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PRINCIPLES OF FRACTAL GEOMETRY AND APPLICATIONS IN ARCHITECTURE AND CIVIL ENGINEERING

Anton Vrdoljak, M.Sc.
Faculty of Civil Engineering, University of Mostar, *anton.vrdoljak@gf.sum.ba*
Kristina Miletić, B.Sc.(Math.)
Faculty of Civil Engineering, University of Mostar, *kristina.miletic@gf.sum.ba*

Abstract: This paper presents a brief overview of fractals and some possible applications of fractal geometry in architecture and civil engineering. The interest in this mathematical discipline has been steadily growing since the end of the 20th century, due to the fascinating beauty of fractals as well as the generally accepted perception that many shapes in nature, if not all, are irregular and rugged, or chaotic (nature is fractal). Fractal geometry, in contrast to Euclidean geometry, offers considerably better methods for describing natural objects, and thereby for achieving harmony with nature, or harmony between clear precision and chaotic imperfection.

Keywords: fractals, self–similarity, fractal geometry, golden ratio, architecture, civil engineering

NAČELA FRAKTALNE GEOMETRIJE I PRIMJENE U ARHITEKTURI I GRAĐEVINARSTVU

Sažetak: U ovom radu prikazan je kraći osvrt na fraktale, te neke moguće primjene fraktalne geometrije u arhitekturi i građevinarstvu. Zanimanje za ovom matematičkom disciplinom je u neprekidnom rastu još od kraja 20. stoljeća, kako zbog same očaravajuće ljepote fraktala, tako i zbog općeprihvaćene spoznaje da je dosta oblika u prirodi, ako ne i svi, nepravilno i neravno, odnosno kaotično (priroda je fraktalna). Fraktalna geometrija, nasuprot euklidskoj, nudi znatno bolje metode za opisivanje prirodnih objekata, te samim time i za postizanje harmonije sa prirodom, tj. sklada između izrazite preciznosti i kaotične nesavršenosti.

Ključne riječi: fraktali, samosličnost, fraktalna geometrija, zlatni rez, arhitektura, građevinarstvo

1. Introduction

Fractals are considered one of the greatest secrets of nature's design, known to people from times immemorial, just not recognized as such. One of the most important fractals - golden ratio (golden section) - is found in ancient cultures, especially in ancient Greek mathematics. Fractals have been used as ornamental elements since distant past, and a documented description of fractals in art is found as far back as 1525 in Albrecht Dürer's *The Painter's Manual*, which describes samples created by using a pentagon (golden ratio). Benoit Mandelbrot, a French-American mathematician of Polish origin, coined the term fractal and defined its meaning in 1975, and fractals have been intensively studied as a mathematical discipline ever since. Although considered the father of fractal geometry, many parts (mathematical concepts) from his book *The Fractal Geometry of Nature* had already been described (constructed) by other mathematicians in the late 19th and early 20th century (Cantor, Hilbert, Koch , Sierpiński...), and some of the roots of the ideas of fractals can be found in the works dating back to the 18th century, of our scientist and philosopher Ruđer Bošković.

2. What are Fractals?

The word fractal was derived from the Latin word *frangere*, which would mean to break, crumble or fragment in free translation [16]. According to Uglešić, in Croatian they could be called *mrvljeniks* [19]. From the Latin language comes the word *fractus*, which is the root of the English word *fractured*, and which is the past participle of the word *frangere*. According to Mandelbrot's definition, these are the geometric objects whose fractal dimension is greater than or equal to the Euclidean dimension [16]. It is simplest to define them as self-similar infinite objects. In other words, they would be geometric objects that provide the same level of detail regardless of the resolution used. Fractals can be magnified an infinite number of times, with each new magnification showing some details that were not visible before the magnification, and with the amount of new details being always approximately the same. Although irregular in shape, jagged and having a repeating form, we can find them everywhere around us.

Thus, the basic idea of fractal geometry is self-similarity - the property of an object to look like itself regardless of which part of it is observed and regardless of how many times it is enlarged, i.e. it can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Roughly speaking, an object is said to be self-similar if it can be broken down into objects that are smaller copies of itself, i.e., if the structure of the whole object is contained in its parts [19]. In addition to self-similarity, it is considered that the basic properties of fractals are also fractal dimension (i.e. degree of irregularity) and formation (construction) by iteration (formation by iteration is the property of generating an object by a mathematical or geometrical process, so that the properties of the generator are iteratively applied on the initial object) [16] .

The fractal dimension would be a value that gives us an insight into the extent to which the fractal fills the space in which it is located. In other words, the fractal dimension is used to express the density at which the object fills the space, or to express how many new parts appear when increasing the resolution. The fractal dimension is not an integer and it is normally greater than the Euclidean dimension [16]. For self-similar objects $\mathfrak{F}$ it is natural to define the self-similarity dimension or scaling dimension $d(\mathfrak{F})$ by the expression:

$$d(\mathfrak{F}) = \frac{\log(N)}{\log(r)}, \tag{1}$$

where N is the number of new copies of the object observed after magnification, and r is the magnification factor.

Fractals can be classified according to the degree of their self-similarity and according to the type of formation [16]. According to the degree of self-similarity, fractals can be divided into:

- exactly self-similar fractals - contain copies of themselves that are similar to the whole fractal (the strongest degree of self-similarity), and are also called geometric fractals;
- quasi self-similar fractals - contain small copies of themselves that are not similar to the whole fractal, and appear in a distorted form, and are also known as algebraic fractals;
- statistically self-similar fractals - do not contain copies of themselves (the lowest degree of self-similarity), but some fractal properties (e.g. fractal dimension) remain the same at different scales.

According to the type of formation, fractals can be divided into:

- iterative fractals - formed by copying and rotating and/or translating the copy, and possibly replacing an element with a copy - these are self-similar fractals;
- recursive fractals - are defined by a recursive mathematical formula that determines whether a given point of space (e.g. complex or Gauss plane) belongs to the set or not - these are quasi-self-similar fractals;
- random fractals - have the lowest degree of self-similarity and are frequently found in nature (coastlines, river branches and flows, mountain ranges, jungles, tree roots and tops, leaves, flowers, clouds, lightnings, climate systems, snowflakes, bacteria, lungs, blood vessel systems...) - these are statistically self-similar fractals.

Examples of exactly self-similar fractals are:

- Cantor set - a subset of separated points remaining after dividing a line segment of length 1 into three equal subsegments, removing all the points from the middle subsegment, dividing each of the remaining subsegments into three, removing all the points from the middle one of these three subsegments and continuing so infinitely many times (Figure 1.a);
- Cantor dust and cloud - multidimensional analogues of the Cantor set;
- Koch curve - is constructed by dividing a line segment of length a into three equal segments, each of length $a/3$, and then adding to the middle segment two more line segments of equal lengths so that they form an equilateral triangle together with the middle segment; then the middle segment is removed and now we have four line segments of equal lengths (each of them being $a/3$); in the second step the procedure from the first step is repeated for each of these four segments, etc. (Figure 1.b);
- Koch snowflake – is obtained when the iteration procedure described for a Koch curve, instead of a line segment of length a, starts with an equilateral triangle with sides of length a (Figure 2);
- Sierpiński sieve (triangle) - construction starts from the initial equilateral triangle, by removing a triangle (also equilateral) obtained by connecting midpoints of the sides of the initial triangle; the remaining three equilateral triangles represent the starting point for the next step, etc. (Figure 3);
- Sierpiński carpet and tetrahedron - are obtained when a similar procedure of iteration described for Sierpiński sieve is applied to square or tetrahedron;
- Menger sponge - a three-dimensional analogue of the Sierpinski carpet; each side of the Menger sponge is a Sierpiński carpet, and each diagonal a Cantor set (Figure 4).

Julia set (Figure 5.a) and Mandelbrot set (represents the unity of simplicity and complexity, Figure 5.b) are the best-known examples of quasi-self-similar fractals, and in many people's opinion the most beautiful fractal forms as well. Some of the examples of statistically self-similar fractals are Perlin noise, predominantly used in computer graphics (for example, when generating computer landscapes), Lorenz attractor, Brownian motion.

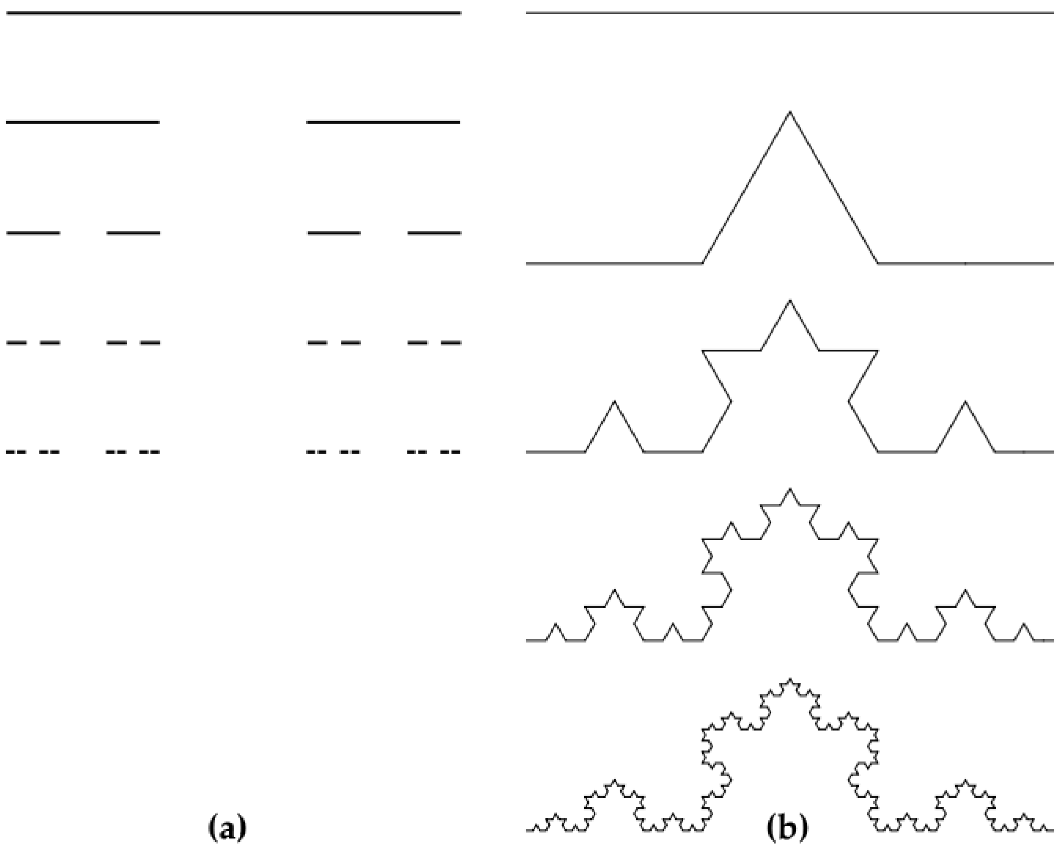

Figure 1. The first four iterations for: (a) Cantor set; (b) Koch curve [36].

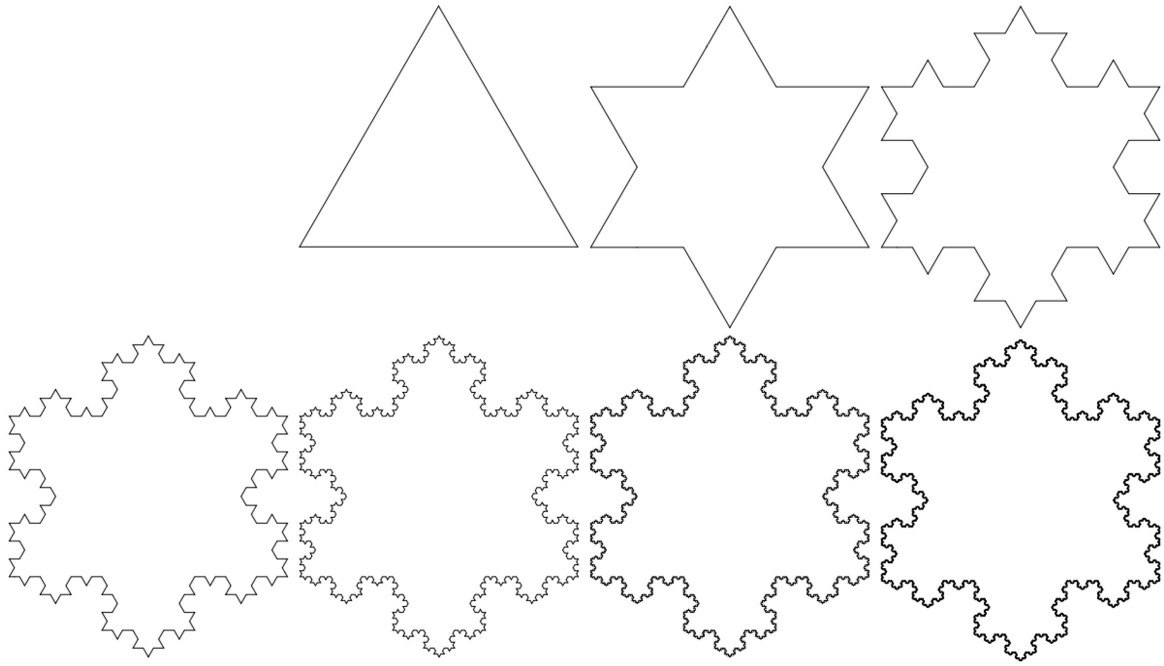

Figure 2. Initial equilateral triangle and the first six iterations for the Koch snowflake [36].

The reason why they are considered one of the greatest secrets of nature's design is best understood by observing Table 1. It is easy to conclude from it that the length of the Koch curve (initial geometric object is a line segment of length a) increases by a factor of 4/3 in each subsequent iteration, or the length of the Koch curve L_n will tend to infinity when the number of repetitions (iterations) n tends to infinity, or

$$L = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \left(\frac{4}{3}\right)^n a \to +\infty. \tag{2}$$

On the other hand, using the well-known equation for the area of an equilateral triangle and properly applying it to bases obtained by the iterative procedure, we can conclude that the area of the plane figure within the Koch curve is a finite, constant value, although this area increases with every new iteration! Namely, the following holds true:

$$P = \frac{\sqrt{3}}{4}\left(\frac{a}{3}\right)^2 + 4\frac{\sqrt{3}}{4}\left(\frac{a}{9}\right)^2 + 16\frac{\sqrt{3}}{4}\left(\frac{a}{27}\right)^2 + \ldots = \frac{a^2\sqrt{3}}{16}\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^k = \frac{a^2\sqrt{3}}{16}\frac{4}{5} = \boxed{\frac{a^2\sqrt{3}}{20}}. \tag{3}$$

The same conclusion can also be assumed for the perimeter of the Koch snowflake (the initial geometric object is an equilateral triangle with sides of length a), i.e. the perimeter of the Koch snowflake is infinite (or tends to infinity). However, its area is finite, i.e. the following holds true:

$$\begin{aligned} P &= \frac{\sqrt{3}}{4}a^2 + 3\frac{\sqrt{3}}{4}\left(\frac{a}{3}\right)^2 + 3\cdot 4\frac{\sqrt{3}}{4}\left(\frac{a}{9}\right)^2 + 3\cdot 4\cdot 4\frac{\sqrt{3}}{4}\left(\frac{a}{27}\right)^2 + 3\cdot 4\cdot 4\cdot 4\frac{\sqrt{3}}{4}\left(\frac{a}{81}\right)^2 + \ldots = \\ &= \frac{\sqrt{3}}{4}a^2 + \frac{3\sqrt{3}}{4}a^2\sum_{k=1}^{\infty}4^{k-1}\frac{1}{9^k} = \frac{\sqrt{3}}{4}a^2 + \frac{3\sqrt{3}}{16}a^2\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^k = \frac{\sqrt{3}}{4}a^2 + \frac{3\sqrt{3}}{16}a^2\frac{4}{5} = \boxed{\frac{2\sqrt{3}}{5}a^2}. \end{aligned} \tag{4}$$

So, the Koch snowflake is a bounded plane figure of finite area (equal to 8/5 of the area of the initial geometric object) with an infinite perimeter!!! The fractal dimensions (similarity dimensions) of the Koch curve, and Koch snowflakes are the same and approximately equal to 1.26, while the fractal dimension of the Cantor set is approximately 0.63, and are greater than their topological dimensions. It is characteristic of the Cantor set that it is equipotent (equinumerous) with the set of real numbers, as well as compact and a perfect set, which is also completely unconnected!!!! Roughly speaking, compactness of a set (space) is a generalization of finiteness. Informal definition would be: a topological space is compact if it has the characteristic that, whenever it is a subset of a union of infinitely many open sets, then it is a subset of a union of finitely many of these sets. Roughly speaking, a connected set is one "that consists of one piece", and a totally disconnected set is one where its only connected subsets are singleton subsets. A perfect set is every set whose points are accumulation points of that set.

Table 1. Lengths of Cantor set and Koch curve after the first four iterations.

Fractal	Length after the 1st iteration	Length after the 2nd iteration	Length after the 3rd iteration	Length after the 4th iteration
Cantor set	2/3 ≈ 66.67%	4/9 ≈ 44.44%	8/27 ≈ 29.63%	16/81 ≈ 19.75%
Koch curve	$L_1 = (4/3)\cdot a \approx 1.33\cdot a$	$L_2 = (16/9)\cdot a \approx 1.78\cdot a$	$L_3 = (64/27)\cdot a \approx 2.37\cdot a$	$L_4 = (256/81)\cdot a \approx 3.16\cdot a$

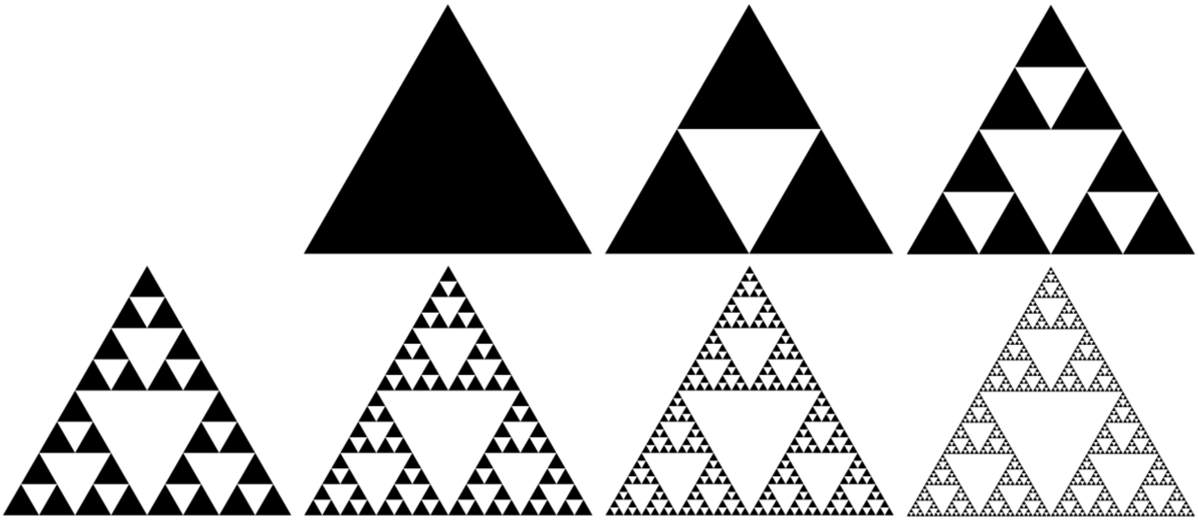

Figure 3. Initial equilateral triangle and the first six iterations of the Sierpiński sieve (triangle) [36].

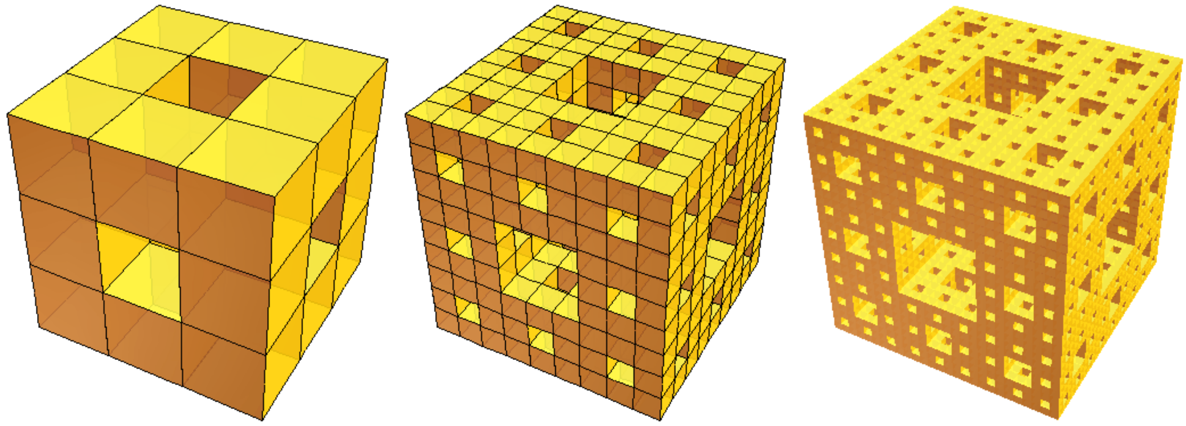

Figure 4. The first three iterations of the Menger sponge – cube with volume 0 and surface area ∞ [36].

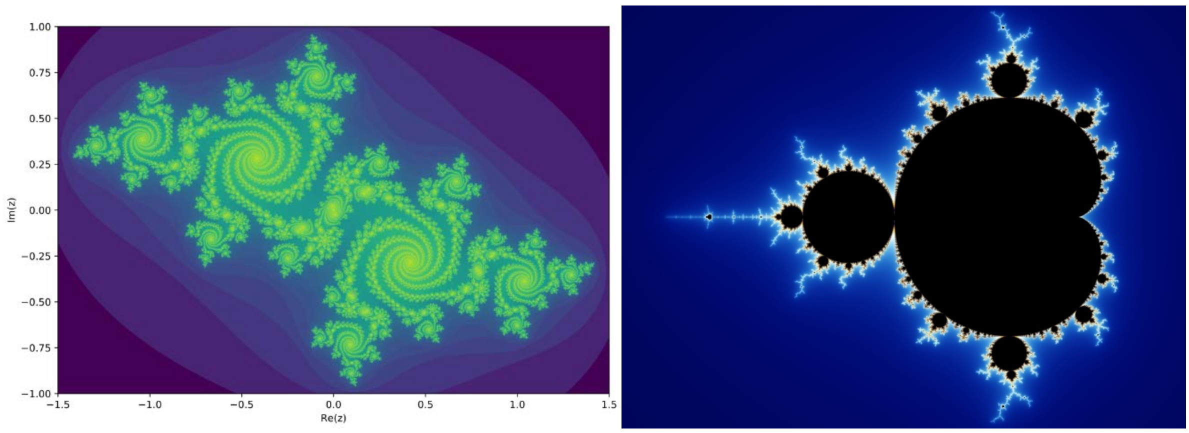

Figure 5. (a) Julia set; (b) Mandelbrot set [source: www.wikipedia.org].

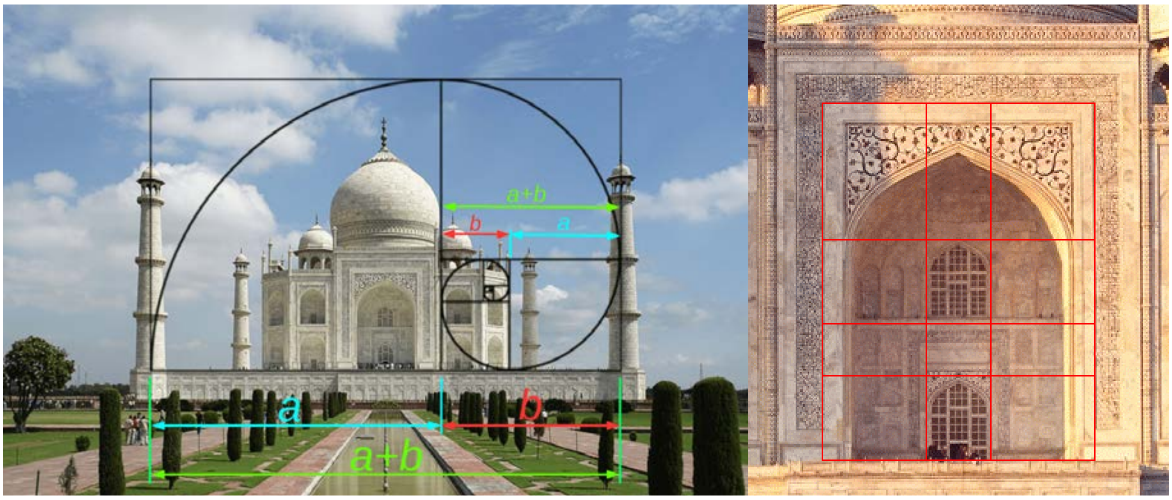

Figure 6. The Taj Mahal Mausoleum in Agra built in the 17th century; the outer frames of the main building as well as the frame of the main gate are golden rectangles [33], [source: www.pinterest.com].

3. Fractal geometry in architecture and civil engineering

Intentionally or unintentionally, architects, builders and other construction experts have been using mathematics and geometry since ancient times as the most basic but nevertheless rather valuable tool in almost all stages of architectural and construction projects. History remembers great builders and architects who were also great mathematicians, and vice versa (Vitruvius, Leonardo Da Vinci,...). Mathematics and architecture have always been close, most of all because of their common aspiration for order and beauty [33]. The discovery of fractal geometry (or the geometry of nature), attributed to the French-American mathematician Benoit Mandelbrot, has inevitably led to a great revolution in natural and technical sciences, and consequently in architecture and civil engineering as fields of technical sciences. The reason why fractals are used so much lies in that many shapes in nature (coastlines, river branches and flows, mountain ranges, jungles, tree roots and tops, leaves, flowers, clouds, lightnings, climate systems, snowflakes, bacteria, lungs, blood vessel systems...) are irregular and rugged, and offer these irregularities on different scales. The characteristics of all these shapes are essentially contrary to the characteristics of regular geometric shapes and bodies of Euclidean geometry (ball, cube, pyramid, cone), but can be considerably better represented using fractals (nature is fractal). In other words, fractal geometry, in contrast to the Euclidean one, offers considerably better methods for describing natural objects. The rugged characteristics of nature are not modeled by smooth shapes of Euclidean geometry, but it is the new approach of fractal complexity that copes with the irregularities of the structure itself. Although complex, a fractal is usually described by a simple algorithm, indicating that there is a law behind the greatest ruggedness and irregularities [16].

Despite the fact that fractal geometry developed only in the late 20th century, fractals have been known to people from times immemorial, they were just not recognized as such (for example, golden ratio was studied by Pythagoras and Euclid in connection with the construction of dodecahedron and icosahedron, i.e. polyhedra bounded by twelve and twenty planes, respectively). Architects and builders have used fractals as decorative elements from ancient times. As examples of ancient architecture in which fractal components are present or dominant, the most remarkable are Egyptian pyramids, Buddhist and Hindu temples, Gothic cathedrals, but also some cathedrals from the earlier period [2], [6], [11]. Historians of mathematics and art, even today, have been unable to determine with certainty when golden ratio appeared for the first time in some of the old civilizations, but they all agree that golden ratio, purposely or not, was applied in ancient Egypt in the construction of the Cheops Pyramid in Giza (the ratio of the height of the facet to half the length of the base edge), one of the seven wonders of the old world that still exists today.

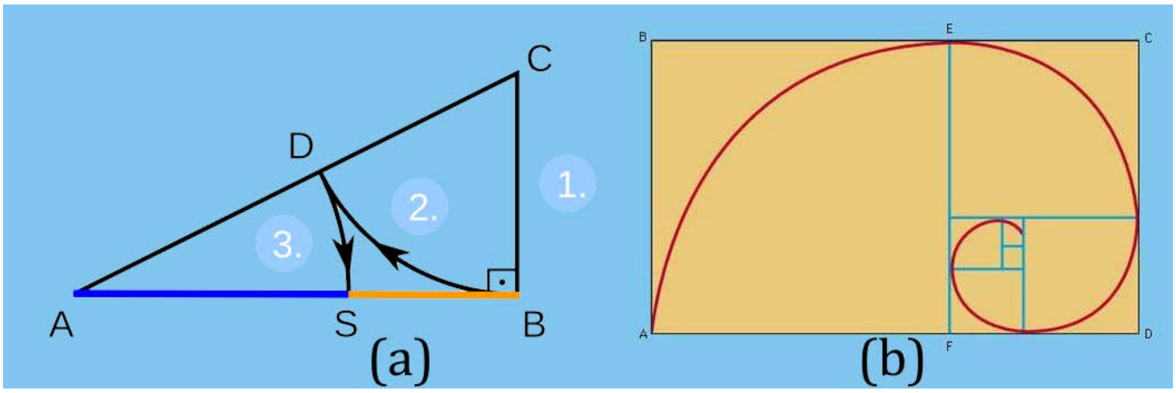

Figure 7. (a) Construction of golden ratio, AB : BC = 2:1; (b) Golden rectangle and golden spiral.

It was precisely the golden ratio that was notably used in ancient and Hindu architecture, Gothic architecture, Renaissance, and later in classicism, usually for the design of faces or ground plans of temples, mausoleums, churches and cathedrals (Parthenon at Athenian Acropolis (432 BC), Taj Mahal in Agra (1653, Figure 6), cathedrals in Anagni (1104), Florence (1436), Milan (Figure 8), Paris (1345), Reims (1275),...). The golden ratio (Lat*sectio aurea*, or divine ratio, Figure 7.a) is the ratio of parts of a line segment in which the whole segment $a+b=AB$ is related to the larger part $b=AS$ as the larger part b is related to the smaller part $a=SB$, or $(a+b):b=b:a=\varphi$, $\varphi=\left(1+\sqrt{5}\right)/2\approx 1,618$. The golden rectangle (Figure 7.b), or a rectangle whose sides are in the golden ratio, is considered to be a fractal because it has the following property of self-similarity: when a square is removed from (or added to) a golden rectangle, a smaller (or larger) rectangle with entirely the same golden ratio always remains, and so indefinitely. The golden spiral (Figure 7.b), or the spiral formed by the arcs connecting opposite corners of inserted squares, is actually a logarithmic spiral, i.e. a spiral for which it is true that any tangent makes the same angle with the spiral radius. The center of the golden spiral is at the intersection of diagonals of golden rectangles, which intersect each other at right angle, at a ratio that involves the golden section again. It is worth mentioning that, as it expands, it changes in size but it never changes in shape, i.e., it carries in itself a growth matrix that is present in nature in many examples. More precisely said, wherever nature needs economical and regular filling of space, there is a golden spiral.

Essentially, it is true that whenever we notice an exceptional beauty and harmony, we will usually reveal the presence of golden ratio [33] so one should not wonder why this concept, which connects mathematics, nature, science, engineering and art in a very unusual and interesting way, is present in all aspects of human life. Human aspiration is to be surrounded by structures and works pleasant to the eye, so it is logical to expect the magic of golden ratio to be found in the pores of mathematics, architecture, painting, sculpture, music and many other scientific disciplines [26]. Numerous experiments (although not all) in which respondents were asked to choose among a certain number of rectangles one that they liked most, show that people prefer a rectangle whose sides are in the ratio φ. The first such experiment was performed by one of the founders of modern psychology, Gustav T. Fechner. A team of colleagues from the Department of Civil Engineering at University North performed similar experiments during classes of history of architecture, and the results of these experiments show that their students choose the golden rectangle as the most pleasing to their eye [33]. The concept shared by the Cheops Pyramid in Giza, the temple of Parthenon at Athenian Acropolis, the Taj Mahal mausoleum in Agra, the Constantine's triumphal arch in Rome, the Cathedral of Notre Dame in Paris, the United Nations headquarters building in New York, a human body, a regular pentagon and a regular pentagram, design of credit cards, a common snail, sunflower ... makes mathematics a universal science. Like no other concept that appeared in the history of mathematics, it inspired many thinkers of various disciplines to discover its presence in various fields of life and periods of human existence [20] - [23], [28], [33].

Figure 8. The Milan Cathedral (built from the late 14th to the middle of 19th century) [2], [6], [11] and Beijing National Aquatic Center (2008) [source: www.pinterest.com].

A considerable number of examples of fractal geometry principles being used in construction and design can be found in modern architecture. Two viewpoints or two approaches to the fractal concept typically stand out: one approach attempts to simulate natural forms or apply similar forms of different sizes at different scales in project design, while the other approach attempts to measure (the measurement is based on calculation of fractal dimension) the complexity of forms [24]. In both approaches, fractal geometry provides a quantitative means to describe the complexity of the physical appearance of an architectural product and the degree to which it qualifies as a "fractal" [6]. The National Aquatic Center in Beijing (Figure 8) or the British Museum in London (Figure 9.a), a typical example of the classic method of opposing the old and the new, and the spiral staircase in the Vatican Museums (Figure 10.a) are just a few modern and magnificent examples of architecture with fractal components. Many other examples of application of fractals in architecture can be found in the following references: [1], [2], [5]–[14], [17], [18], [21], [24], [25], [30]–[32].

In the last two decades, primarily thanks to the rapid development of computer technologies and computer graphics, civil engineers have been provided with powerful tools for modeling and analysis of structures with highly nonlinear structure (e.g. hyperbolic paraboloid), modeling and regulation of rivers and sea shores, techniques they knew from before, but also for creating something completely new, which normally relies on disruption of the established and usual order of things. Even Mandelbrot used the example of coastline as a fractal - inlets look like bays, capes look like peninsulas; if we came closer, every rock would look like a peninsula... Today, there are other specialized software solutions that enable engineers to apply fractal geometry in the design of grid or reticulated shell structures, but also in the regulation of rivers and sea shores, or to make models of simple repeating forms, that are close to the fractal concept, applying a set of rules of non-linear transformations in the process [3], [4], [10], [13], [15].

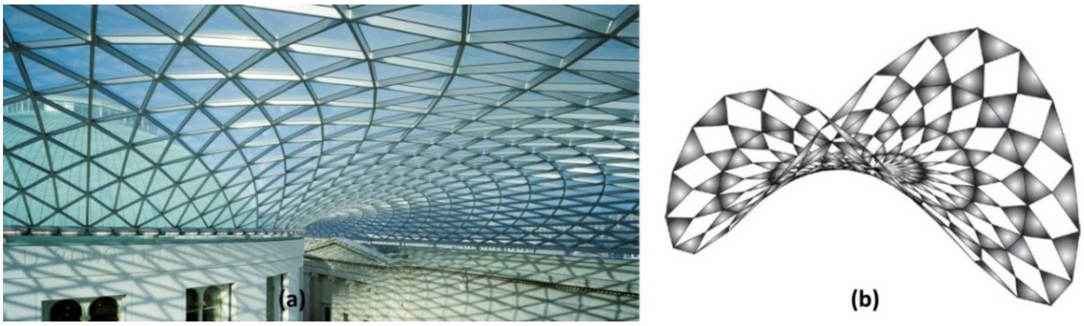

Figure 9. (a) The British Museum in London [3]; (b) Hyperbolic paraboloid [3], [27], [29].

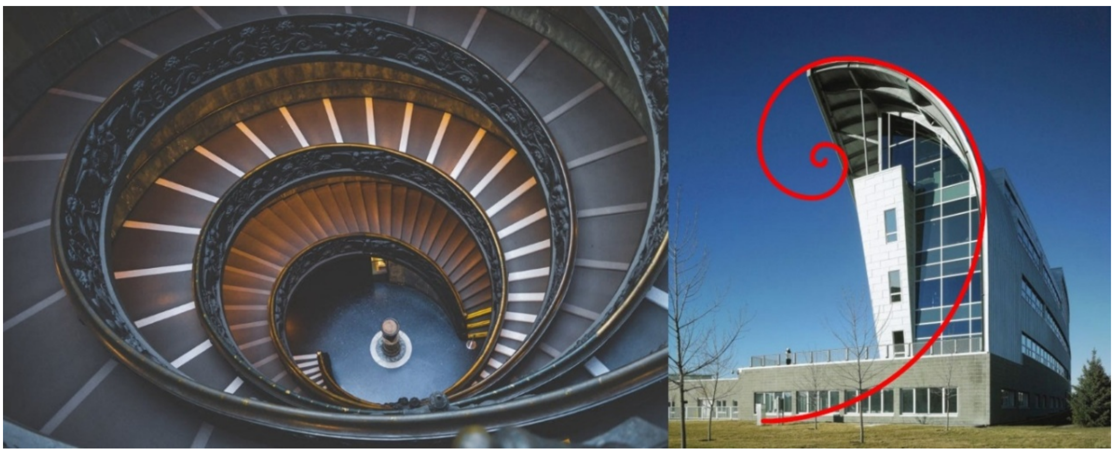

Figure 10. (a) Spiral staircase in the Vatican Museums (1932); (b) City of Calgary Water Centre (2008) [source: www.pinterest.com].

Undoubtedly, there are many more examples of application of fractal geometry in modern architecture and civil engineering, and there is almost not a branch of these two technical sciences in which fractals are not applied. Furthermore, with high confidence we may claim that not all fractals have been discovered yet, and that there are still many architectural and civil engineering objects and shapes with fractal analysis not done. That is why application areas of fractal geometry will continue to be a subject of research of new researchers, who will, like their predecessors and contemporaries, aim at finding harmony between clear precision and chaotic imperfection. At the end, let us take a look at a very important fact closely related to the use of fractal geometry in design. Namely, if we would apply fractal geometry for the design of mega-panels consisting of sub-panels, which contain elements that have the same shape, we would achieve a considerable saving in prototype production, which would facilitate and speed up the production of these prototypes, as well as their installation.

4. Conclusion

The paper presents a brief overview of one of the greatest secrets of nature's design: rugged, irregular, self-similar infinite objects (developed through multiple repetition of, often a simple, algorithm). Whether we classify them according to the degree of their self-similarity or according to the mode of their formation, we can conclude that there really is a large number of different types of fractals, so the paper lays emphasis on just a few of the most interesting representatives from the aspect of their application in architecture and civil engineering. Viewed from this point, fractal geometry is used and applied in many ways: unintentionally and intentionally. Fractal geometry can help us understand and analyze the complexities we can find in cities of the Antiquity and Middle Ages, but also in temples, cathedrals, mausoleums, and other structures built to these days by the civilizations that preceded us. Ensuring a balance between the old and the new is most often of great importance in architectural structures and design. Exactly fractal geometry is the convenient approach that can be used in the process of supporting creativity in ideas of new forms, which can be of great help when defining modern architectural models as well as testing the harmony between the old and the new. Furthermore, from the viewpoint of a civil engineer, fractal geometry is necessary, for example, for modeling and regulation of seemingly chaotic structures like rivers and sea shores, as well as for analysis of structures, modeling of shell elements or spatial lattices. It is also suitable for modeling the structure and hydraulic properties of unsaturated soils and for developing models of flow through porous media.

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