300 Years of Leonhard Euler

(1707 - 1783)

The 300th anniversary of the birth of one of the most productive mathematicians of all times, Leonhard Euler, was celebrated in 2007. For this occasion, some found an inspiration and wrote an article (for example Dakić 2007a and b, Sruk 2007) or a book (Bradley and Sandifer 2007), and also a scientific workshop was held with a book in preparation (Grattan-Ginness and Pulte 2007). In this article, Euler's life and work will be presented with special emphasis on his contribution to cartography.

He was born on April 15, 1707 in Swiss town of Basel, in a family of a Lutheran priest. One of close family friends was Johann Bernoulli, an already affirmed mathematician in Europe of that time. It is believed that friendship was crucial for the life path of Leonhard Euler. He enrolled the University of Basel in order to obtain general education in 1720. Johann Bernoulli soon discovered his great potential for mathematics and was giving him private tuition. Euler completed the study of philosophy in 1723 and in autumn of the same year he began studying theology. Despite profound religiousness, Euler transferred to the study of mathematics, which he finished in 1726 with doctor dissertation on sound dispersion (*De Sono*). A year later he wrote a paper on the best arrangement of masts on a ship, which was recommended for the Paris Academy reward. Although the Paris Academy awarded him another prize that time, he later won it as many as 12 times.

In the same year, he was offered a position in St. Petersburg, at the Russian Royal Academy of Sciences. The Russian Academy was attractive to young and ambitious European scientists in that time because of emperor's abundant money support and work conditions. He arrived to St. Petersburg in May 1727 and started working at the medical department of the Academy. In 1731, he was named professor of physics, and in 1733 he accepted the chair of mathematics. In the following year, he married Katarina, daughter of Swiss painter Gsell, and they had thirteen children, of which five made it through childhood, and only three outlived their father. Euler loved children and, according to his words, he made some of his greatest mathematic discoveries holding a child in his arms, while others were playing at his feet.

Euler's portrait (by Jakob Emanuel Handmann, 1753) (URL1)

Eulerov portret (Jakob Emanuel Handmann, 1753.) (URL1)

He left St. Petersburg in 1740. Invited by Prussian king Frederik the Great, he went to the Berlin Academy of Sciences, where he spent next 25 years as a leader of the Department of Mathematics. He performed many duties in Berlin, like supervising observatories and botanical gardens, staff selection, care about publications of calendars and geographical maps, and he was also an advisor for lottery, insurance, and many other things. Despite all his duties, he managed to publish 380 scientific articles during that period, as well as his two great works: *Introducio in analysis infinitorum* (*An introduction to analysis of the infinite*) and *Institutiones calculi differentialis* (*Basis of differential calculations*). Most of his works from calculus of variations, special functions theory, differential equations, astronomy, and mechanics were also made in Berlin. In 1747, he became a member of London's Royal Society, and he was also a member of almost all important academies in Europe and was awarded many rewards and acknowledgments.

Invited by Russian empress Catherine II, he returned to St. Petersburg in 1766, where he stayed to his death. Problems with Euler's health began as early as 1735, when he went through a hard fever and nearly died, and a few years later he could not use his right eye. It was not long before cataract covered his other eye. After a failed surgery in 1771, Euler became completely blind, but thanks to his brilliant mind and incredible memory as well as to his ability to perform complicated mathematical operations mentally, he continued his work with help of two of his sons and other mathematicians.

Euler's home was burnt in a great fire that caught St. Petersburg in the same year, but his mathematical manuscripts were saved. He died on September 18 1783, and was buried at the Lazarevsky cemetery, Pskov. The St. Petersburg Academy had continued to publish Euler's unpublished work for 50 years after his death.

The volume of Euler's mathematical achievements is enormous and its significance invaluable, so just a short overview of his creative work is given here. All of Euler's work (from books and articles to his correspondence) from 1911 was published by the Euler Committee of the Swiss Academy of Sciences under the title *Leonhardi Euleri Opera Omnia*. 84 volumes of encyclopaedic format were published in a hundred years. Three volumes of Euler's gathered works *Opera omnia* are devoted to the theory of numbers and he is rightly considered the founder of the modern theory of numbers. Four volumes of *Opera Omnia* are devoted to geometry, especially to analytical and differential.

Mathematical analysis had been the central focus of his work and interest for

300 godina Leonharda Eulera

(1707 - 1783)

Godine 2007. obilježena je tristota godišnjica rođenja jednog od najproduktivnijih matematičara svih vremena Leonharda Eulera. Tim povodom neki su našli inspiraciju i napisali članak (npr. Dakić 2007a i b, Sruk 2007) ili knjigu (Bradley i Sandifer 2007), a održana je i znanstvena radionica s knjigom u pripremi (Grattan-Ginness i Pulte 2007). U ovom članku bit će prikazan Eulerov život i rad s posebnim osvrtom na njegov doprinos kartografiji.

Rođen je 15. travnja 1707. u švicarskom gradu Baselu, u obitelji luteranskog svećenika. Jedan od bliskih prijatelja obitelji bio je Johann Bernoulli, u to vrijeme u Europi već afirmirani matematičar. Smatra se da je to prijateljstvo bilo presudno za životni put Leonharda Eulera. Godine 1720. upisuje se na baselsko Sveučilište kako bi stekao opće obrazovanje. Uskoro Johann Bernoulli otkriva njegov veliki potencijal za matematiku i daje mu privatnu poduku. Studij filozofije Euler završava 1723. godine, a ujesen iste godine započinje studij teologije. Unatoč dubokoj religioznosti, Euler prelazi na studij matematike koji dovršava 1726. doktorskom disertacijom o širenju zvuka (*De Sono*). Godinu dana kasnije piše rad o najboljem rasporedu jarbola na brodu, kojim je predložen za nagradu Pariške akademije. Iako mu je tada Pariška akademija dodijelila drugu nagradu, kasnije ju je dobio čak 12 puta.

Iste godine ponuđeno mu je mjesto u St. Petersburgu na Ruskoj carskoj akademiji znanosti. U to vrijeme Ruska je akademija, zbog obilate novčane carske potpore i uvjeta rada bila privlačna za mlade i ambiciozne europske znanstvenike. U St. Petersburg stigao je u svibnju 1727. godine te se zaposlio na medicinskom odjelu Akademije. Godine 1731. imenovan je profesorom fizike, a 1733. preuzima katedru matematike. Iduće godine ženi se Katarinom, kćerkom švicarskoga slikara Gsella, i dobiva trinaestero djece, od kojih je petoro preživjelo djetinjstvo, a samo je troje nadživjelo oca. Euler je volio djecu i prema njegovim riječima, do nekih od svojih najvećih matematičkih dostignuća došao je držeći jedno dijete u naručju, dok su se druga igrala oko njegovih nogu.

St. Petersburg napušta 1740. godine. Na poziv pruskog kralja Fridriha Velikog odlazi u Berlinsku akademiju znanosti, gdje kao predvodnik Matematičkog odjela provodi sljedećih 25 godina. U Berlinu izvršava brojne obveze, kao što su nadgledanje opservatorija i botaničkih vrtova, izbor osoblja, briga o publikaciji kalendara i geografskih karata, a bio je i savjetnik za lutriju, osiguranje i još mnogo toga. Unatoč obvezama, u tom periodu uspijeva objaviti oko 380 znanstvenih članaka i svoja dva velika djela: *Introducio in analysis infinitorum* (*Uvod u analizu beskonačnosti*) i *Institutiones calculi differentialis* (*Osnove diferencijalnog računa*). U Berlinu je nastala i većina njegovih radova iz računa varijacija, teorije specijalnih funkcija, diferencijalnih jednadžbi, astronomije, mehanike. Godine 1747. postaje članom Londonskoga kraljevskoga društva, a također je član gotovo svih značajnih akademija u Europi i dobitnik brojnih priznanja i nagrada.

U St. Petersburg vraća se 1766. na poziv ruske carice Katarine II, gdje ostaje do kraja života. Problemi s Eulerovim zdravljem počeli su još 1735. godine kad je prebolio tešku vrućicu i zamalo umro, a par godina kasnije oslijepio je na desno oko. Nije prošlo dugo vremena a mrena je prekrila i njegovo drugo oko. Nakon neuspjele operacije 1771. godine Euler je potpuno oslijepio, no zahvaljujući briljantnom umu i nevjerojatnom pamćenju i sposobnosti obavljanja složenih računskih operacija napamet, nastavlja s radom uz pomoć dvojice svojih sinova te drugih matematičara.

Iste godine u velikom požaru koji je zahvatio St. Petersburg izgorio mu je dom, ali su spašeni njegovi matematički rukopisi. Umro je 18. rujna 1783. godine, a sahranjen je u Pskovu na Lazarevskom groblju. Još punih 50 godina nakon Eulerove smrti Sanktpetersburška akademija tiskala je njegove neobjavljene radove.

Opseg Eulerovih matematičkih dostignuća je ogroman, a značaj neprocjenjiv te je ovdje dan tek kratki osvrt na njegovo stvaralaštvo. Cjelokupno Eulerovo djelo (od knjiga i članaka do njegove korespondencije) od 1911. godine izdaje Eulerov odbor Švicarske akademije znanosti pod naslovom *Leonhardi Euleri Opera omnia*. U stotinjak godina objavljena su 84 sveska enciklopedijskog formata. Tri sveska Eulerovih sabranih djela *Opera omnia* posvećena su teoriji brojeva i s pravom ga smatraju utemeljiteljem moderne teorije brojeva. Četiri sveska *Opera omniae* posvećena su geometriji, naročito analitičkoj i diferencijalnoj.

Matematička analiza dugo je bila središnja točka njegovog rada i interesa, a svoje najznačajnije djelo *Introducio in analysin infinitorum* (*Uvod u analizu beskonačnosti*) objavljuje 1748. godine. U tom djelu Euler definira funkciju kao analitički izraz sastavljen nekom metodom od promjenjive vrijednosti i brojeva ili od konstantnih vrijednosti, definira polinome, trigonometrijske funkcije, eksponencijalne funkcije, te njihov inverz – logaritamske funkcije.

Euler definira eksponencijalnu funkciju i na skupu kompleksnih brojeva, te je povezuje s trigonometrijskim funkcijama. Za bilo koji realni broj *vrijedi* $e^{ix} = \cos x + i \sin x$. Iz te formule proistječe čuvena jednakost $e^{i\pi} + 1 = 0$, koju mnogi, budući da ona povezuje pet krucijalnih matematičkih konstanti *e*, 1, *i*, π i 0, drže *najljepšom jednakošću cijele matematičke znanosti*.

Uz Eulerovo ime veže se čitav niz pojmova. Euler, koji je u pisanju bio neuobičajeno produktivan, uobličio je matematički zapis koji je zaživo te se uz manje promjene rabi i danas. Osim oznake *f*(*x*) za standardni zapis realne funkcije (1734), uveo je još oznaku *i* za drugi korijen iz -1 (1777), slovo *e* za zapis poznatog *Eulerovog broja* (1727), oznaku Σ za zbrajanje (1755), oznake Δ, *sin*, *cos* i mnoge druge. Iako mu se to pripisuje, on nije uveo oznaku π za omjer opsega i promjera kružnice, ali je dosljednom uporabom pridonio da bude prihvaćena. Evo nekih od formula i teorema koje dugujemo a long time, and he published his most significant work *Introducio in analysis infinitorum* (*An introduction to analysis of the infinite*) in 1748. In this work, Euler defines function as an analytical expression composed by a method from variables and numbers or from constants; he defines polynomials, trigonometric functions, exponential functions and their inverse – logarithmic functions.

Euler also defined an exponential function in a set of complex numbers, and he connects it to trigonometric functions. For any real number x, it follows $e^{ix} = \cos x + i \sin x$. From this formula comes the famous equation $e^{i\pi} + 1 = 0$, which is considered *the most beautiful equation of the whole mathematical science* since it connects five crucial mathematical constants e, 1, i, π and 0.

A whole series of terms is often connected to Euler's name. Euler, who was incredibly productive in writing, shaped mathematical notation that survived and is still used with minor changes even today. Besides the standard notation *f(x)* for a real function (1734), he also introduced *i* for square root from -1 (1777), the letter *e* for notation of the famous *Euler's number* (1727), notation Σ for summation (1755), notations Δ, *sin*, *cos* and many others. Although it is often attributed to him, he did not introduce the notation π for proportion of perimeter and diameter of a circle, but he contributed to its acceptance by using it persistently. Here are some formulae and theorems we owe to Euler:

$$e^{i\pi} + 1 = 0$$

Euler's function, Euler's numbers, Euler's straight line, Euler's formula for homogenous functions, Euler's formula for curvature of the plane, Euler integral of the second kind or gamma – function, Euler integral of the first kind or beta – function, Euler angles: nutation angle, precession angle and rotation angle; they are connected to transformations of rectangular coordinates and many other things.

Another Euler's simple yet imposing result belongs to the list of great mathematical discoveries on the basis of topology. It is *Euler's formula* – equation that connects the number of points (V), the number of edges (B) and the number of sides (S) of every convex polyhedron: $V - B + S = 2$. The consequence of Euler's polyhedral formula is the existence of exactly five regular polyhedra. These are (regular) tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron.

Euler is remembered for solving (perhaps it is better to say for proving insolvability) problem of the *Köninsberg bridges*. He published his work under the title *Solutio problematis ad geometriam situs pertinentis* that represents the earliest example of the use of graphs and topology. Namely, the town of Köningsberg, today Kaliningrad, is situated on the banks of river Prögel and on two river islands. Different points of the town were connected by seven bridges. The question was if it was possible to go around the town, and in doing so only to cross every bridge just one time. Euler produced a simple solution to the problem and demonstrated that such a walk was not possible.

Euler is also known as a promoter of mathematics. He laid out his attitudes to teaching mathematics in his famous *Letters to a German princess*. These were 200 letters that Euler wrote to Frederik's niece, princess of Anhalt-Dessau he was tutoring. These letters were published in

Map of the Europe from Euler's Atlas, edition from 1760 (URL3)

Karta Europe iz Eulerovog atlasa, izdanje iz 1760. godine (URL3)

Euleru: *Eulerova funkcija*, *Eulerovi brojevi*, *Eulerov pravac*, *Eulerova formula za hommogene funkcije*, *Eulerova formula zakrivljenosti plohe*, *Eulerov integral druge vrste ili gama-funkcija*, *Eulerov integral prve vrste ili beta-funkcija*, *Eulerovi kutovi*: kut nutacije, kut precesije i kut rotacije; vezani su uz transformacije pravokutnih koordinata i još mnogo toga.

Još jedan Eulerov jednostavan, ali impozantan rezultat pripada popisu velikih matematičkih otkrića koja su u temeljima topologije. Riječ je o *Eulerovoj formuli* – jednakosti koja povezuje broj vrhova (V), broj bridova (B) i broj strana (S) svakog konveksnog poliedra: $V - B + S = 2$. Posljedica Eulerove poliedarske formule je postojanje točno pet pravilnih poliedara. To su (pravilni) tetraedar, heksaedar (kocka), oktaedar, dodekaedar i ikosaedar.

Euler je ostao zapamćen i po rješavanju (možda je bolje reći po dokazivanju nerješivosti) problema *Köningsberških mostova*. Rad je objavio pod nazivom *Solutio problematis ad geometriam situs pertinentis* koji ujedno predstavlja i najraniji primjer upotrebe teorije grafova ili topologije. Naime, grad Köningsberg, danas Kaliningrad smješten je na obalama rijeke Prögel i na dvama riječnim otocima. Različite točke grada povezivalo je sedam mostova. Pitanje je bilo je li moguće obići grad, a pritom proći svakim mostom samo jednom. Euler je jednostavnim rješenjem postavljenog problema pokazao da takva šetnja nije moguća.

$$V - B + S = 2$$

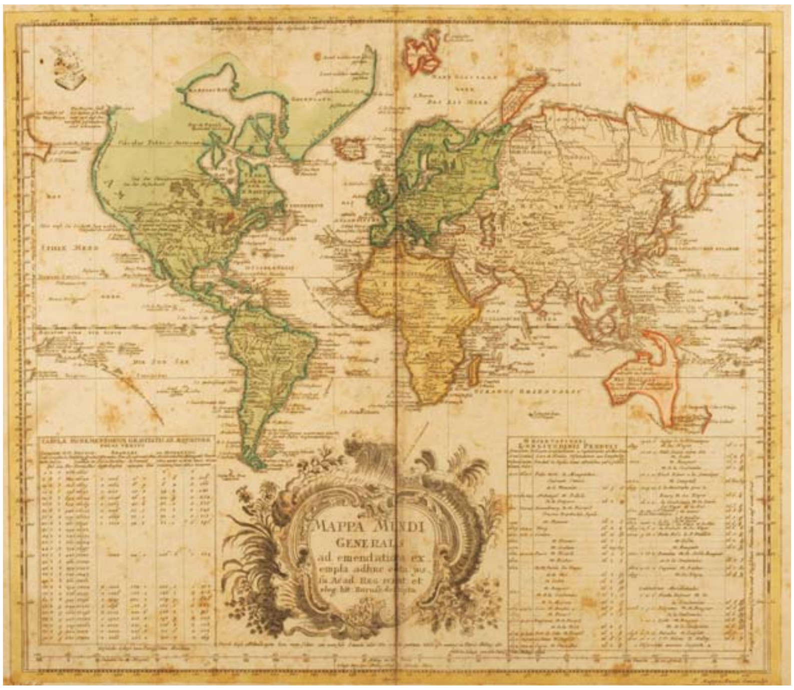
Euler je poznat i kao popularizator matematike. Svoje stavove o poduci matematike iznio je u poznatim *Pismima jednoj njemačkoj princezi*. Radi se o 200 pisama što ih je Euler pisao Frederikovoj nećakinji, princezi od Anhalt-Dessaua, kojoj je davao poduke. Ta su pisma objavljena u knjizi koja je doživjela veliki uspjeh. U knjizi Euler izlaže i analizira niz problema iz matematike i fizike, a zbog dubokog razumijevanja cjeline prirodnih znanosti, knjigu krasi sistematičan stil i pristupačan način izlaganja.

U zabavnoj matematici poznat je kao prvi koji je izumio kvadrat veličine 9 redaka i 9 stupaca u koji se po horizontali i vertikali mogu upisati brojevi od jedan do devet, a da se ni jednom u pojedinoj liniji ne ponovi isti broj. To se smatra pretečom danas sve popularnijih sudoku-slagalica koje su nastale u Japanu u drugoj polovici 20. stoljeća. Zanimljivo je da je Euler bio vrlo pobožan i da je dobar dio života tražio savršen, magični poredak brojeva koji bi posvjedočio opstojnost Boga.

Osim u matematici do značajnih rezultata Euler je došao i u fizici. Uveo je diferencijalne jednadžbe za gibanje krutog tijela pod utjecajem vanjskih sila te diferencijalne jednadžbe za gibanje idealnih tekućina. Razvio je teoriju turbina, proučavao širenje zvuka i svjetlosti itd. Njegova *Mehanica* iz 1736. može se smatrati i matematičkom knjigom jer predstavlja Newtonove zakone u analitičkom obliku.

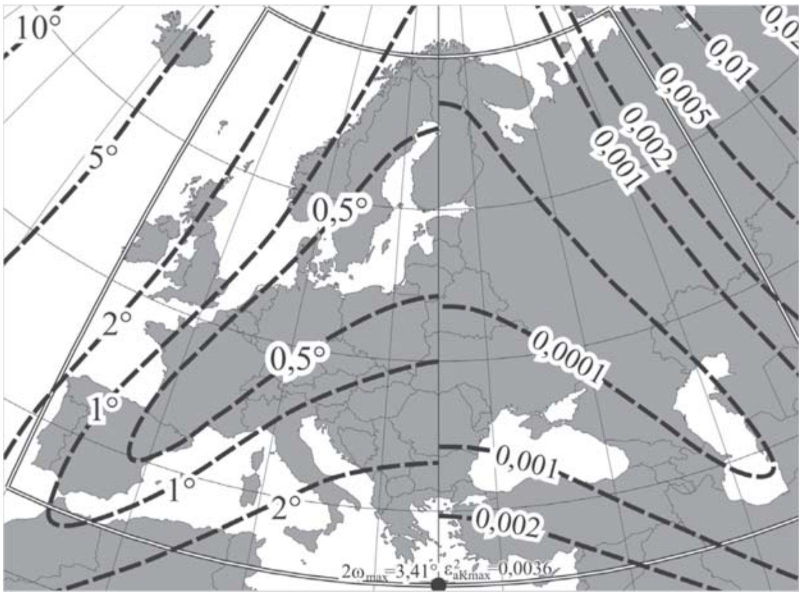
Kao astronom, Euler objavljuje radove iz nebeske mehanike, a dao je doprinos razvoju teorije gibanja planeta i kometa.

Kroz brojeve i njihove omjere proučavao je i teoriju glazbe. Godine 1739. objavio je *Tentamen novae theoriae musicae* – djelo koje je predstavljalo pokušaj da se povežu matematika i glazba.



Map of the world from Euler's Atlas, edition from 1760 (URL3)

Karta svijeta iz Eulerovog atlasa, izdanje iz 1760. godine (URL3)



Approximated Euler projection of Europe with minimal angular distortion - left: isocols of the maximum angular distortion and right: the Kavrayskiy criterion (Györffy 2006)

Aproksimacija Eulerove projekcije Europe s minimalnom deformacijom kuteva - lijevo: izokole maksimalnih deformacija kuteva i desno: kriterij Kavrajskoga (Györffy 2006)

a book that achieved a great success. In the book, Euler states and analyses a series of problems from mathematics and physics, and the book is systematical and accessible because of deep understanding of natural sciences.

In entertaining mathematics, he is known as the first who invented the square with nine rows and nine columns in which numbers can be written in from one to nine horizontally and vertically without the same number repeating in any particular line. It is considered the forerunner of all today's sudoku-puzzles created in Japan in the second half of the 20th century. It is interesting to say that Euler was very religious and he spent a great deal of his life looking for the perfect, magical order of numbers in order to confirm the existence of God.

Besides mathematics, Euler also reached important results in physics. He introduced differential equations for hard body motion under the influence of outer force and differential equations for perfect liquids motion. He developed a theory of turbines, studied dispersion of sound and light, etc. His *Mechanica* from 1736 can be considered mathematical book as well since it shows Newton's laws in analytical form.

As an astronomer, Euler published works from celestial mechanics, and he also gave his contribution to the development of the theory of planet and comet movements.

Through numbers and their proportions He also studied music theory, through numbers and their proportions. In 1739, he published *Tentamen novae theoriae musicae* – a work that represents an attempt to connect mathematics and music.

Euler and Cartography

Cartography was another area Euler became involved in during his stay in St. Petersburg. In 1753, he was appointed director of the St. Petersburg Academy's geography section, and his task was to help French astronomer and cartographer Joseph Nicolas De l'Isle in preparing a map of the whole Russian Empire titled *Mappa Generalis Totius Imperii Russici* (1:8,9 Mill.). In his autobiographical writings, Euler says his eyesight problems began in 1738 with overstrain due to his cartographic work. However, it is believed that blindness was consequence of poisoning because of a boil. Despite everything, he continued with his work with great passion and devoted himself to the creation of an atlas. Atlas of the Russian Empire (*De L'Isle – Atlas Rvssicvs … Vastissimvm Imperivm Rvssicvm cum adiacentibvs Regionibvs*), consisting of 20 maps, was published in 1745 and in spite of several shortcomings (small number of astronomically determined points, low degree of accuracy of the maps, etc.); it represents an important contribution to Russian cartography.

Euler was involved in cartography during his stay at the Royal Academy of Sciences in Berlin. He is the author of a school atlas first published in 1753 under the title *Atlas geographicus omnes orbis terrarum regiones in XLI tabulis*. The next edition of the atlas with a title and foreword in German, French and Latin, was published in 1760 and it contained 44 maps.

Euler continued his cartographic work during his second stay in St. Petersburg. In 1777, the St. Petersburg Academy of Sciences published three important Euler's works on cartography: *De repraesentatione superficiei sphaericae super plano, De projectione geographica superficiei sphaericae* and *De projectione geographica de Lisliana in mappa generali Imperii Russici usitata*. These works were later translated into German and were published in vol. 93 of Ostwald's Classics of Exact Sciences (*Ostwald's Klassiker der Exakten Wissenschaften*) in Leipzig in 1898 under the title *Drei Abhandlungen über Kartenprojection* (Three discussions on cartographic projections) and later on also into Russian (Eyler 1959).

As it is well known, P. Chebyshev (1821 – 1894) was the first mathematician

Euler i kartografija

Kartografija je još jedno područje kojim se Euler bavio za vrijeme svog boravka u St. Petersburgu. Godine 1735. imenovan je ravnateljem geografske sekcije Sanktpetersburške akademije, a zadatak mu je bio pomoći francuskom astronomu i kartografu Josephu Nicolasu De l'Isleu pri izradi karte cjelokupnoga Ruskog Carstva *Mappa Generalis Totius Imperii Russici* (1:8,9 mil.). U autobiografskim zapisima Euler piše kako su njegovi problemi s vidom počeli 1738. zbog prvelikog zamora tijekom kartografskog rada. Ipak, smatra se da je sljepoća bila posljedica trovanja zbog gnojnog čira. Usprkos svemu nastavio je raditi s velikim žarom i posvetio se izradi atlasa. Atlas Ruskog Carstva (*De L'Isle – Atlas Rvssicvs … Vastissimvm Imperivm Rvssicvm cum adiacentibvs Regionibvs*), sastavljen od 20 karata, izašao je 1745. godine, te unatoč manjim nedostacima (mali broj astronomski određenih točaka, mala točnost karata i sl.), predstavlja znatan doprinos razvoju ruske kartografije.

I za vrijeme svog boravka na Kraljevskoj akademiji znanosti u Berlinu Euler se bavio kartografijom. Autor je školskog atlasa čije je prvo izdanje izašlo 1753. godine pod naslovom *Atlas geographicus omnes orbis terrarum regiones in XLI tabulis*. Iduće izdanje atlasa s naslovom i predovorom na njemačkom, francuskom i latinskom, izašlo je već 1760. godine, a sadrži 44 karte.

Euler se nastavio baviti kartografijom i za vrijeme svog drugog boravka u St. Petersburgu. Godine 1777. Sanktpetersburška akademija znanosti objavila je tri značajna Eulerova rada o kartografiji: *De repraesentatione superficiei sphaericae super plano, De projectione geographica superficiei sphaericae* i *De projectione geographica de Lisliana in mappa generali Imperii Russici usitata*. Ti su radovi kasnije prevedeni na njemački jezik i pod naslovom *Drei Abhandlungen über Kartenprojection* (Tri rasprave o kartografskim projekcijama) objavljeni u 93. svesku Ostwaldovih klasika egzaktnih znanosti (*Ostwald's Klassiker der Exakten Wissenschaften*) u Leipzigu 1898, te također i na ruski jezik (Eyler 1959).

Kao što je vrlo dobro poznato, P. Čebišev (Chebyshev) (1821-1894) bio je prvi matematičar koji je značajnije istraživao problem najbolje jednolike aproksimacije (best uniform approximation), a koji se u klasičnom zapisu može izreći ovako:

Neka je f neprekidna funkcija, $a, b \in R$, $n \in N$. Traži se polinom stupnja najviše n tako da

$$\max_{x \in [a,b]} \left| f(x) - p(x) \right|$$

bude minimalno za sve polinome stupnja ne većeg od n.

Najvažnije svojstvo rješenja toga problema je činjenica da ono daje procjenu pogreške aproksimacije za svaku točku iz intervala [a, b]. No, zbog teškoće konkretnog rješavanja tog problema prvi rezultati bili su prikazani tek u 19. st. Sam Čebišev dao je nužan uvjet rješenja, kazavši da moraju postojati barem n+2 točke u kojima pogreška funkcije f-p postiže svoj maksimum, ali nije eksplicitno spomenuo da u tim točkama odstupanja postižu maksimalne vrijednosti s promjenjivim predznakom. Dakle, teorem alternacije predznaka koji karakterizira rješenje nije dokazao Čebišev. Upotrebom Čebiševljevih rezultata Kirchberger je učinio prvi pokušaj koji su kompletirali Borel i Young. Algoritam za računanje najbolje aproksimacije prvi je prikazao Remez tek 1934. Tu je svojstvo alternacije krucijalno za definiranje iterativnog postupka koji generira rješenje.

Ipak, neki specijalni slučajevi su razmatrani prije Čebiševa, a prvi od njih bio je Eulerov kartografski problem s kojim se bavio za vrijeme svojeg drugog boravka u St. Petersbrugu. Naime, 1777. godine Euler je analizirao lokalnu i globalnu točnost De l'Isleove konusne projekcije. Taj članak bio je posljednji od tri članka koja se bave kartografijom. Objavljen je 1777. i čini se da je izrađen na temelju njegova prethodnog rada na kartografskom odjelu Akademije znanosti u St. Petersburgu. Euler je najprije dokazao činjenicu da se dio sfere ne može projicirati u ravninu uz čuvanje mjerila u obje dimenzije. Zatim je razmatrao pitanje o najpogodnijoj projekciji za kartu Rusije. Razmatrajući nekoliko vrsta projekcija, npr. stereografsku i polarne projekcije, utvrdio je da treba upotrijebiti De l'Isleovu projekciju zbog njezinih važnih svojstva:

1. paralele i meridijani sijeku se u projekciji pod pravim kutom

2. lokalno daje dobru aproksimaciju.

Karta se može upotrijebiti za procjenu udaljenosti između bilo kojih dviju točaka.

Matematički problem koji je Euler riješio bila je aproksimacija funkcije $\cos x$ linearnom funkcijom. Uočio je da najbolja aproksimacija može biti karakterizirana činjenicom da moraju postojati tri točke u kojima pogreška $a - bx - c\cos x$ dostiže maksimalnu vrijednost alternirajući u predznaku – teorem o alterniranju u njegovu najjednostavnijem obliku. S tim postavkama mogao je odrediti parametre za, u tom smislu, najbolju kartu. Taj je rad definirao Eulerovu projekciju koja se upotrebljavala za kartu cijele Rusije sve do početka 20. st. (Steffens, 2007).

Eulerove projekcije su ekvivalentna preslikavanja kod kojih se mreža meridijana i paralela preslikava u ortogonalnu mrežu u ravnini projekcije. Od svih ekvivalentnih projekcija one su najbliže po karakteru deformacija konformnim projekcijama, pa su se zbog toga njima bavili mnogi znanstvenici kao što su L. Euler, D. A. Grave, N. A. Urmaev, G. A. Meščerjakov (Meshcheryakov), K. Frankich, J. Györffy i dr.

Ako označimo s R označimo polumjer sfere, tada se Eulerove projekcije mogu opisati sljedećim diferencijalnim jednadžbama:

$$\left| \frac{dx}{d\varphi} \cdot \frac{dy}{d\lambda} - \frac{dy}{d\varphi} \cdot \frac{dx}{d\lambda} \right| = R^2 \cos\varphi$$

(ekvivalentnost)

$$\left| \frac{dx}{d\varphi} \cdot \frac{dx}{d\lambda} + \frac{dy}{d\varphi} \cdot \frac{dy}{d\lambda} \right| = 0$$

(ortogonalnost kartografske mreže)

gdje su φ i λ geografske koordinate točke na sferi, a x, y odgovarajuće koordinate u ravnini projekcije. Rješavanjem tog sustava parcijalnih diferencijalnih jednadžbi mogu se dobiti različiti tipovi Eulerovih projekcija. Nije teško pokazati da su ekvivalentne uspravne cilindrične, konusne i azimutalne projekcije također Eulerove projekcije.

Iako zanimljive s teorijskog stajališta, Eulerove projekcije također su privukle znanstvenike na traženje takvih projekcija za određena područja. Tako se npr. Meščerjakov (1968) bavi Eulerovom projekcijom za kartu svijeta, Frankich (1982) za područje Kanade, a Györffy (2006) za Europu.

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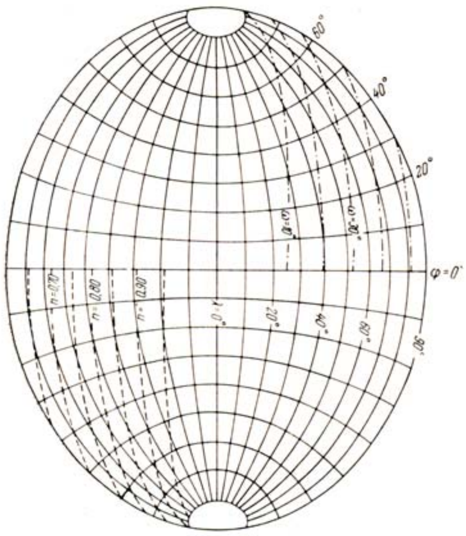

Euler projection according to Meshcheryakov, 1968

Eulerova projekcija po Meščerjakovu, 1968.

who investigated the problem of best uniform approximation in depth, and which can be expressed in classical notation in this way:

Let f be continuous function, $a, b \in R$, $n \in N$. Find a polynomial of degree at most n so that

$$\max_{x\in[a,b]} |f(x) - p(x)|$$

will be minimal for all polynomials of degree at most n.

The most important property of the solution of this problem is the fact that it gives an estimation of the error of approximation for every point of the interval [a, b]. But, due to the difficulty of this problem, the first results were presented only in 19th century. Chebyshev himself gave a necessary condition for the solution, stating that there must be at least n+2 points where error function f–p reaches its maximum, but he did not explicitly mention that these deviation points reach the maximum values with an alternating sign. So, the alternation theorem that characterizes the solution was not proven by Chebyshev. Using Chebyshev's results, Kirchberger made the first attempt that was completed by Borel and Young. An algorithm to calculate the best approximation was firstly presented by Remez only in 1934. Here, the alternation property is crucial for defining the iterating procedure that generates the solution.

Still, some special cases had been discussed before Chebyshev, and first of them was Euler's cartographic problem he dealt with during his second stay in St. Petersburg. Namely, Euler analysed local and global accuracy of the De l'Isle conic projection in 1777. This paper was the last of three papers dealing with cartography. It was published in 1777 and it seems that it was made based on his former work at the cartographic department of the Academy of Sciences in St. Petersburg. Euler first proved the fact that a part of the sphere cannot be projected onto a plane while preserving scale in both dimensions. Then he considered the issue of the most convenient projection for the map of whole Russia. Considering several kinds of projections, for example stereographic and polar projections, he determined that De l'Isle projection should be used because of the following important properties:

1. Parallels and meridians intersect in the projection perpendicularly
2. It gives a good approximation locally

A map can be used for estimating the distance between any two points.

The mathematical problem that Euler solved was approximating the function cos$x$ by a linear function. He noted that the best approximation can be characterized by the fact that there must be three points in which error $a - bx - c\cos x$ reaches the maximum value alternating in sign – the alternation theorem in its simplest form. With these settings, he was able to determine parameters for, in that sense, the best map. This work defined the Euler projection that was used for a map of whole Russia until the beginning of the 20th century (Steffens, 2007).

Euler's projections are equivalent mappings in which the graticule is being mapped to an orthogonal graticule in the projection plane. Of all equivalent projections, they are closest by character of deformation to conformal projections, so many scientists worked with them, like L. Euler, D. A. Grave, N. A. Urmaev, G. A. Meshcheryakov, K. Frankich, J. Györffy and others.

If we use R to denote the radius of a sphere, then Euler's projections can be described by following differential equations:

$$\left|\frac{dx}{d\varphi}\cdot\frac{dy}{d\lambda} - \frac{dy}{d\varphi}\cdot\frac{dx}{d\lambda}\right| = R^2\cos\varphi$$

(equivalency)

$$\left|\frac{dx}{d\varphi}\cdot\frac{dx}{d\lambda} + \frac{dy}{d\varphi}\cdot\frac{dy}{d\lambda}\right| = 0$$

(orthogonality of the graticule)

where φ and λ are geographical coordinates of a point on the sphere, and x and y are corresponding coordinates in the projection plane. Different types of Euler's projections can be obtained by solving this system of partial differential equations. It is not hard to show that equivalent normal aspect cylindrical, conical and azimuthal projections are also Euler's projections.

Although interesting from a theoretical viewpoint, Euler's projections also attracted scientists to study these kinds of projections for specific areas. So, for example, Meshcheryakov (1968) deals with Euler's projection for a world map, Frankich (1982) for Canada, and Györffy (2006) for Europe.

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