximal class and $M_1 \cap M_2 = \mathrm{Z}(M_1)$. Describe also the group $\mathrm{Aut}(G)$.

4889. Find the Schur multiplier and the representation groups of a p-group of maximal class with an abelian subgroup of index p.

4890. Let G be a nonabelian p-group with an abelian subgroup A of index p. Compare the Schur multipliers $\mathrm{M}(G)$ and $\mathrm{M}(A)$.

4891. Study the p-groups containing a self centralizing cyclic subgroup.

4892. Study the p-groups all of whose minimal nonabelian subgroups, except one, are metacyclic (non-metacyclic).

4893. Let $G = T_1 \cdots T_n$ be a product of n pairwise permutable minimal nonabelian subgroups $T_1, \dots, T_n$. Is is true that $\mathrm{dl}(G) \le 2n$?

Below we consider the case $n = 2$ only.

Theorem A10. *Let a p-group $G = ST$, where S and T are minimal nonabelian. Then $dl(G) \le 4$.*

Proof. If $|G : S| = p$, then $G' < S$ is abelian so $\mathrm{dl}(G) = 2$. Next assume that $|G : S| > p$. Let $S < F \in \Gamma_1$. Then, by the modular law, $F = S(F \cap T)$, where $F \cap T$ as a maximal subgroup of T is abelian. Let $F \cap T < H$, where H is a maximal subgroup of F. Then, by the modular law again, $H = (F \cap T)(H \cap S)$ is a product of two permutable abelian subgroups $F \cap T$ and $H \cap S$. By Ito's Theorem (see [B1, Exercise 1.63]) we have $\mathrm{dl}(H) \le 2$. As $|F : H| = p$, it follows that $\mathrm{dl}(F) \le \mathrm{dl}(H) + 1 \le 3$. As $|G : F| = p$, it follows that $\mathrm{dl}(G) \le \mathrm{dl}(H) + 1 \le 4$. ∎

4894. Estimate the derived length of a product of n pairwise permutable metacyclic p-groups.

4895. Estimate the derived length of a product of n pairwise permutable abelian p-groups.

4896. Let a p-group $G = B_1 \dots B_n$, where all B_i contain an abelian subgroup of index p. Estimate the derived length of G (see #4893).

4897. Estimate the derived length of a product of n normal metacyclic p-groups.

4898. Estimate the derived length of a product of n normal (pairwise permutable metabelian) p-groups.

4899. Let a p-group $G = AB$, where $A, B < G$. Is it possible to estimate $|\mathrm{dl}(G) - (\mathrm{dl}(A) + \mathrm{dl}(B))|$? A similar question for groups of exponent p.

4900. Estimate the derived length of a p-group $G = A_1 \dots A_n$, where all pairwise permutable factors are either abelian or minimal nonabelian subgroups. Consider a more general case when all factors are metabelian. Consider also a case when the derived subgroups of all factors have orders $\leq p$.

4901. Let a p-group $G = AB$, where $A, B < G$. Is it true that the number $||G'| - (|A'| + |B'|)|$ is unbounded?

4902. Suppose that a group G of exponent p is a product of n factors of order $\leq p^3$. Is it possible to estimate the derived length of G, independenr on n?

4903. Study the group $\mathrm{Aut}(G)$, where G is a group of maximal class and order $\leq p^{p+1}$.

4904. How many there are groups of maximal class of order p^p and exponent p?

4905. Study the p-groups G such that, whenever minimal nonabelian subgroups $S, T < G$ have equal exponent, then $\mathrm{c}_k(S) = \mathrm{c}_k(T)$ for all k.

4906. Study the p-groups G such that, whenever minimal nonabelian subgroups $S, T < G$ have equal order, then $\mathrm{s}_k(S) = \mathrm{s}_k(T)$ for all k.

4907. Study the groups G of exponent p such that, whenever nonabelian subgroups $F, H \leq G$ have the same order, then $\alpha_1(F) = \alpha_1(H)$. Consider in detail also the case when G is metacyclic.

4908. Describe the group $\mathrm{Aut}(S \times M)$, where S is minimal nonabelian p-group and M is a metacyclic p-group.

4909. Study the 2-groups all of whose minimal non-metacyclic subgroups have order 2^5.

4910. Study the p-groups all of whose minimal irregular subgroups have the same order p^{p+1}.

4911. Study the automorphism group of a two-generator p-group with elementary abelian (of exponent p) Frattini subgroup. Consider also the case where the Frattini subgroup is replaced by the derived subgroup.

4912. Study the p-groups all of whose two-generator subgroups are either minimal nonabelian or metacyclic.

4913. Let $U < V$ be p-groups with $|V : U| = p$. Compare $|\mathrm{Aut}(U)|$ and $|\mathrm{Aut}(V)|$.

4914. Given n, find a p-group G of order p^n such that $|\mathrm{Aut}(G)|$ is maximal possible.

4915. Describe the maximal abelian subgroups of G all of whose maximal cyclic subgroups have the same order.

4916. Study the p-groups G of exponent $p^e > p$ such that $\mathrm{Aut}(G)$ has exactly $e+1$ orbits on G.

4917. Study the p-groups G such that the group $\mathrm{Aut}(G)$ is transitive on the set $\mathcal{A}_1(G)$ of all $\mathcal{A}_1$-subgroups of G.

4918. Estimate the minimal possible number of $\mathrm{Aut}(G)$-orbits on a nonabelian group G of order p^n.

4919. Here we solve Problem #4860.

THEOREM A11 (Jan). *Let G be a nonabelian p-group all of whose nonabelian two-generator subgroups are minimal nonabelian. Then $\mho_1(G) \leq \mathrm{Z}(G)$. Hence, if $p = 2$, then* $\mathrm{cl}(G) = 2$.

PROOF. Take $g \in G$ such that $g^p \notin \mathrm{Z}(G)$. Then $[x, g^p] \neq 1$ for some $x \in G$. and so the nonabelian subgroups $\langle x, g^p \rangle$ and $\langle x, g \rangle$ are minimal nonabelian, by hypothesis. As $\langle x, g^p \rangle < \langle x, g^p \rangle$, we get a contradiction. Thus, $\mho_1(G) \leq \mathrm{Z}(G)$. In particular, $\exp(G/\mathrm{Z}(G)) = p$. Therefore, if $p = 2$, then the group $G/\mathrm{Z}(G)$ is elementary abelian so that $\mathrm{cl}(G) = 2$. □

4920. Does there exist a nonabelian p-group G such that for any its minimal nonabelian subgroup S it contains a minimal nonabelian subgroup T satisfying $S \cap T = \{1\}$. Estimate minimal possible $|G|$.

4921. Study the p-groups all of whose absolutely regular subgroups are abelian (in that case, if $p > 3$, then such groups are abelian, by [B1, Exercise 1.8a]).

4922. Study the groups of exponent p all of whose noncyclic two-generator subgroups of equal order (i) are isomorphic, (ii) have equal class.

4923. Study the metacyclic p-groups all of whose noncyclic subgroups of equal order (i) are isomorphic. (ii) have equal class.

4924. Classify the p-groups all of whose nonabelian subgroups are normal.

4925. Study the groups all of whose nonabelian subgroups of index $> p$ are nonnormal.

4926. Describe the p-groups G such that $H^\phi = H$ for all $H < G$ and all p-automorphisms $\phi \in \mathrm{Aut}(G)$.

4927. Describe the p-groups all of whose proper subgroups have elementary abelian Frattini subgroups (derived subgroups).

4928. Describe the p-groups G such that for all $H \leq G$ one has (i) $|H/H_G| \leq p^2$, (ii) $|H^G : H| \leq p^2$.

4929. Describe the p-groups G, $p > 2$, such that $G/\mho_1(G)$ is extraspecial.

4930. Study the p-groups G such that $|G : H^G| = p$ for all nonnormal $H < G$.

4931. Study the non-Dedekindian p-groups G such that $|G : S^G| = p$ for all minimal nonabelian $S < G$.

4932. Classify the irregular p-groups all of whose proper subgroups are absolutely regular (it is easy to see that such groups have order p^{p+1}; indeed, our groups has no normal subgroup of order p^p and exponent p so our group is of maximal class). Describe $\mathrm{Aut}(G)$.

4933. Study the irregular p-groups G such that $|G : \mathrm{N}_G(A)| = p$ for any maximal absolutely regular $A < G$.

4934. Study the irregular p-groups G such that $|G : \mathrm{N}_G(R)| = p$ for any maximal regular $R < G$. In particular, describe the irregular p-groups all of whose maximal regular subgroups have index p. For the same R describe the irregular p-groups G such that $|G : R^G| = p$.

4935. Given n, study the irregular p-groups all of whose subgroups of index p^n are irregular.

4936. Study the p-groups all of whose subgroups of index p^3 are metacyclic (absolutely regular).

4937. Given $p > 2$ and $n > p + 1$, does there exist a minimal irregular subgroup of order p^n?

4938. Study the p-groups in which the normalizer of any maximal cyclic subgroup is metacyclic.

4939. Study the nonabelian p-groups G such that, whenever $H \leq G$ is nonabelian and $A < H$ is a maximal abelian subgroup of G, then either $A \triangleleft G$ or $|G : A^G| = p$.

4940. Study the irregular p-groups G such that, whenever $H \leq G$ is irregular and $A < H$ is a maximal regular subgroup of G, then either $A \triangleleft G$ or $|G : A^G| = p$ (for $p = 2$ this problem coincides with the previous one).

4941. Study the p-groups G such that $|G : \mathrm{N}_G(H)| \leq p^2$ for all (all cyclic, abelian, minimal nonabelian) $H < G$.

4942. Study the p-groups with two conjugacy classes of non-quasinormal subgroups (minimal nonabelian subgroups (two problems)).

4943. Study the p-groups all of whose minimal nonabelian subgroups are non-metacyclic.

4944. Study the p-groups all of whose nonabelian metacyclic subgroups are minimal nonabelian.

4945. Study the p-groups all of whose $\mathcal{A}_2$-subgroups are metacyclic.

4946. Describe $\mathrm{Aut}(G)$, where a p-group G is an $\mathcal{A}_2$-group.

4947. Let a p-group $G = M_1 * M_2$ (central product), where M_1, M_2 are metacyclic $\mathcal{A}_2$-groups, $M_1 \cap M_2 = \Omega_1(M_1') = \Omega_1(M_2')$. Describe the group $\mathrm{Aut}(G)$.

4948. Study the two-generator p-groups containing a normal elementary abelian subgroup E such that G/E is a group of maximal class. Is it true that the subgroup E is characteristic in G?

4949. Study a p-group G containing a normal elementary abelian subgroup E such that the quotient group G/E is metacyclic. Is it true that the subgroup E is characteristic in G?

4950. Find $\mathrm{c}_k(\Sigma_{p^n})$ for all k and n.

4951. Find $|\mathrm{Aut}(\Sigma_{p^n})|$. Consider in detail the case $p = 2$ (is it true that that group is prime power?).

4952. Given $n > 3$, study the groups G of order p^n and class $n - 2$. Consider in detail the case $\mathrm{d}(G) = 2$.

4953. Describe the p-groups all of whose subgroups of equal (different) orders are permutable.

4954. Study the p-groups all of whose maximal abelian subgroups are permutable with all minimal nonabelian subgroups.

4955. Given a minimal nonabelian p-group S find the minimal order of homocyclic p-group H such that $\mathrm{Aut}(H)$ contains a subgroup isomorphic with S. Do this for an $\mathcal{A}_2$-subgroup S.

4956. Study the p-groups G such that, whenever $S, T < G$ are distinct isomorphic $\mathcal{A}_1$-subgroups, then $[S, T] = \{1\}$.

4957. Study the nonabelian p-groups G such that, whenever $N \triangleleft G$ is the greatest normal subgroup of G with nonabelian quotient group G/N. If $\mathrm{d}(G) = 2$, then G/N is minimal nonabelian. Consider the case when $\mathrm{d}(G) > 2$.

4958. Study the two-generator irregular p-groups G such that, whenever $N \triangleleft G$ is the greatest normal subgroup of G with irregular quotient group G/N. Is it true that then G/N is minimal irregular?

4959. Study the p-groups such that, whenever $S, T < G$ are distinct minimal nonabelian, then $S \cap T$ is maximal either in S or in T. Study also the case when S, T are arbitrary non-incident subgroups.

4960. Study the group $\mathrm{Aut}(G)$, where a nonabelian p-group G contains an abelian subgroup of index p and its center is cyclic. What can be said on the Schur multiplier and the representation group of G?

4961. Describe the minimal nonabelian subgroups of the representation group of a given abelian p-group.

4962. Let G be an absolutely regular p-group with $|\Omega_1(G)| < p^{p-1}$. Is it true that a Sylow p-subgroup of the group $\mathrm{Aut}(G)$ is regular (note that the fundamental subgroup G_1 of a p-group G of maximal class is absolutely regular so that a Sylow p-subgroup of the group $\mathrm{Aut}(G_1)$ is irregular)? Describe the absolutely regular p-groups G such that a Sylow p-subgroup of $\mathrm{Aut}(G)$ is irregular.

4963. Study the automorphiam group $\mathrm{Aut}(G)$ and its Sylow p-subgroup, where G is an irregular p-group of maximal class.

4964. Classify the p-groups of order $p^n > p^2$ admitting an automorphism of order p^{n-2}.

4965. Study the p-groups admitting an automorphism ϕ such that the set Γ_1 is a ϕ-orbit (in that case, all members of the set Γ_1 are isomorphic).

4966. Does there exists a nonabelian p-group G admitting an automorphism ϕ such that the set Γ_2 is a ϕ-orbit?

4967. Let a group G be of order p^n and exponent p^e. Estimate the number of sizes of $\mathrm{Aut}(G)$-orbits on G.

4968. Describe the nonabelian p-groups G such that $\mathrm{N}_G(S)/S$ is cyclic (abelian) for any minimal nonabelian $S \le G$.

4969. Describe the irregular p-groups G such that $\mathrm{N}_G(R)/R$ is cyclic (metacyclic, abelian) for any maximal regular $R < G$.

4970. Study the irregular p-groups G such that $|R/R_G| \le p$ for all maximal regular $R < G$.

4971. Study the nonabelian p-groups G such that $|A/A_G| \le p$ for all maximal abelian $A < G$.

4972. Describe the non-metacyclic p-groups G such that $|\mathrm{N}_G(M)/M| \le p$ for all maximal metacyclic $M < G$.

4973. Describe the irregular p-groups G such that $\mathrm{N}_G(R)/R$ is cyclic (metacyclic, abelian) for any maximal regular $R < G$.

4974. Describe the nonabelian (irregular) p-groups G such that $|H^G : H| \le p$ for any maximal abelian (maximal regular) subgroup $H < G$.

4975. Describe the nonabelian p-groups G such that $|S^G : S| \le p$ for any minimal nonabelian subgroup $S < G$. Consider also the case where $S^G = \mathrm{N}_G(S)$ for all nonnormal minimal nonabelian $S < G$.

4976. Describe the non-metacyclic p-groups G such that $|M^G : M| \le p$ for any maximal metacyclic subgroup $M < G$. Consider also the case where $M^G = \mathrm{N}_G(M)$ for all nonnormal maximal metacyclic $M < G$.

4977. Study the irregular p-groups all of whose (i) maximal regular subgroups, (ii) minimal irregular subgroups are quasinormal.

4978. Describe the p-groups all of whose (i) maximal cyclic (maximal abelian) subgroups are quasinormal, (ii) cyclic (abelian) subgroups of maximal order are quasinormal.

4979. Study the minimal irregular p-groups all of whose maximal subgroups are isomorphic.

4980. Study the irregular p-groups all of whose maximal regular subgroups are isomorphic.

4981. Study the metacyclic p-groups all of whose maximal subgroups are isomorphic.

4982. Describe the p-groups G such that $|\mathrm{N}_G(H) : H| = p$ for all nonnormal $H < G$ (see #4475).

4483. Describe the p-groups G such that $\alpha_1(H^G) = \alpha_1(H) + p$ for all nonnormal $H < G$.

4984. Let G be a p-group of maximal class. Describe the subgroup of those automorphisms of G that fix all members of the set Γ_1.

4985. Construct a nonabelian group G of exponent p such that for any minimal nonabelian $S < G$ there exists a minimal nonabelian subgroup $T < G$ satisfying $S \cap T = \{1\}$. Do this also for a p-group G of an arbitrary exponent.

4986. Describe the p-groups in which the centralizer of any noncentral subgroup is abelian. In particular, consider the p-groups in which every maximal, by inclusion, noncentral subgroup contains its centralizer.

4987. Study the p-groups G such that, whenever A is a maximal abelian subgroup of G and $B < A$ with $B \not\leq \mathrm{Z}(G)$, then $|\mathrm{C}_G(B) : A| = p$.

4988. Describe the nonabelian p-groups G all of whose nonabelian subgroups are pairwise non-isomorphic (in that case all nonabelian subgroups are characteristic in G). Is it true that the orders of such groups are bounded? Consider also a partial case when all minimal nonabelian subgroups are pairwise non-isomorphic.

4989. Study the p-groups of class > 2 that have no normal subgroups of class 2 (a group of maximal class and order $> p^4$ with an abelian subgroup of index p; moreover, given $n > 2$, study the p-groups without normal subgroups of class n).

4990. Study the irregular p-groups G satisfying $\mathrm{c}_1(G) = 1 + p + \cdots + p^{p-1}$ (see [B1, §13]).

4991. Given $n > 1$, describe the p-groups G satisfying $\mathrm{c}_n(G) = p^{p-1}$ (see [B1, §13]).

4992. Study the p-groups without normal maximal metabelian subgroups.

4993. Study the p-groups G satisfying (i) $\mathrm{cl}(G) = \mathrm{cl}(\Gamma)$, where Γ is a representation group of G, (ii) $\mathrm{cl}(G) = \mathrm{cl}(P)$, where P is a Sylow p-subgroup of the group $\mathrm{Aut}(G)$.

4994. Describe the groups of order $p^n > p^3$ admitting an automorphism of order p^{n-3}.

4995. (Old problem) Study the p-groups G, $p > 2$, such that $\mathrm{Aut}(G)$ is a p-group.

4996. Given n, describe the set of numbers $\mathrm{O}_n - n$, where O_n is the number of $\mathrm{Aut}(G)$-orbits on G and G runs over the set of all groups of order p^n.

4997. (i) Study the p-groups in which any two elements of different orders are permutable. (ii) Classify the p-groups in which any two elements of equal order are permutable.

4998. Study the p-groups in which any two subgroups of equal order (different orders) are permutable.

4999. Describe the p-groups of exponent $> p$ without quasinormal subgroups of class 2 (note that quasinormal subgroups of groups of exponent p are normal).

5000. Describe the p-groups H such that any p-group G containing H as a subgroup of index p satisfies $\mathrm{cl}(G) = \mathrm{cl}(H)$ (example: $H = \mathrm{SD}_{2*n}$).

References

[B1] Y. Berkovich, Groups of Prime Power Order, Vol. 1, Walter de Gruyter, Berlin, 2008.

[BJ2] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 2, Walter de Gruyter, Berlin, 2008.

[BJ3] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 3, Walter de Gruyter, Berlin, 2011.

[BJ4] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 4, Walter de Gruyter, Berlin, 2016.

[BJ5] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 5, Walter de Gruyter, Berlin, 2016.

[BJ6] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 6, Walter de Gruyter, Berlin, 2018.

[Cos] J. Cossey, *Special maximal subgroups of p-groups*, Bull. Aust. Math. Soc. **89** (2014), 415–419.

[Hor1] M.W. Horosevskii, *The automorphism groups of finite p-groups*, Algebra i Logika **10** (1971), 81–86 (in Russian).

[Hor2] M.W. Horosevskii, *Automorphisms of finite groups*, Mat. Sb. (N.S.) **93** (1974), 576–587 (in Russian).

Y. Berkovich
Department of Mathematics
University of Haifa
Haifa
Israel
E-mail: berkov@math.haifa.ac.il

Z. Janko
Department of Mathematics
University of Heidelberg
Heidelberg
Germany
E-mail: janko@mathi.uni-heidelberg.de

Received: 27.12.2018.