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# MULTIVALUED ANISOTROPIC PROBLEM WITH FOURIER BOUNDARY CONDITION INVOLVING DIFFUSE RADON MEASURE DATA AND VARIABLE EXPONENTS

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Abstract. We study a nonlinear anisotropic elliptic problem under Fourier type boundary condition governed by a general anisotropic operator with variable exponents and diffuse Radon measure data which does not charge the sets of zero $p(\cdot)$-capacity. We prove an existence and uniqueness result of entropy or renormalized solution.

## 1. Introduction

Let $\Omega$ be an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ such that $\text{meas}(\Omega) > 0$. The study of various mathematical problems with variable exponent has received considerable attention in recent years. These problems are very interesting from the purely mathematical point of view. On the other hand, their study is motivated by various applications where such equations appear in the most natural way. These problems arise in many applications as the modeling of electro-rheological fluids which are characterized by their ability to change the mechanical properties under the influence of the exterior electro-magnetic field ([12, 14, 28, 29, 30]), reaction-diffusion systems, modeling of propagation of epidemic disease ([2]). Another important application is the image processing where the anisotropy and nonlinearity of the diffusion operator and convection terms are used to underline the borders of the distorted image and to eliminate the noise ([1, 10]). In this paper, we are

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interested in the following nonlinear multivalued elliptic anisotropic problem:

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) + \beta(u) \ \ni \ \mu & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, \frac{\partial u}{\partial x_i}) \cdot \eta_i + \lambda u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\beta$ is a maximal monotone graph on $\mathbb{R}$ such that $0 \in \beta(0)$, $\mu$ a bounded Radon diffuse measure, $|\mu|(\Omega)$ (the total variation of $\mu$) a bounded positive measure on $\Omega$, $g \in L^1(\partial\Omega)$, $\lambda > 0$, $\overrightarrow{\eta} = (\eta_1, \ldots, \eta_N)$ the outward unit normal to $\partial\Omega$.

Note that the space in which we work is the anisotropic Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, where $\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$ is a vector with variable components (for $i = 1, \ldots, N$, $p_i(\cdot)$ is a continuous function defined below).

We set $\overline{\operatorname{dom}(\beta)} = [m,\ M]$ with $m \leq 0 \leq M$ and denote by

$$p_M(x) := \max(p_1(x), \ldots, p_N(x)) \quad \text{and} \quad p_m(x) := \min\left(p_1(x), \ldots, p_N(x)\right).$$

In the classical Lebesgue and Sobolev spaces with constant exponent, many authors have studied problems with a maximal monotone graph and measure data (see [3, 4, 5, 11, 13, 19]). These problems have been extended to the Sobolev spaces with variable exponent in the context of isotropic operators (see [25, 27]). In this paper, we extend the study of problems with maximal monotone graph and measure data to the Sobolev spaces with variable exponents in the context of anisotropic operators. It is not a surprise to meet new difficulties when passing from isotropic variable exponents to anisotropic variable exponent. The most difficult is the appropriate choice of components in order to obtain necessary a priori estimates. To overcome these difficulties, we combine the classical techniques with the recent techniques that have appeared when treating anisotropic problems with variable exponents.

This paper is focused on the anisotropic elliptic strongly nonlinear equation with variable exponent in which the $\overrightarrow{p}(\cdot)$-Laplacian is general. All previous works treating problems like (1.1) considered particular cases of the maximal monotone graph $\beta$ and data $\mu$. Indeed, in [6], Koné et *al*. used the minimization technique to prove the existence of weak solution when $\beta$ is a power ($\beta(t) = |t|^{p_M(x)-2}t$) and $\mu$ is an $L^1$ function. In [18], Ibrango and Ouaro used the technique of monotone operators in Banach spaces to obtain the existence and uniqueness of entropy solution of problem (1.1) when $\beta$ is a continuous, surjective and nondecreasing function such that $\beta(0) = 0$ and $\mu \in L^1(\Omega)$.

Our aim is to extend the main result of authors in [18]. More precisely we prove the existence and uniqueness of renormalized or entropy solution to the general elliptic problem (1.1). The novelty in our work is that we are dealing with general non-linearities $\beta$ and measure data.

We denote by $\mathcal{L}^N$ the $N$-dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{M}_b(X)$ the space of bounded Radon measures in $X$, equipped with its standard norm $\|.\|_{\mathcal{M}_b(X)}$. Given $\mu \in \mathcal{M}_b(X)$, we say that $\mu$ is diffuse with respect to the capacity $W^{1,p(\cdot)}(X)$ ($p(\cdot)$-capacity for short) if $\mu(A) = 0$, for every set $A$ such that $Cap_{p(\cdot)}(A, X) = 0$, where the Sobolev $p(\cdot)$-capacity of $A$ with respect to $X$ is defined by

$$Cap_{p(\cdot)}(A, X) = \inf_{u \in S_{p(\cdot)}(A)} \int_X \left( |u|^{p(x)} + |\nabla u|^{p(x)} \right) dx,$$

with

$$S_{p(\cdot)}(A) = \{u \in W_0^{1,p(\cdot)}(X) : u \geq 1 \text{ in an open set containing } A \text{ and } u \geq 0 \text{ in } X\}.$$

In the case $S_{p(\cdot)}(A) = \emptyset$, we set $Cap_{p(\cdot)}(A, X) = +\infty$.

The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(\cdot)}(X)$.

Note that, since we are dealing with the Fourier boundary condition, we cannot work with the common space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. However, the common space is $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, so we cannot use directly the argument of decomposition of measure, since the second part of the measure is in $W^{-1,p'_m(\cdot)}(\Omega)$ (the dual of $W_0^{1,p_m(\cdot)}(\Omega)$).

To overcome this difficulty, we use the same ideas as authors in [27]. We consider a smooth domain $\Omega$ in order to work with the space $W_0^{1,\tilde{p}_m(\cdot)}(U_\Omega)$, where $\tilde{p}_m(\cdot) : U_\Omega \longrightarrow (1, \infty)$ is a continuous function such that $\tilde{p}_m(x) = p_m(x)$ for all $x \in \overline{\Omega}$, and return after to the space $W^{1,p_m(\cdot)}(\Omega)$. More precisely, $\Omega$ is assumed to be a bounded domain in $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class $C^1$. Then, $\Omega$ is an extension domain (see [8]), so we can fix an open bounded subset $U_\Omega$ of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, and there exists a bounded linear operator

$$E \colon W^{1,p_m(\cdot)}(\Omega) \to W_0^{1,\tilde{p}_m(\cdot)}(U_\Omega),$$

for which

i) $E(u) = u$ a.e. in $\Omega$ for each $u \in W^{1,p_m(\cdot)}(\Omega)$,

ii) $\|E(u)\|_{W_0^{1,\tilde{p}_m(\cdot)}(U_\Omega)} \leq C\|u\|_{W^{1,p_m(\cdot)}(\Omega)}$, where $C$ is a constant depending only on $\Omega$.

We define

$$\mathfrak{M}_b^{p_m(\cdot)}(\Omega) := \{\mu \in \mathcal{M}_b^{\tilde{p}_m(\cdot)}(U_\Omega) : \mu \text{ is concentrated on } \Omega\}.$$

This definition is independent of the open set $U_\Omega$. Note that for $u \in W^{1,p_m(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p_m(\cdot)}(\Omega)$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u d\mu.$$

On the other hand, as $\mu$ is diffuse, there exist (see [25, 27]) $f \in L^1(U_\Omega)$ and $F \in L^{\tilde{p}'_m(\cdot)}(U_\Omega)^N$ such that $\mu = f - div(F)$ in $\mathcal{D}'(U_\Omega)$. Therefore, we can also write

$$\langle \mu, E(u)\rangle = \int_{U_\Omega} fE(u)dx + \int_{U_\Omega} F \cdot \nabla E(u)dx.$$

Before presenting our main result, we first give the following hypotheses.

Let $\overrightarrow{p}(\cdot) = (p_1(\cdot), \dots, p_N(\cdot))$ be such that for any $i = 1, \dots, N$, $p_i(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function with

$$1 < p_i^- := \inf_{x\in\overline{\Omega}} p_i(x) \le p_i^+ := \sup_{x\in\overline{\Omega}} p_i(x) < \infty. \tag{1.2}$$

The operator $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. $a_i(x,\xi)$ is continuous in $\xi$ for a.e. $x \in \Omega$ and measurable in $x$ for every $\xi \in \mathbb{R}$) satisfying:

- there exists a positive constant $C_1$ such that
$$|a_i(x,\xi)| \le C_1\left(j_i(x) + |\xi|^{p_i(x)-1}\right), \tag{1.3}$$
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where $j_i$ is a non-negative function in $L^{p'_i(\cdot)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p'_i(x)} = 1$;
- for $\xi,\ \eta \in \mathbb{R}$ with $\xi \ne \eta$ and for every $x \in \Omega$, there exists a positive constant $C_2$ such that
$$(a_i(x,\xi) - a_i(x,\eta))(\xi-\eta) \ge \begin{cases} C_2|\xi-\eta|^{p_i(x)} & \text{if } |\xi-\eta| \ge 1, \\ C_2|\xi-\eta|^{p_i^-} & \text{if } |\xi-\eta| < 1; \end{cases} \tag{1.4}$$
- there exists a positive constant $C_3$ such that
$$a_i(x,\xi)\cdot\xi \ge C_3|\xi|^{p_i(x)}, \tag{1.5}$$
for $\xi \in \mathbb{R}$ and almost every $x \in \Omega$.

The hypotheses on $a_i$ are classical in the study of nonlinear PDEs (see [6, 7, 17]).

Throughout this paper, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\overline{p}-N}{\overline{p}(N-1)} \tag{1.6}$$

and

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1, \tag{1.7}$$

where $\dfrac{N}{\overline{p}} = \displaystyle\sum_{i=1}^N \frac{1}{p_i^-}$, and for all $x \in \partial\Omega$,

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

A prototype example that is covered by our assumption is the following anisotropic $\overrightarrow{p}(\cdot)$-Laplacian problem: setting

$$a_i(x,\xi) = |\xi|^{p_i(x)-2}\xi, \text{ where } p_i(x) \geq 2 \text{ for any } i = 1, \dots, N,$$

we obtain the problem

$$\text{(1.8)} \qquad \begin{cases} \beta(u) - \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) \ni \mu & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x, \frac{\partial u}{\partial x_i}) \cdot \eta_i + \lambda u = g & \text{on } \partial\Omega. \end{cases}$$

For any $l_0 > 0$, we consider a function $h_0$ such that

(i) $h_0 \in C_c^1(\mathbb{R})$, $h_0(r) \geq 0$, for all $r \in \mathbb{R}$,
(ii) $h_0(r) = 1$ if $|r| \leq l_0$ and $h_0(r) = 0$ if $|r| \geq l_0 + 1$.

If $\gamma$ is a maximal monotone operator defined on $\mathbb{R}$, we denote by $\gamma_0$ the main section of $\gamma$; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \gamma(s) & \text{if } \gamma(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap \text{dom}(\gamma) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap \text{dom}(\gamma) = \emptyset. \end{cases}$$

We give a useful convergence result (see [25]).

LEMMA 1.1. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x,y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$. We suppose that: $\forall n \geq 1, w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then,*

$$z \in \text{dom}(\beta).$$

The rest of the paper is organized as follows. In Section 2, we introduce some fundamental preliminary results which are useful in this work. Then, we study the existence and uniqueness of entropy or renormalized solution in Section 3.

## 2. Preliminary results

We recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces with variable exponents. Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ a measurable function such that } \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

The $p(\cdot)$-modular of the space $L^{p(\cdot)}(\Omega)$ is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(\cdot)}(\Omega)$, the following inequality (see [15, 16]) will be used later:

$$\min\left\{|u|^{p^-}_{p(\cdot)};\ |u|^{p^+}_{p(\cdot)}\right\} \leq \rho_{p(\cdot)}(u) \leq \max\left\{\ |u|^{p^-}_{p(\cdot)};\ |u|^{p^+}_{p(\cdot)}\right\}. \tag{2.1}$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for any $x \in \Omega$, we have the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\ |u|_{p(\cdot)}|v|_{q(\cdot)}. \tag{2.2}$$

If $\Omega$ is bounded and $p,\ q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous (see [23, Theorem 2.8]).

Herein, we need the following anisotropic Sobolev space with variable exponent:

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) := \left\{u \in L^{p_M(\cdot)}(\Omega) :\ \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N\right\},$$

which is a separable and reflexive Banach space (see [24]) under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = |u|_{p_M(\cdot)} + \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}.$$

We need the following embedding and trace results.

Theorem 2.1 ([15, Corollary 2.1]). *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded open set and for all $i = 1, \ldots, N, p_i \in L^\infty(\Omega), p_i(x) \geq 1$ a.e. $x \in \Omega$. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$ a.e. $x \in \Omega$ such that*

$$\operatorname{ess}\inf_{x\in\Omega}(p_M(x) - q(x)) > 0,$$

*we have the compact embedding*

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega). \tag{2.3}$$

Theorem 2.2 ([7, Theorem 6]). *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded open set with smooth boundary and let $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N, r \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min\{p_1^\partial(x), \ldots, p_N^\partial(x)\}, \quad \forall x \in \partial\Omega. \tag{2.4}$$

*Then, there exists a compact boundary trace embedding*

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega).$$

*In particular,*

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{1}(\partial\Omega).$$

We introduce now the numbers

$$q = \frac{N(\overline{p}-1)}{N-1} \text{ and } q^* = \frac{N(\overline{p}-1)}{N-\overline{p}} = \frac{Nq}{N-q}.$$

The following result is due to Troisi (see [31]).

THEOREM 2.3. *Let* $p_1, \dots, p_N \in [1,\infty)$*;* $g \in W^{1,(p_1,\dots,p_N)}(\Omega)$ *and*

$$\begin{cases} q = (\overline{p})^* & \text{if } (\overline{p})^* < N, \\ q \in [1,\infty) & \text{if } (\overline{p})^* \geq N. \end{cases}$$

*Then, there exists a constant* $C_4 > 0$ *depending on* $N, p_1, \dots, p_N$ *if* $\overline{p} < N$ *and also on* $q$ *and* $\operatorname{meas}(\Omega)$ *if* $\overline{p} \geq N$ *such that*

$$\|g\|_{L^q(\Omega)} \leq C_4 \prod_{i=1}^{N} \left[ \|g\|_{L^{p_M}(\Omega)} + \left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}, \tag{2.5}$$

*where* $\frac{1}{\overline{p}} = \sum_{i=1}^{N} p_i$ *and* $(\overline{p})^* = \dfrac{N\overline{p}}{N-\overline{p}}$.

In this paper, we will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)(1 < q < +\infty)$ as the set of measurable function $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution

$$\lambda_g(k) := \operatorname{meas}(\{x \in \Omega : |g(x)| > k\}),\ k \geq 0 \tag{2.6}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \quad \text{for some finite constant } C > 0. \tag{2.7}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$.

$$\|g\|_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q},\ \forall k > 0\}. \tag{2.8}$$

Finally, we will use through the paper, the truncation function $T_k$ $(k > 0)$, defined by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.9}$$

It is clear that $\lim_{k\to+\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.

For any $v \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we use $v$ instead of $v|_{\partial\Omega}$ for the trace of $v$ on $\partial\Omega$. Set $\mathcal{T}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the set of the measurable functions $u \colon \Omega \longrightarrow \mathbb{R}$ such that for any $k > 0$, $T_k(u) \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. We define the space $\mathcal{T}_{tr}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ satisfying:

i) $u_n \longrightarrow u$ a.e. in $\Omega$ as $n \to +\infty$,
ii) $\frac{\partial T_k(u_n)}{\partial x_i} \longrightarrow \frac{\partial T_k(u)}{\partial x_i}$ in $L^1(\Omega)$, for all $k > 0$ as $n \to +\infty$,
iii) there exists a measurable function $v$ on $\partial\Omega$ such that $u_n \longrightarrow v$ a.e. on $\partial\Omega$ as $n \to +\infty$.

We need the following lemma proved in [6].

Lemma 2.4. *Let $g$ be a nonnegative function in $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Assume $\overline{p} < N$ and there exists a constant $C > 0$ such that*

$$\int_\Omega |T_k(g)|^{p_M^-} dx + \sum_{i=1}^N \int_{\{|g|\le k\}} \left|\frac{\partial g}{\partial x_i}\right|^{p_i^-} dx \le C(k+1), \tag{2.10}$$

*for every $k > 0$. Then, there exists a constant $D$, depending on $C$ such that*

$$\|g\|_{\mathcal{M}^{q^*}(\Omega)} \le D,$$

*where $q^* = \dfrac{N(\overline{p}-1)}{N-\overline{p}}$.*

## 3. Statement of the main results

The notion of renormalized solution to problem (1.1) where the data $\mu$ belongs to $\mathfrak{M}_b^{p_m(\cdot)}(\Omega)$ is the following.

Definition 3.1. *For any $\mu \in \mathfrak{M}_b^{p_m(\cdot)}(\Omega)$ and $g \in L^1(\partial\Omega)$, a renormalized solution of problem (1.1) is a couple $(u,\ b) \in \mathcal{T}_{tr}^{1,\overrightarrow{p}(\cdot)}(\Omega) \times L^1(\Omega)$, $u \in \operatorname{dom}(\beta)$ $\mathcal{L}^N$- a.e. in $\Omega$, $b \in\ \beta(u)$ $\mathcal{L}^N$- a.e. in $\Omega$, $tr(u) \in L^1(\partial\Omega)$, there exists $\nu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$ such that $\nu \perp \mathcal{L}^N$,*

$$\nu^+ \text{ is concentred on } [u = M],\ \nu^- \text{ is concentred on } [u = m]$$

*and*

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i} dx + \int_\Omega b\varphi dx + \int_\Omega \varphi d\nu + \lambda\int_{\partial\Omega} u\varphi d\sigma \\ = \int_\Omega \varphi d\mu + \int_{\partial\Omega} g\varphi d\sigma,\end{aligned} \tag{3.1}$$

*for any $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Moreover,*

$$\lim_{n\to+\infty} \int_{[n\le|u|\le n+1]} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx = 0, \text{ for } i = 1,\ldots,N. \tag{3.2}$$

Theorem 3.2. *Assume that (1.2)-(1.7) hold, $\mu \in \mathfrak{M}_b^{p_m(\cdot)}(\Omega)$ and $g \in L^1(\partial\Omega)$. Then, problem (1.1) admits a renormalized solution.*

PROOF. The proof is done in three steps.

STEP 1 (the approximate problem). For every $\epsilon > 0$, we consider the Yosida regularization $\beta_\epsilon$ of $\beta$ (see [9]), given by

$$\beta_\epsilon = \frac{1}{\epsilon}(I - (I + \epsilon\beta)^{-1}).$$

Thanks to [9], there exists a non negative, convex and l.s.c. function $j$ defined on $\mathbb{R}$ such that

$$\beta = \partial j.$$

To regularise $\beta$, we consider

$$j_\epsilon(s) = \min_{r\in\mathbb{R}} \left\{ \frac{1}{2\epsilon}|s-r|^2 + j(r) \right\}, \ \forall\ s \in \mathbb{R},\ \forall \epsilon\ > 0.$$

By [9, Proposition 2.11] we have

$$\begin{cases} \mathrm{dom}(\beta) \subset \mathrm{dom}(j) \subset \overline{\mathrm{dom}(j)} = \overline{\mathrm{dom}(\beta)}, \\ j_s(s) = \dfrac{\epsilon}{2}|\beta_\epsilon(s)|^2 + j(J_\epsilon) \text{ where } J_\epsilon := (I+\epsilon\beta)^{-1}, \\ j_\epsilon \text{ is a convex, Frechet-differentiable function and } \beta_\epsilon = \partial j_\epsilon, \\ j_\epsilon \uparrow j \text{ as } \epsilon \downarrow 0. \end{cases}$$

Moreover, for any $\epsilon > 0$, $\beta_\epsilon$ is a nondecreasing and Lipschitz-continuous function (see [27]).

Since $\mu \in \mathcal{M}_b^{\tilde{p}_m(\cdot)}(U_\Omega)$, recall that $\mu = f - div(F)$ in $\mathcal{D}'(U_\Omega)$ with $f \in L^1(U_\Omega)$ and $F \in (L^{\tilde{p}'_m(\cdot)}(U_\Omega))^N$ where $U_\Omega$ is the open subset of $\mathbb{R}^N$ which extends $\Omega$ via the operator $E$.

We regularize $f$, $g$ and $\mu$ respectively as follows. For any $\epsilon > 0$ and $x \in U_\Omega$, we define the functions

$$f_\epsilon(x) = T_{\frac{1}{\epsilon}}(f(x))\chi_\Omega(x), \qquad g_\epsilon(x) = T_{\frac{1}{\epsilon}}(g(x))\chi_{\partial\Omega}(x).$$

Let $(F_\epsilon)_{\epsilon>1} \subset C_0^\infty(U_\Omega)$ be a sequence such that $F_\epsilon \to F$ strongly in $(L^{\tilde{p}'_m(\cdot)}(U_\Omega))^N$. For any $\epsilon > 0$, we set $\tilde{F}_\epsilon = \chi_\Omega F_\epsilon$ and $\mu_\epsilon = f_\epsilon - div(\tilde{F}_\epsilon)$. For any $\epsilon > 0$, one has

- $\mu_\epsilon \in \mathfrak{M}_b^{p_m(\cdot)}(\Omega)$, $\mu_\epsilon \rightharpoonup \mu$ in $\mathcal{M}_b(U_\Omega)$ and $\mu_\epsilon \in L^\infty(\Omega)$,
- $(f_\epsilon)_{\epsilon>0}$ and $(g_\epsilon)_{\epsilon>0}$ are sequences of bounded functions which converges to $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$ respectively.

Moreover,

$$\|f_\epsilon\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)},\ \forall\ \epsilon > 0, \quad \|g_\epsilon\|_{L^1(\partial\Omega)} \le \|g\|_{L^1(\partial\Omega)}, \qquad \forall\ \epsilon > 0$$

and

$$\left| \int_\Omega T_k(\varphi)d\mu_\epsilon \right| \le kC(\mu,\Omega),\ \forall\ k > 0,\ \forall\ \varphi \in \mathcal{T}^{1,p_m(\cdot)}(\Omega). \tag{3.3}$$

We have the following lemma (see [27, Lemma 4.1]).

LEMMA 3.3. *The Yosida regularization $\beta_\epsilon$ is a surjective operator.*

Now, we consider the following approximating scheme problem

(3.4)
$$P(\beta_\epsilon,\mu_\epsilon)\begin{cases} -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial u_\epsilon}{\partial x_i})+\beta_\epsilon(u_\epsilon)+\epsilon|u_\epsilon|^{P_M(x)-2}u_\epsilon=\mu_\epsilon & \text{in } \Omega,\\ \sum_{i=1}^{N}a_i(x,\frac{\partial u_\epsilon}{\partial x_i})\cdot\eta_i+\lambda u_\epsilon=g_\epsilon & \text{on } \partial\Omega,\end{cases}$$

where $\epsilon>0$.

Theorem 3.4. *The problem (3.4) admits at least one weak solution $u_\epsilon$ in the sense that $u_\epsilon\in W^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap L^1(\partial\Omega)$, $\beta_\epsilon(u_\epsilon)\in L^\infty(\Omega)$ and $\forall\varphi\in W^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$,*

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial\varphi}{\partial x_i}dx+\int_\Omega\beta_\epsilon(u_\epsilon)\varphi dx+\epsilon\int_\Omega|u_\epsilon|^{P_M(x)-2}u_\epsilon\varphi dx$$
$$+\lambda\int_{\partial\Omega}u_\epsilon\varphi d\sigma=\int_\Omega\varphi d\mu_\epsilon+\int_{\partial\Omega}g_\epsilon\varphi d\sigma. \tag{3.5}$$

Proof. If $b$ is a surjective, continuous and nondecreasing function with $b(0)=0$ and $\Upsilon\in L^\infty(\Omega)$, for any $k>0$, the following problem

$$P(T_k(b),\Upsilon)\begin{cases} -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial u}{\partial x_i})+T_k(b(u))+\epsilon|u|^{P_M(x)-2}u=\Upsilon & \text{in } \Omega,\\ \sum_{i=1}^{N}a_i(x,\frac{\partial u}{\partial x_i})\cdot\eta_i+\lambda T_k(u)=g & \text{on } \partial\Omega\end{cases}$$

admits at least one solution $u_k\in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that for all $\varphi\in W^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$,

(3.6)
$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_k}{\partial x_i}\right)\frac{\partial\varphi}{\partial x_i}dx+\int_\Omega T_k(b(u_k))\varphi dx+\epsilon\int_\Omega|u_k|^{P_M(x)-2}u_k\varphi dx$$
$$+\lambda\int_{\partial\Omega}T_k(u_k)\varphi d\sigma=\int_\Omega\Upsilon\varphi dx+\int_{\partial\Omega}g\varphi d\sigma.$$

Furthermore,

$$\forall k>\|\Upsilon\|_\infty,\quad |b(u_\epsilon)|\le\|\Upsilon\|_\infty \text{ a.e. in } \Omega. \tag{3.7}$$

Indeed, we define an operator $A_k$ by

$$\langle A_k(u),\varphi\rangle=\langle A(u),\varphi\rangle+\int_{\partial\Omega}T_k(b(u))\varphi dx+\lambda\int_{\partial\Omega}T_k(u)\varphi d\sigma,\ \forall\ u,\varphi\in X_0,$$

where

$$\langle A(u), \varphi\rangle = \sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i}dx + \epsilon\int_\Omega |u|^{p_M(x)-2}u\varphi dx.$$

We also define the reflexive space

$$E := W^{1,\overrightarrow{p}(\cdot)}(\Omega) \times L^{p_M(\cdot)}(\partial\Omega).$$

Let $X_0$ be the subspace of $E$ defined by

$$X_0 = \{(u, v) \in E : v = \tau(u)\},$$

where $\tau(u)$ is the trace of $u \in \mathcal{T}_{tr}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the usual sense, since $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. In the sequel, we will identify an element $(u, v) \in X_0$ with its representative $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

The operator $A_k$ is onto (see [17, 18]). Therefore, by setting

$$\langle F, \varphi\rangle = \int_\Omega \Upsilon\varphi dx + \int_{\partial\Omega} g\varphi dx,$$

it follows that $F \in E' \subset X_0'$. Then, we can deduce the existence of a function $u_k \in X_0$ such that

$$\langle A_k(u_k), \varphi\rangle = \langle F, \varphi\rangle, \quad \text{for all } \varphi \in X_0.$$

We can reason like authors in [26] (see also [18]) to obtain

$$|b(u_k)| \le \|\Upsilon\|_{L^\infty(\partial\Omega)}$$

and

$$|u_k| \le \frac{1}{\lambda}\|g_\epsilon\|_{L^\infty(\Omega)}.$$

Since $|g_\epsilon| \le |g| \Rightarrow \|g_\epsilon\|_\infty \le |g|$, we have

$$\operatorname{meas}(\partial\Omega) \times \|g_\epsilon\|_{L^\infty(\Omega)} \le \|g\|_{L^1(\partial\Omega)}.$$

Hence, we deduce that

$$\|g_\epsilon\|_{L^\infty(\partial\Omega))} \le \frac{\|g\|_{L^1(\partial\Omega)}}{\operatorname{meas}(\partial\Omega)}.$$

Now, we fix $k = \max\left(\|\Upsilon\|_{L^\infty(\partial\Omega)}, \dfrac{\|g\|_{L^1(\partial\Omega)}}{\lambda\operatorname{meas}(\partial\Omega)}\right) + 1$ in $P(T_k(b), \Upsilon)$ and set $\Upsilon = \mu_\epsilon$, $b = \beta_\epsilon$ to end the proof of Theorem 3.4. □

LEMMA 3.5. *Let $u_\epsilon$ be a weak solution of $P(\beta_\epsilon, \mu_\epsilon)$. Then, there exists a positive constant $C(\mu, \Omega)$ such that for any $k > 0$,*

$$\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)} dx \le k\left(\frac{C(\mu, g, \Omega) + \|g\|_{L^1(\partial\Omega)}}{C_3}\right), \tag{3.8}$$

$$\int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \le k(C(\mu, g, \Omega) + \|g\|_{L^1(\partial\Omega)}) \tag{3.9}$$

*and*

$$\|u_\epsilon\|_{L^1(\partial\Omega)} \le \frac{C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}}{\lambda}. \tag{3.10}$$

Proof. We begin by proving (3.8) and (3.9). By taking $\varphi = T_k(u_\epsilon)$ as test function in (3.5), we get

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial T_k(u_\epsilon)}{\partial x_i}dx &+ \int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \\ + \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon)dx &+ \lambda\int_{\partial\Omega} u_\epsilon T_k(u_\epsilon)d\sigma \\ = \int_\Omega T_k(u_\epsilon)d\mu_\epsilon &+ \int_{\partial\Omega} g_\epsilon T_k(u_\epsilon)d\sigma.\end{aligned}$$

Then, taking into account that $\int_{\partial\Omega} g_\epsilon T_k(u_\epsilon)d\sigma \le k\|g\|_{L^1(\partial\Omega)}$, we use (1.5) and (3.3) in the last equality to get

$$\begin{aligned}C_3\sum_{i=1}^N \int_{|u_\epsilon|\le k}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx &+ \int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \\ + \lambda\int_{\partial\Omega} u_\epsilon T_k(u_\epsilon)d\sigma &+ \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon)dx \\ &\le k(C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}).\end{aligned} \tag{3.11}$$

Since $T_k,\ \beta_\epsilon, s\mapsto |s|^{r(\cdot)-2}s$ are nondecreasing and $\beta_\epsilon(0) = T_k(0) = 0$, all the integrals in (3.11) are nonnegative. Therefore, we deduce from (3.11) that

$$\sum_{i=1}^N \int_\Omega \left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx \le k\left(\frac{C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}}{C_3}\right)$$

and

$$\int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \le k(C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}).$$

Let us prove (3.10). We use the fact that all integrals in (3.11) are nonnegative to obtain

$$\lambda\int_{\partial\Omega} u_\epsilon T_k(u_\epsilon)d\sigma \le k(C(\mu,g,\Omega) + \|g\|_{L^1(\partial\Omega)}). \tag{3.12}$$

We deduce from (3.12) that

$$\int_{\partial\Omega} u_\epsilon \frac{1}{k}T_k(u_\epsilon)d\sigma \le \frac{C(\mu,g,\Omega) + \|g\|_{L^1(\partial\Omega)}}{\lambda}. \tag{3.13}$$

Finally, we pass to the limit as $k$ goes to 0 in (3.13) by using Fatou's lemma to get (3.10). ∎

We have the following lemma (we refer to [27, Proposition 4.2]).

LEMMA 3.6. (i) *The sequence* $(\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$.

(ii) *For any* $k > 0$, *the sequence* $(\beta_\epsilon(T_k(u_\epsilon))_{\epsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$.

PROOF.

(i) Dividing the terms in (3.9) by $k > 0$ and letting $k$ goes to 0, we get

$$\lim_{k\to 0}\int_\Omega \beta_\epsilon(u_\epsilon)\frac{1}{k}T_k(u_\epsilon)dx = \int_\Omega \beta_\epsilon(u_\epsilon)sign_0(u_\epsilon)dx.$$

As

$$\int_\Omega \beta_\epsilon(u_\epsilon)sign_0(u_\epsilon)dx = \int_\Omega |\beta_\epsilon(u_\epsilon)|dx \le C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}, \tag{3.14}$$

then, (i) follows.

(ii) Assertion (ii) follows from (i). Since for any $k > 0$,

$$\int_\Omega |\beta_\epsilon(T_k(u_\epsilon))|dx \le \int_\Omega |\beta_\epsilon(u_\epsilon)|dx.$$

□

PROPOSITION 3.7. *Let* $u_\epsilon$ *be a weak solution of (3.4). Then, for all* $k > 0$,

$$\sum_{i=1}^N \int_{|u_\epsilon|\le k} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i^-} dx \le N\,\text{meas}(\Omega) + k\left(\frac{C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}}{C_3}\right) \tag{3.15}$$

*and*

$$\int_{\partial\Omega} |T_k(u_\epsilon)|d\sigma \le \frac{C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}}{\lambda}. \tag{3.16}$$

PROOF. Let us prove (3.15).

$$\begin{aligned}
\sum_{i=1}^N \int_{\{|u_\epsilon|\le k\}} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i^-} dx &= \sum_{i=1}^N \int_{\{|u_\epsilon|\le k, |\frac{\partial u_\epsilon}{\partial x_i}|\le 1\}} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i^-} dx \\
&\quad + \sum_{i=1}^N \int_{\{|u_\epsilon|\le k, |\frac{\partial u_\epsilon}{\partial x_i}|> 1\}} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i^-} dx \\
&\le N\,\text{meas}(\Omega) + \sum_{i=1}^N \int_{\{|u_\epsilon|\le k, |\frac{\partial u_\epsilon}{\partial x_i}|\ge 1\}} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)} dx \\
&\le N\,\text{meas}(\Omega) + \sum_{i=1}^N \int_{\{|u_\epsilon|\le k\}} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)} dx \\
&\le N\,\text{meas}(\Omega) + \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)} dx
\end{aligned}$$

$$\leq N\,\text{meas}(\Omega) + k\left(\frac{C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}}{C_3}\right).$$

Since $|T_k(u_\epsilon)| \leq |u_\epsilon|$, we have

$$\int_{\partial\Omega} |T_k(u_\epsilon)| d\sigma \leq \|u_\epsilon\|_{L^1(\partial\Omega)}. \tag{3.17}$$

Then, we deduce from (3.10) and (3.17) that

$$\int_{\partial\Omega} |T_k(u_\epsilon)| d\sigma \leq \frac{C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)}}{\lambda}$$

∎

LEMMA 3.8. *If $u_\epsilon$ is a weak solution of (3.4), then there exists a constant $D$ which depends on $\mu$ and $\Omega$ such that*

$$\text{meas}\{|u_\epsilon| > k\} \leq \frac{D}{\min(b_\epsilon(k), |b_\epsilon(-k)|)},\ \forall\ k > 0 \tag{3.18}$$

*and a constant $D'$ which depends on $\mu$ and $\Omega$ such that*

$$\text{meas}\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > k\right\} \leq \frac{D'}{k^{\frac{1}{(p_M^-)'}}},\ \forall k \geq 1. \tag{3.19}$$

PROOF. For the proof of (3.18), we refer to [20, 27]. For the proof of (3.19), we refer to [6]. ∎

We need the following lemma (see [6, 17, 18, 22]).

LEMMA 3.9. *For any $k > 0$, there exists some positive constants $C$ and $C'$ such that*

(i) $\|u_\epsilon\|_{\mathcal{M}^{q^*}(\Omega)} \leq C$;

(ii) $\left\|\frac{\partial u_\epsilon}{\partial x_i}\right\|_{\mathcal{M}^{p_i^- \frac{q}{p}}(\Omega)} \leq C', \quad \forall\ i = 1, \ldots, N.$

STEP 3 (Convergence results). In order to pass to the limit, the following convergence results are necessary (see [6] and [22]).

LEMMA 3.10. *For $i = 1, \ldots, N$, as $\epsilon \to +\infty$, we have*

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ in } L^1(\Omega) \text{ a.e. } x \in \Omega. \tag{3.20}$$

PROPOSITION 3.11. *There exists a measurable function $u : \Omega \longrightarrow \mathbb{R}$ such that $u \in \text{dom}(\beta)$ a.e. in $\Omega$ and*

$$u_\epsilon \longrightarrow u \text{ in measure and a.e. in } \Omega \text{ as } \epsilon \longrightarrow 0. \tag{3.21}$$

Proof. For the proof of (3.21), we refer to [6] (see also [22]).

As for $k > 0$, $T_k$ is continuous, then $T_k(u_\epsilon) \to T_k(u)$ a.e. in $\Omega$. Finally, using Lemma 1.1 we deduce that for all $k > 0$, $T_k(u) \in \mathrm{dom}(\beta)$ a.e. in $\Omega$. Since $T_k(u) \in \mathrm{dom}(\beta)$, we get $u \in \mathrm{dom}(\beta)$ a.e. in $\Omega$ and as $\mathrm{dom}(\beta)$ is bounded, then $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. ∎

Proposition 3.12. *Assume (1.2)-(1.7). If* $u_\epsilon \in E$ *is a weak solution of* (3.4) *then*

(i) *for all* $i = 1, \ldots, N$, $\dfrac{\partial u_\epsilon}{\partial x_i}$ *converges in measure to the weak partial gradient of* $u$;

(ii) *for all* $i = 1, \ldots, N$ *and* $k > 0$, $a_i\left(x, \frac{\partial}{\partial x_i} T_k(u_\epsilon)\right)$ *converges to* $a_i\left(x, \frac{\partial}{\partial x_i} T_k(u)\right)$ *in* $L^1(\Omega)$ *strongly and in* $L^{p_i'(\cdot)}(\Omega)$ *weakly;*

(iii) *for* $i = 1, \ldots, N$, $a_i\left(x, \dfrac{\partial u_n}{\partial x_i}\right) \dfrac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \dfrac{\partial u}{\partial x_i}\right) \frac{\partial u}{\partial x_i}$ *in* $L^1(\Omega)$ *and a.e.* $x \in \Omega$.

Proof. For the proof of (i) and (ii) we refer to [6].

(iii) The continuity of $a_i(x, \xi)$ with respect to $\xi \in \mathbb{R}$ gives us

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ a.e. } x \in \Omega.$$

Therefore,

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial u}{\partial x_i} \text{ a.e. } x \in \Omega.$$

Setting $y_\epsilon = a_i\left(x, \dfrac{\partial u_\epsilon}{\partial x_i}\right) \dfrac{\partial u_\epsilon}{\partial x_i}$ and $y = a_i\left(x, \dfrac{\partial u}{\partial x_i}\right) \dfrac{\partial u}{\partial x_i}$, for $i = 1, \ldots, N$, we have

$$\begin{cases} y_\epsilon \geq 0, y_\epsilon \to y \text{ a.e. in } \Omega,\ y \in L^1(\Omega), \\ \displaystyle\int_\Omega y_\epsilon dx \to \int_\Omega y dx \end{cases}$$

and as

$$\int_\Omega |y_\epsilon - y| dx = 2 \int_\Omega (y - y_\epsilon)^+ dx + \int_\Omega (y_\epsilon - y) dx$$

and $(y - y_\epsilon)^+ \leq y$, it follows by using Lebesgue dominated convergence theorem, that

$$\lim_{\epsilon \to 0} \int_\Omega |y_\epsilon - y| dx = 0,$$

which means that

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial u}{\partial x_i} \text{ in } L^1(\Omega) \text{ strongly.}$$

∎

We have the following lemmas (see [21, 25, 27]).

LEMMA 3.13. *For any* $h \in C_c^1(\mathbb{R})$ *and* $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, *for any* $i = 1, \ldots, N$,

$$\frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi) \longrightarrow \frac{\partial}{\partial x_i}(h(u)\varphi) \text{ strongly in } L^1(\Omega) \text{ as } \epsilon \to 0.$$

LEMMA 3.14. *We have*

$$\lim_{\epsilon \to 0} \int_\Omega h(u_\epsilon)\xi d\mu_\epsilon = \int_\Omega h(u)\xi d\mu, \tag{3.22}$$

$$\lim_{\epsilon \to 0} \epsilon \int_\Omega |u_\epsilon|^{p_M(x)-2} u_\epsilon h(u_\epsilon)\xi dx = 0 \tag{3.23}$$

*and*

$$\lim_{\epsilon \to 0} \lambda \int_{\partial\Omega} u_\epsilon h(u_\epsilon)\varphi d\sigma = \lambda \int_{\partial\Omega} uh(u)\varphi d\sigma, \tag{3.24}$$

*for any* $h \in C_c^1(\mathbb{R})$ *and* $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Now, we pass to the limit in $\beta_\epsilon(u_\epsilon)$.

Since, for any $k > 0$, $(h_k(u_\epsilon)\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k \in \mathcal{M}_b(\Omega)$ such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon) \rightharpoonup z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, for any $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned} \int_\Omega \varphi dz_k = &- \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h_k(u)\varphi]dx \\ &- \lim_{\epsilon \to 0} \epsilon \int_\Omega \varphi |u|^{P_M(x)-2} u h_k(u) dx \\ &- \lambda \int_{\partial\Omega} \varphi u h_k(u) d\sigma + \int_\Omega \varphi h_k(u) d\mu + \int_{\partial\Omega} \varphi g h_k(u) d\sigma. \end{aligned}$$

Since

$$- \lim_{\epsilon \to 0} \epsilon \int_\Omega \varphi |u|^{P_M(x)-2} u h_k(u) dx = 0,$$

we have

$$\begin{aligned} \int_\Omega \varphi dz_k = &- \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h_k(u)\varphi]dx - \lambda \int_{\partial\Omega} \varphi u h_k(u) d\sigma \\ &+ \int_\Omega \varphi h_k(u) d\mu + \int_{\partial\Omega} \varphi g h_k(u) d\sigma, \end{aligned}$$

which implies that $z_k \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$ and, for any $k \le l$,

$$z_k = z_l \text{ on } [|T_k(u)| < k].$$

Let us consider the Radon measure defined by

$$\left\{ \begin{array}{l} z = z_k, \quad \text{on } [|T_k(u)| < k] \text{ for } k \in \mathbb{N}^*, \\ z = 0 \quad \text{on } \displaystyle\bigcap_{k \in \mathbb{N}^*} [|T_k(u)| = k]. \end{array} \right. \tag{3.25}$$

For any $h \in \mathcal{C}_c(\mathbb{R})$, $h(u) \in L^\infty(\Omega, d|z|)$ and

$$\begin{aligned} \int_\Omega h(u)\varphi dz = & -\int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u)\varphi]dx - \lambda \int_{\partial\Omega} uh(u)\varphi d\sigma \\ & + \int_\Omega h(u)\varphi d\mu + \int_{\partial\Omega} gh(u)\varphi d\sigma, \end{aligned}$$

for any $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Indeed, let $k_0 > 0$ be such that $\text{supp}(h) \subseteq [-k_0, k_0]$,

$$\begin{aligned} & \int_\Omega h(u)\xi dz = \int_\Omega h(u)\xi dz_{k_0} \\ & = -\lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx \\ & \quad - \lim_{\epsilon\to 0} \epsilon \int_\Omega |u_\epsilon|^{p_M(x)-2} u_\epsilon h(u_\epsilon)\varphi dx - \lim_{\epsilon\to 0} \lambda \int_{\partial\Omega} u_\epsilon h(u_\epsilon)\varphi d\sigma \\ & \quad + \lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon + \lim_{\epsilon\to 0} \int_{\partial\Omega} g_\epsilon h(u_\epsilon)\varphi d\sigma \\ & = -\lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx \\ & \quad - \lim_{\epsilon\to 0} \epsilon \int_\Omega |u_\epsilon|^{p_M(x)-2} u_\epsilon h(u_\epsilon)\varphi dx - \lim_{\epsilon\to 0} \lambda \int_{\partial\Omega} u_\epsilon h(u_\epsilon)\varphi d\sigma \\ & \quad + \lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon + \lim_{\epsilon\to 0} \int_{\partial\Omega} g_\epsilon h(u_\epsilon)\varphi d\sigma \\ & = -\int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u)\xi]dx - \lambda \int_{\partial\Omega} uh(u)\varphi d\sigma \\ & \quad + \int_\Omega h(u)\varphi d\mu + \int_{\partial\Omega} gh(u)\varphi d\sigma. \end{aligned}$$

Moereover, we have the following lemma (see [25, Lemma 4.7]).

Lemma 3.15. *The Radon-Nikodym decomposition of the measure $z$ given by* (3.25) *with respect to $\mathcal{L}^N$,*

$$z = b\mathcal{L}^N + \nu \quad \textit{with } \nu \perp \mathcal{L}^N$$

*satisfies the following properties*

$$\begin{cases} b \in \beta(u)\mathcal{L}^N - a.e. \ in \ \Omega, \ b \in \ L^1(\Omega), \ \nu \ \in \mathcal{M}_b^{p_i(\cdot)}(\Omega), \\ \nu^+ \ is \ concentrated \ on \ \ [u = M], \\ \nu^- \ is \ concentrated \ on \ \ [u = m]. \end{cases}$$

To end the proof of Theorem 3.2, we consider $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\varphi$ as a test function in (3.5) to get

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\varphi dx \\ &\quad + \epsilon \int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon h(u_\epsilon)\varphi dx + \lambda \int_{\partial\Omega} u_\epsilon h(u_\epsilon)\varphi d\sigma \\ &= \int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon + \int_{\partial\Omega} g_\epsilon h(u_\epsilon)\varphi d\sigma. \end{aligned} \tag{3.26}$$

The first term of (3.26) can be written as

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial T_{l_0+1}(u_\epsilon)}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx \\ &\qquad = \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx, \end{aligned}$$

for some $l_0 > 0$; so, by Lemmas 3.13 and 3.14, we have

$$\begin{aligned} &\lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx \\ &\quad = \lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial T_{l_0+1}(u_\epsilon)}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx \\ &\quad = \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial T_{l_0+1}(u)}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u)\xi]dx \\ &\quad = \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u)\xi]dx. \end{aligned}$$

By using convergence results in (3.26), we get

$$\begin{aligned} \lim_{\epsilon\to 0} \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx &= \int_\Omega h(u)\xi d\mu + \int_{\partial\Omega} gh(u)\varphi d\sigma - \lambda \int_{\partial\Omega} uh(u)\varphi d\sigma \\ &\quad - \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u)\xi]dx \end{aligned}$$

$$= \int_\Omega h(u)\varphi dz = \int_\Omega bh(u)\varphi dx + \int_\Omega h(u)\varphi d\nu.$$

Passing to the limit in (3.26) as $\epsilon \to 0$, we get

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u)\varphi]dx + \int_\Omega bh(u)\varphi dx + \int_\Omega h(u)\varphi d\nu + \lambda\int_{\partial\Omega} uh(u)\varphi d\sigma = \int_\Omega h(u)\varphi d\mu + \int_{\partial\Omega} gh(u)\varphi d\sigma. \tag{3.27}$$

Letting $\epsilon$ goes to 0 in (3.26) it yields that $(b, u)$ is a solution of the problem (1.1). To end the proof of Theorem 3.2, we prove (3.2). We take $\xi = T_1(u_\epsilon - T_n(u_\epsilon))$ as a test function in (3.5) to get

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}(T_1(u_\epsilon - T_n(u_\epsilon)))dx + \int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx + \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx + \lambda\int_{\partial\Omega} u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))d\sigma = \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon + \int_\Omega g_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx. \tag{3.28}$$

Since

$$\int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx + \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx + \lambda\int_{\partial\Omega} u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))d\sigma \geq 0$$

and

$$\frac{\partial}{\partial x_i}(T_1(u_\epsilon - T_n(u_\epsilon))) = \frac{\partial u_\epsilon}{\partial x_i}\chi_{[n<|u_\epsilon|<n+1]},$$

we have from equality (3.28),

$$\sum_{i=1}^N \int_{[n<|u_\epsilon|<n+1]} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i}dx \leq \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon + \int_\Omega g_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx. \tag{3.29}$$

We have (see [27])

$$\lim_{n\to+\infty}\lim_{\epsilon\to 0}\int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon = 0$$

and

$$\lim_{n\to+\infty}\lim_{\epsilon\to 0}\int_\Omega g_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx = 0.$$

Then, using (1.5), and letting $n \to +\infty$, $\epsilon \to 0$ respectively in (3.12), we get

$$\begin{aligned}\lim_{n\to+\infty}\lim_{\epsilon\to 0}\frac{1}{C}\sum_{i=1}^{N}\int_{[n<|u_\epsilon|<n+1]}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx\\ = \lim_{n\to+\infty}\frac{1}{C}\sum_{i=1}^{N}\int_{[n<|u|<n+1]}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx \le 0.\end{aligned}$$

□

The connection between our notion of solution and the entropic formulation is given in the following Theorem. In particular, as the domain of $\beta$ is bounded, this equivalent formulation is very useful for the proof of uniqueness of solution to problem (1.1). We reason as in [25] to get the following results.

THEOREM 3.16. *If $(u, b)$ is a solution of (1.1) in the sense of Theorem 3.2, then $(u, b)$ is a solution in the following sense: for any $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi \in \operatorname{dom}\beta$ and for any $k > 0$*

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-\varphi)dx + \int_\Omega bT_k(u-\varphi)dx\\ &\quad + \lambda\int_{\partial\Omega} uT_k(u-\varphi)d\sigma\\ &\le \int_\Omega T_k(u-\varphi)d\mu + \int_{\partial\Omega} gT_k(u-\varphi)d\sigma.\end{aligned} \tag{3.30}$$

The result of the uniqueness of solution to problem (1.1) is the following.

THEOREM 3.17. *Let $(u_1, b_1)$ and $(u_2, b_2)$ be two solutions of (1.1). Then*

$$\begin{cases} u_1 = u_2 \text{ a.e.} & \text{in } \Omega,\\ b_1 = b_2 \text{ a.e.} & \text{in } \Omega.\end{cases}$$

PROOF. For $u_1$, we choose $\varphi = u_2$ as a test function in (3.30) to get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_1}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u_1-u_2)dx + \int_\Omega b_1T_k(u_1-u_2)dx\\ &\quad + \lambda\int_{\partial\Omega} u_1T_k(u_1-u_2)d\sigma \le \int_{\partial\Omega} gT_k(u_1-u_2)d\sigma + \int_\Omega T_k(u_1-u_2)d\mu.\end{aligned}$$

Similarly for $u_2$, we choose $\varphi = u_1$ as a test function in (3.30) to get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_2}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u_2-u_1)dx + \int_\Omega b_2T_k(u_2-u_1)dx\\ &\quad + \lambda\int_{\partial\Omega} u_2T_k(u_2-u_1)d\sigma \le \int_\Omega T_k(u_2-u_1)d\mu + \int_{\partial\Omega} gT_k(u_2-u_1)d\sigma.\end{aligned}$$

Adding these two inequalities yields

$$\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial u_1}{\partial x_i})-a_i(x,\frac{\partial u_2}{\partial x_i})\right)\frac{\partial}{\partial x_i}T_k(u_1-u_2)dx$$
$$+\int_{\Omega}(b_1-b_2)T_k(u_1-u_2)dx+\lambda\int_{\partial\Omega}(u_1-u_2)T_k(u_1-u_2)d\sigma\le 0.$$

Therefore, as in [18] the result follows. □

## References


[1] L. Alvarez, P.-L. Lions, and J.-M. Morel, *Image selective smoothing and edge detection by nonlinear diffusion. II*, SIAM J. Numer. Anal. **29** (1992), 845–866.

[2] M. Bendahmane, M. Langlais and M. Saad, *On some anisotropic reaction- diffusion systems with $L^1$-data modeling the propagation of an epidemic disease,* Nonlinear Anal. **54** (2003), 617–636.

[3] L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. **87** (1989), 149–169.

[4] L. Boccardo and T. Gallouët, *Nonlinear elliptic equations with right hand side measures*, Comm. Partial Differential Equations **17** (1992), 641–655.

[5] L. Boccardo, T. Gallouët and L. Orsina, *Existence and uniqueness of entropy solution for nonlinear elliptic equations with measure data*, Ann. Inst. Henri Poincaré Anal. Non Linéaire **13** (1996), 539–551.

[6] B. Koné, S. Ouaro and F.D.Y. Zongo, *Nonlinear elliptic anisotropic problem with Fourier boundary condition*, Int. J. Evol. Equ. **8** (2013), 305–328.

[7] M.-M. Boureanu and V. D. Radulescu, *Anisotropic Neumann problems in Sobolev spaces with variable exponent*, Nonlinear Anal. **75** (2012), 4471–4482.

[8] H. Brézis, Analyse fonctionnelle. Théorie et applications, Masson, Paris, 1983.

[9] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North Holland, Amsterdam, 1973.

[10] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration,* SIAM J. Appl. Math. **66** (2006), 1383–1406.

[11] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 741–808.

[12] L. Diening, Theoretical and numerical results for electrorheological fluids, PhD. thesis, University of Frieburg, Germany, 2002.

[13] G. Dolzmann, N. Hungerbühler and S. Müller, *Non-linear elliptic systems with measure-valued right-hand side*, Math Z. **226** (1997), 545–574.

[14] F. Ettwein and M. Ruzicka, *Existence of strong solutions for electrorheological fluids in two dimensions: steady Dirichlet problem,* in: Geometric Analysis and Nonlinear Partial Differential Equations, Springer-Verlag, Berlin, 2003, 591–602.

[15] X. Fan, *Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p}(\cdot)$-Laplacian equations*, Complex Var. Elliptic Equ. **56** (2011), 623–642.

[16] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), 424–446.

[17] I. Ibrango and S. Ouaro, *Entropy solutions for anisotropic nonlinear problems with homogeneous Neumann boundary condition,* J. Appl. Anal. Comput. **6** (2016), 271–292.

[18] I. Ibrango and S. Ouaro, *Entropy solution for doubly nonlinear elliptic anisotropic problems with Fourier boundary conditions*, Discuss. Math. Differ. Incl. Control Optim. **35** (2015), 123–150.

[19] N. Igbida, S. Ouaro and S. Soma, *Elliptic problem involving diffuse measure data*, J. Differential Equations **253** (2012), 3159–3183.

[20] I. Konaté and S. Ouaro, *Good Radon measure for anisotropic problems with variable exponent,* Electron. J. Differential Equations **2016** (2016), No. 221, 19 pp.

[21] I. Konaté and S. Ouaro, *Nonlinear multivalued problems with variable exponent and diffuse measure data in anisotropic space*, Gulf J. Math. **6** (2018), 13–30.

[22] B. Koné, S. Ouaro and S. Soma, *Weak solutions for anisotropic nonlinear elliptic problem with variable exponent and measure data*, Int. J. Evol. Equ. **5** (2010), 327–350.

[23] O. Kovacik and J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41(116)** (1991), 592–618.

[24] M. Mihailescu, P. Pucci and V. Radulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Anal. Appl. **340** (2008), 687–698.

[25] I. Nyanquini, S. Ouaro and S. Soma, *Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data*, An. Univ. Craiova Ser. Mat. Inform. **40** (2013), 174–198.

[26] I. Nyanquini, S. Ouaro, *Entropy solution for nonlinear elliptic problem involving variable exponent and Fourier type boundary condition*, Afr. Mat. **23** (2012), 205–228.

[27] S. Ouaro, A. Ouédraogo and S. Soma, *Multivalued homogeneous Neumann problem involving diffuse measure data and variable exponent*, Nonlinear Dyn. Syst. Theory **16** (2016), 102–114.

[28] C. Pfeiffer, C. Mavroidis, Y. Bar-Cohen, B. Dolgin, *Electrorheological fluid based force feedback device,* in Proc. SPIE 3840, Telemanipulator and Telepresence Technologies VI, 1999, 88–99.

[29] K. Rajagopal and M. Ružička, *Mathematical modelling of electro-rheological fluids,* Cont. Mech. Therm. **13** (2001), 59–78.

[30] M. Ružička, Electrorheological fluids: modelling and mathematical theory, Springer, Berlin, 2000.

[31] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Recherche. Mat **18** (1969), 3–24.

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