PARTNERICE MERCATOROVE PROJEKCIJE
U spomen professoru emeritusu Waldi R. Tobleru (1930. – 2018.)

COMPANIONS OF THE MERCATOR PROJECTION
In memory of Professor Emeritus Waldo R. Tobler (1930–2018)

MILJENKO LAPAINE

Sveučilište u Zagrebu, Geodetski fakultet, Kačićeva 26, 10000 Zagreb, Hrvatska / *Unversity of Zagreb, Faculty of Geodesy, Zagreb, Croatia*, email: mlapaine@geof.hr

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Pokojni professor emeritus Waldo Tobler objavio je 2017. kratki članak *A companion for Mercator* (Tobler, 2017.). Inspirirala ga je neodgovarajuća upotreba Mercatorove projekcije i predložio je alternativu u obliku ekvivalentne projekcije. Jednadžbe Tobler-Mercatorove kartografske projekcije čuvaju raspone Mercatorove projekcije između paralela, uz prilagođen razmak između meridijana. Sva područja, iako drastično deformirana po obliku, imaju točne površine. Inspirirana Toblerovim pristupom, u ovome se radu predlaže alternativna ekvidistantna projekcija. Jednadžbe te projekcije također čuvaju raspone Mercatorove projekcije između paralela, uz prilagođene razmake između meridijana. Na taj način dobivena je nova partnerica Mercatorove projekcije, pseudocilindrična projekcija ekvidistantna uzduž paralela. Izvedene su i formule koje opisuju distorzije Mercatorove projekcije i distorzije njezinih partnerica.
Ključne riječi: kartografska projekcija, projekcije partnerice, Mercator

The late Professor Emeritus Waldo Tobler recently published a short paper, *A Companion for Mercator* (Tobler, 2017). He was inspired by the often inappropriate use of the Mercator projection and suggested an alternative equal-area projection. The equations of the Tobler-Mercator map projection maintain the Mercator projection ranges between the parallels, with adjusted forms of the meridians. All territories, though drastically deformed by shape, cover their exact surface areas. Inspired by Tobler's approach, this paper proposes an alternative equidistant projection. The equations of this projection also preserve the Mercator projection ranges between the parallels, with the customized forms of meridians. Thus, a new companion of the Mercator projection has been created -a pseudocylindrical projection equidistant along the parallels. Formulas that describe the distortions of the Mercator projection and its companions have also been derived.
Key words: map projection, companion projection, Mercator

UVOD

Kartografska projekcija je preslikavanje zakrivljene plohe, npr. sfere ili elipsoida, u ravninu (LAPAINE, 2017.). U toj definiciji termin preslikavanje preuzet je iz matematike, a često se teorija kartografskih projekcija naziva matematičkom kartografijom. Zbog toga, kada se bavimo kartografskim projekcijama, moramo upotrijebiti matematiku.

Cilj proučavanja kartografskih projekcija je stvaranje matematičke osnove za izradu karata i rješavanje teorijskih i praktičnih zadataka u kartografiji, geodeziji, geografiji, astronomiji, navigaciji i drugim srodnim znanostima (FRANČULA, 2004.). Fizičku Zemljinu površinu u kartografiji aproksimiramo plohom rotacijskog elipsoida. Budući da spljoštenost takvog elipsoida nije velika, u nekim ga slučajevima možemo zamijeniti sferom. Pri izradi karata najprije se točke sa Zemljine površine prenose po određenim pravilima na plohu elipsoida, što je zadatak geodezije. Zatim se elipsoid preslikava u ravninu, a to je zadatak kartografije, posebno njezina dijela koji se bavi kartografskim projekcijama.

Budući da kartografskih projekcija teorijski ima beskonačno mnogo, moramo ih na neki način podijeliti, odnosno klasificirati. Uočimo najprije da ne postoji kartografska projekcija koja bi bila preslikavanje bez distorzija. Tu je činjenicu prvi dokazao L. Euler (1777.). Stoga je uobičajena podjela kartografskih projekcija na konformne (čuvaju kutove), ekvivalentne (čuvaju površine), ekvidistantne (čuvaju izvjesne udaljenosti) i sve ostale.

Ono što prvo pada u oči kad gledamo neku kartu je kartografska mreža, to jest mreža slika paralela i meridijana. Na temelju oblika kartografske mreže kartografske projekcije dijelimo na cilindrične, konusne, azimutne, pseudokonusne, pseudocilindrične, polikonusne i druge. Naglasimo da se projekcija ne naziva cilindričnom zato što se Zemljina sfera ili elipsoid preslikavaju na plašt cilindra, nego zato što se karta izrađena u takvoj projekciji može saviti u plašt cilindra. Analogno tumačenje vrijedi i za konusne projekcije. Nadalje, uobičajena je podjela kartografskih projekcija prema aspektu na uspravne, poprečne i kose

INTRODUCTION

A map projection is mapping from a curved surface, for example a sphere or ellipsoid, into a plane (LAPAINE, 2017). In this definition, the term mapping was taken from mathematics, which is not surprising, because the theory of map projections is often called mathematical cartography. This is why we use mathematics when dealing with map projections.

The aim of map projections is to create a mathematical basis for mapping and solving theoretical and practical tasks in cartography, geodesy, geography, astronomy, navigation and other related scientific fields (FRANČULA, 2004). The physical surface of the earth is approximated by the rotating ellipsoid surface in cartography. Since the flattening of the ellipsoid is small, in some cases it may be replaced by a sphere. When producing maps, points on the Earth's surface are transmitted by certain rules onto the ellipsoid surface, and this is the task of geodesy. Then the ellipsoid or sphere is mapped into the plane, which is the task of cartography, especially map projections.

Since theoretically there is an infinite number of map projections, we have to make a classification. We should first notice that there is no map projection that can map without distortion. This was first proved by L. Euler (1777). Therefore, map projections are usualy classified as conformal (keeping angles), equal-area (keeping areas), equidistant (keeping a certain distances) and the rest.

When we first look at a map, the eye is drawn to the graticule, a network of images of parallels and meridians. Based on the shape of the graticule, map projections are classified as cylindrical, conical, azimuthal, pseudoconical, pseudocylindrical, polyconical and other. A projection is not called cylindrical because the Earth's sphere or ellipsoid is mapped onto a cylindrical surface, but because the map made is a projection which can be bent into a cylindrical surface. Analogous interpretation applies to conical projections. It is common to divide map projections according to aspect into normal, transverse and oblique (FRANČULA, 2004). For simplicity, (Frančula, 2004.). U ovom ćemo se radu radi jednostavnosti baviti uspravnim cilindričnim i pseudocilindričnim projekcijama sfere.

KARTOGRAFSKE PROJEKCIJE

Najprije definirajmo geografsku parametrizaciju sfere polumjera $R > 0$ sa središtem u ishodištu koordinatnog sustava kao preslikavanje

$$x = R\cos\varphi\cos\lambda,\ y = R\cos\varphi\sin\lambda,\ z = R\sin\varphi \quad (1)$$

gdje je

$$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \lambda \in [-\pi, \pi].$$

Parametar φ je geografska širina, a λ je geografska dužina, kao što je uobičajeno. Nije teško dobiti prvu diferencijalnu formu preslikavanja (1)

$$ds^2 = R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2. \quad (2)$$

Kartografska projekcija obično se definira kao preslikavanje u ravninu s pomoću formula

$$x = x(\varphi, \lambda),\ y = y(\varphi, \lambda), \quad (3)$$

gdje su x i y koordinate u pravokutnom (matematičkom, desno orijentiranom) koordinatnom sustavu u ravnini, a

$$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \lambda \in [-\pi, \pi].$$

Prva diferencijalna forma preslikavanja (3) je

$$ds'^2 = Ed\varphi^2 + 2Fd\varphi d\lambda^2 + Gd\lambda^2, \quad (4)$$

gdje su koeficijenti

$$E = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2,\ E = \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda},\ G = \left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2. \quad (5)$$

Faktor lokalnoga linearnog mjerila c za preslikavanje sfere u ravninu definira se u teoriji kartografskih projekcija relacijom

$$c = \frac{ds'}{ds}, \quad (6)$$

in this paper we will deal with the normal aspect of cylindrical and pseudocylindrical projections of the sphere.

MAP PROJECTIONS

First, we will define a geographic parameterisation of a sphere with a radius $R > 0$ and the centre in the origin of the coordinate system as mapping

$$x = R\cos\varphi\cos\lambda,\ y = R\cos\varphi\sin\lambda,\ z = R\sin\varphi \quad (1)$$

where

$$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \lambda \in [-\pi, \pi].$$

Parameter φ is geographic latitude, and λ geographic longitude, as usual. It is not difficult to derive the first differential form of mapping (1)

$$ds^2 = R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2. \quad (2)$$

A map projection is usually defined as mapping into a plane by using the formulae

$$x = x(\varphi, \lambda),\ y = y(\varphi, \lambda), \quad (3)$$

where x and y are coordinates in a rectangular (mathematical, right oriented) coordinate system in a plane, and

$$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \lambda \in [-\pi, \pi].$$

The first differential form of the mapping (3) reads

$$ds'^2 = Ed\varphi^2 + 2Fd\varphi d\lambda^2 + Gd\lambda^2, \quad (4)$$

where the coefficients are

$$E = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2,\ E = \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda},\ G = \left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2. \quad (5)$$

The local linear scale factor c of the mapping (3) is defined in map projection theory as

$$c = \frac{ds'}{ds}, \quad (6)$$

a važan je pri određivanje distorzija ili deformacija nastalih zbog projekcije.

CILINDRIČNE PROJEKCIJE

Cilindrične projekcije u uspravnom aspektu ili uspravne cilindrične projekcije su takve projekcije kod kojih su slike paralela normalne mreže međusobno paralelni pravci, a slike meridijana, pravci okomiti na slike paralela (Tobler, 1962.) (Sl. 1.). Uočimo da ne postavljamo uvjet da su slike meridijana udaljene od slike srednjeg meridijana područja preslikavanja proporcionalno razlici njihovih geografskih dužina, kao što je to uobičajeno u literaturi o kartografskim projekcijama. Cilindrične projekcije kod kojih su slike meridijana udaljene od slike srednjeg meridijana područja preslikavanja proporcionalno razlici njihovih geografskih dužina zvat ćemo konvencionalnim cilindričnim projekcijama (Sl. 2.).

Dakle, cilindrična projekcija je preslikavanje zadano formulama

$$x = x(\lambda),\ y = y(\varphi, \lambda), \tag{7}$$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$ a funkcije u (7) su neprekidne i monotono rastuće. Pri tome su, kao i kod svake kartografske projekcije, x i y koordinate točke u pravokutnom (matematičkom, desnom)

and is important when determining distortion or deformation due to projection.

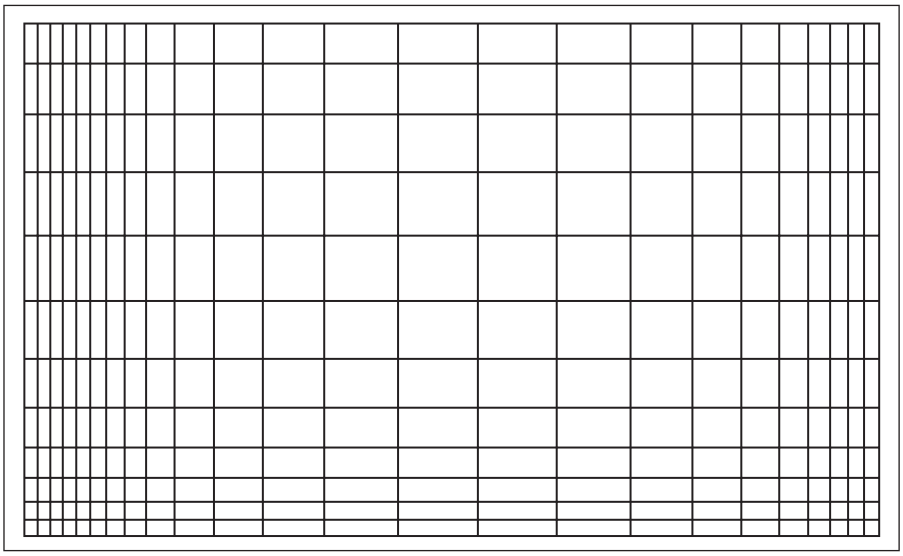
CYLINDRICAL PROJECTIONS

Normal aspect cylindrical projections or normal cylindrical projections are projections where images of parallels of latitude are parallel straight lines, and images of meridians are also parallel lines perpendicular to the images of parallels of latitude (Tobler, 1962) (Fig. 1). There is no condition that the distance of the images of meridians to the image of the central meridian is proportional to the difference of their longitudes, as is commonly found in the literature on map projections. Cylindrical projections for which the images of meridians are at a distance from the image of central meridian that is proportional to the difference of their longitudes, are called conventional cylindrical projections (Fig. 2).

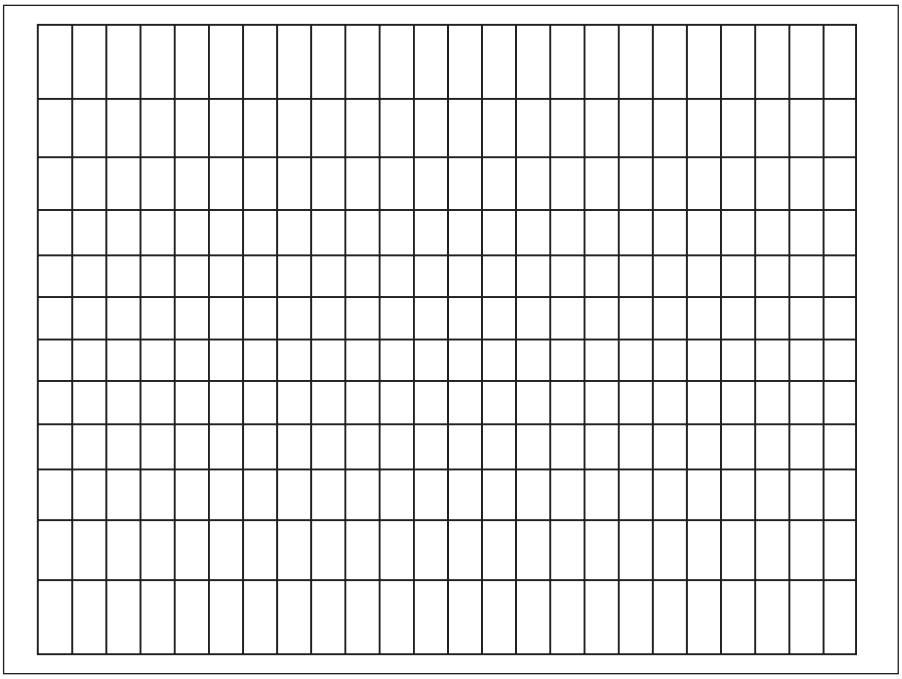
Thus, the cylindrical projection is a mapping given by the formulae

$$x = x(\lambda),\ y = y(\varphi, \lambda), \tag{7}$$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, and functions in (7) are continuous and monotonically increasing. As with any map projection, x and y are coordinates in a rectangular (mathematical, right oriented) co-



Slika 1. *Mreža meridijana i paralela u općoj uspravnoj cilindričnoj projekciji*
Figure 1 *The graticule in general normal aspect cylindrical projection*



Slika 2. *Mreža meridijana i paralela u konvencionalnoj uspravnoj cilindričnoj projekciji*
Figure 2 *The graticule in conventional normal aspect cylindrical projection*

koordinatnom sustavu u ravnini. Kao što se lijepo vidi, riječ je o preslikavanju u ravninu, a ne na plašt cilindra. Za takvo preslikavanje je

$$E = \left(\frac{dy}{d\varphi}\right)^2, \ F = 0, \ G = \left(\frac{dx}{d\lambda}\right)^2, \tag{8}$$

prva diferencijalna forma

$$ds'^2 = \left(\frac{dy}{d\varphi}\right)^2 d\varphi^2 + \left(\frac{dx}{d\lambda}\right)^2 d\lambda^2, \tag{9}$$

pa je

$$c^2 = \frac{\left(\frac{dy}{d\varphi}\right)^2 d\varphi^2 + \left(\frac{dx}{d\lambda}\right)^2 d\lambda^2}{R^2(d\varphi^2 + \cos^2\varphi d\lambda^2)}. \tag{10}$$

Lokalni linearni faktori mjerila c uzduž meridijana, odnosno paralela bit će

$$h = \mathrm{h}(\varphi) = \frac{1}{R}\frac{dy}{d\varphi}, \ k = k(\varphi,\lambda) = \frac{1}{R\cos\varphi}\frac{dy}{d\lambda}. \tag{11}$$

Za konvencionalne cilindrične projekcije jednadžbe su malo jednostavnije od onih u (7)

$$x = K(\lambda - \lambda_0), \ y = y(\varphi), \tag{12}$$

gdje je K konstanta proporcionalnosti, a λ_0 geografska dužina srednjeg meridijana područja preslikavanja. Za takve cilindrične projekcije i ako

ordinate system in a plane. As anyone can see, this is mapping to the plane, and not to a cylindrical surface. For such mapping, we have

$$E = \left(\frac{dy}{d\varphi}\right)^2, \ F = 0, \ G = \left(\frac{dx}{d\lambda}\right)^2, \tag{8}$$

the first differential form

$$ds'^2 = \left(\frac{dy}{d\varphi}\right)^2 d\varphi^2 + \left(\frac{dx}{d\lambda}\right)^2 d\lambda^2, \tag{9}$$

and

$$c^2 = \frac{\left(\frac{dy}{d\varphi}\right)^2 d\varphi^2 + \left(\frac{dx}{d\lambda}\right)^2 d\lambda^2}{R^2(d\varphi^2 + \cos^2\varphi d\lambda^2)} \tag{10}$$

The local linear scale factor c along the meridians, and parallels will be respectively

$$h = \mathrm{h}(\varphi) = \frac{1}{R}\frac{dy}{d\varphi}, \ k = k(\varphi,\lambda) = \frac{1}{R\cos\varphi}\frac{dy}{d\lambda}. \tag{11}$$

For conventional cylindrical projections, the equations are a little simpler than those given in (7)

$$x = K(\lambda - \lambda_0), \ y = y(\varphi), \tag{12}$$

where K is a constant of proportionality, and λ_0 longitude of the chosen central meridian. For such cylindrical projections, and if we map the unit preslikavamo jediničnu sferu, čime ne smanjujemo općenitost izvoda, (11) se pojednostavljuje i glasi

$$h = \mathrm{h}(\varphi) = \frac{dy}{d\varphi}, \; k = k(\varphi) = \frac{K}{\cos\varphi}. \qquad (13)$$

Uz uvjete $h = k$, i da se ekvator preslika bez distorzije na koordinatnu os x, a srednji meridijan na koordinatnu os y dobit ćemo jednadžbe konformne cilindrične projekcije

$$x = \lambda, \; y = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right), \qquad (14)$$

gdje je φ geografska širina, λ geografska dužina. Konformna cilindrična projekcija svijeta, koja se još naziva i Mercatorovom projekcijom (prema Gerhardu Kremeru zvanom Mercator), prikazana je na Slici 3a.

Uz uvjete $h = 1$, i da se ekvator preslika na koordinatnu os x, a srednji meridijan na koordinatnu os y dobit ćemo jednadžbe cilindrične projekcije ekvidistantne uzduž meridijana

$$x = \lambda, \; y = \varphi, \qquad (15)$$

gdje je φ geografska širina, λ geografske dužine. Ekvidistantna cilindrična projekcija svijeta prikazana je na Slici 3b.

Uz uvjete $hk = 1$, i da se ekvator preslika na ko-

sphere, which does not reduce the generality of the derivation, (11) is simplified and reads

$$h = \mathrm{h}(\varphi) = \frac{dy}{d\varphi}, \; k = k(\varphi) = \frac{K}{\cos\varphi}. \qquad (13)$$

Under the condition that $h = k$, that the equator is mapped without distortion on coordinate axis x, and the central meridian on coordinate axis y, we will get the equations of the conformal cylindrical projection

$$x = \lambda, \; y = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right), \qquad (14)$$

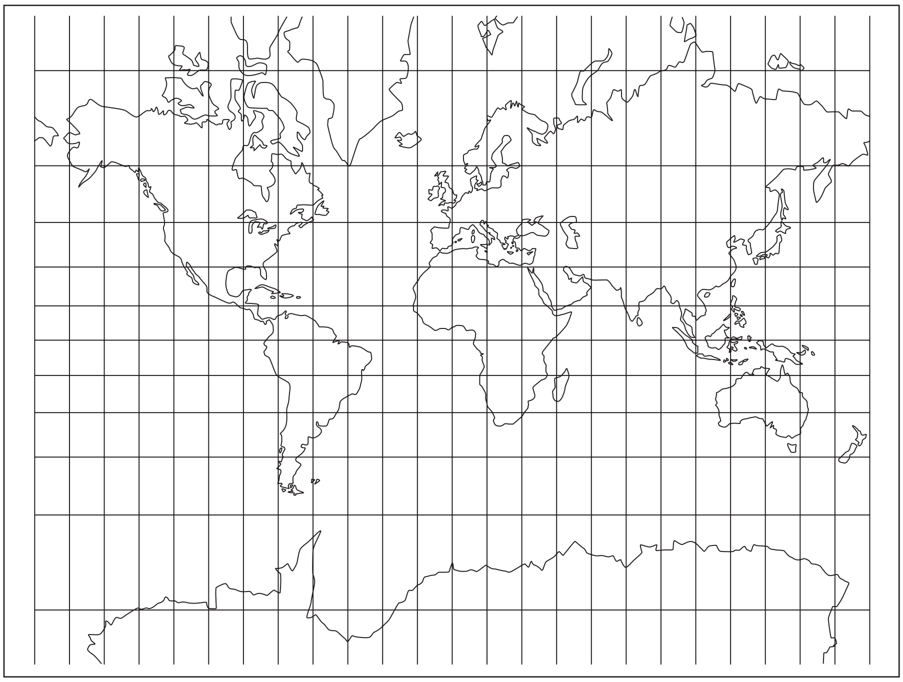
where φ is latitude, and λ longitude. The conformal cylindrical projection of the world is shown in Figure 3a. It is also known as the Mercator projection, after the famous cartographer Gerhard Kremer, commonly known as Mercator (1512–1594).

Under the condition that $h = 1$ that the equator is mapped on coordinate axis x and the central meridian on coordinate axis y, we get the equations of the cylindrical projection equidistant along meridians

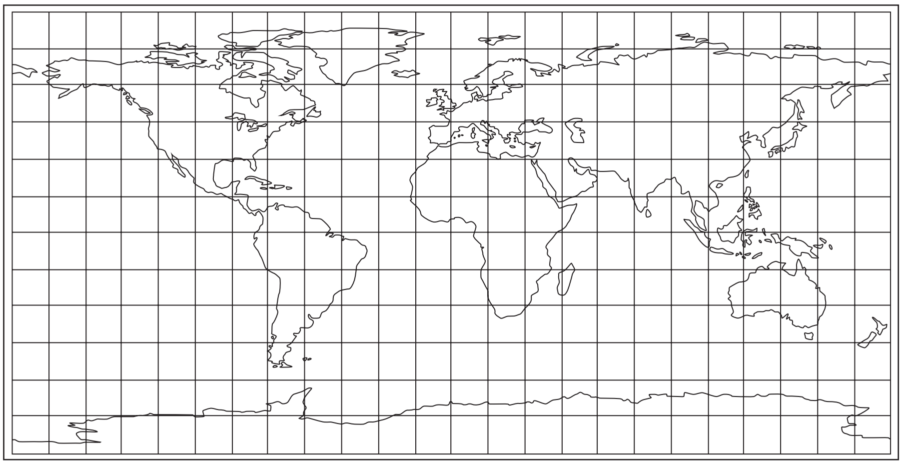
$$x = \lambda, \; y = \varphi, \qquad (15)$$

An equidistant cylindrical projection of the world is shown in Figure 3b.

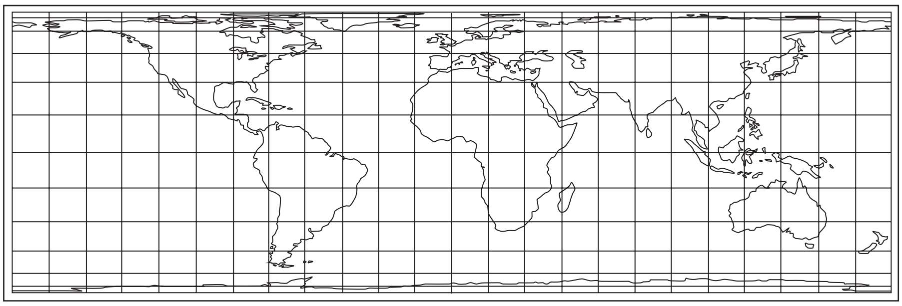
Under the condition that $hk = 1$ and that the



(a)



(b)



(c)

Slika 3. *(a) Mercatorova konformna projekcija. Polovi na Mercatorovoj karti svijeta nalaze se u beskonačnosti, pa se ne mogu prikazati; stoga se ograničavamo na područje preslikavanja između 80 stupnjeva sjeverne i 80 stupnjeva južne geografske širine; (b) ekvidistantna uzduž meridijana cilindrična projekcija; (c) ekvivalentna cilindrična projekcija*
Figure 3 *(a) The Mercator conformal cylindrical projection. The poles on the Mercator world map are at infinity, so they cannot be displayed. Therefore, the mapping area is limited to latitude between 80 degrees N and 80 degrees S; (b) Cylindrical projection equidistant along meridians. (c) Equal–area cylindrical projection*

ordinatnu os x, a srednji meridijan na koordinatnu os y dobit ćemo jednadžbe ekvivalentne cilindrične projekcije

$$x = \lambda,\ y = \sin\varphi, \tag{16}$$

gdje je φ geografska širina, λ geografska dužina. Ekvivalentna cilindrična projekcija svijeta, koja se još naziva i Lambertovom projekcijom (prema Johannu Heinrichu Lambertu, 172 8– 1777), prikazana je na Slici 3c.

PARTNERICE MERCATOROVE PROJEKCIJE UZDUŽ MERIDIJANA

Pogledamo li slike 3a-c lako ćemo uočiti da su na njima slike meridijana jednako razmaknute. To je posljedica jednadžbi za funkciju $x = x(\lambda)$ koje

equator is mapped on the coordinate axis x while the central meridian is mapped on the coordinate axis y, we get the equations of an equal-area cylindrical projection

$$x = \lambda,\ y = \sin\varphi, \tag{16}$$

An equal-area cylindrical projection of the world is shown in Figure 3c. It is known as the Lambert projection, after the famous mathematician Johann Heinrich Lambert (1728 – 1777).

COMPANIONS ALONG MERIDIANS OF THE MERCATOR PROJECTION

Looking at Figures 3a-c, it is easily noted that the images of meridians are equally spaced. This is the result of the equation $x = x(\lambda)$ for the cylindrical su za cilindrične projekcije iz prethodnog poglavlja uvijek iste $x = \lambda$. Takve projekcije možemo nazvati *partnericama uzduž meridijana*. Sve takve projekcije prijateljice imaju jednadžbe

$$x = \lambda \qquad (17)$$
$$y = y(\varphi),$$

gdje se funkcija $y = y(\varphi)$ može definirati na različite načine. Na tome se nećemo zadržavati jer su cilindrične projekcije dobro obrađene u mnogim udžbenicima o kartografskim projekcijama. Jedino ćemo reći da se vizualnom usporedbom uspravne Mercatorove (konformne) projekcije s ekvidistantnom i ekvivalentnom cilindričnom projekcijom lako može uočiti da Mercatorova projekcija znatno deformira kako udaljenosti tako i površine (Sl. 3a, b, c). Ono što je novo je naziv „partnerice“, a obrazloženje tog naziva slijedi u sljedećem poglavlju.

PARTNERICE MERCATOROVE PROJEKCIJE UZDUŽ PARALELA

Posljedica Eulerova dokaza da ne postoji kartografska projekcija bez distorzija je da ne postoji kartografska projekcija koja bi istodobno bila konformna i ekvivalentna. Budući da su konformnost i ekvivalentnost podjednako poželjne karakteristike kartografskih projekcija, na razne se načine pokušavalo spojiti nespojivo. Tako je primjerice W. Tobler (1978.) predložio ekvivalentnu projekciju cijelog svijeta na podlozi Mercatorove projekcije. Rezultat je karta svijeta koja nije neprekidna, svako područje je odvojeno od susjednih. U istom radu Tobler je dao jednadžbe neprekidne ekvivalentne projekcije koja ima međusobne udaljenosti paralela jednake onima kod Mercatorove projekcije. O toj projekciji Tobler je malo podrobnije izvijestio u kraćem prilogu objavljenom 2018. godine, u kojem je tu projekciju nazvao partnericom Mercatorove projekcije (*a companion for Mercator*).

Da bi se dobila ekvivalentna projekcija uz zadržavanje jednakog razmještaja u smjeru geografskih širina (Sl. 4c), bilo je potrebno promijeniti jed-

projections from the previous section, which is always the same, $x = \lambda$. Such projections may be called *companions along meridians*. All such projection companions have the equations

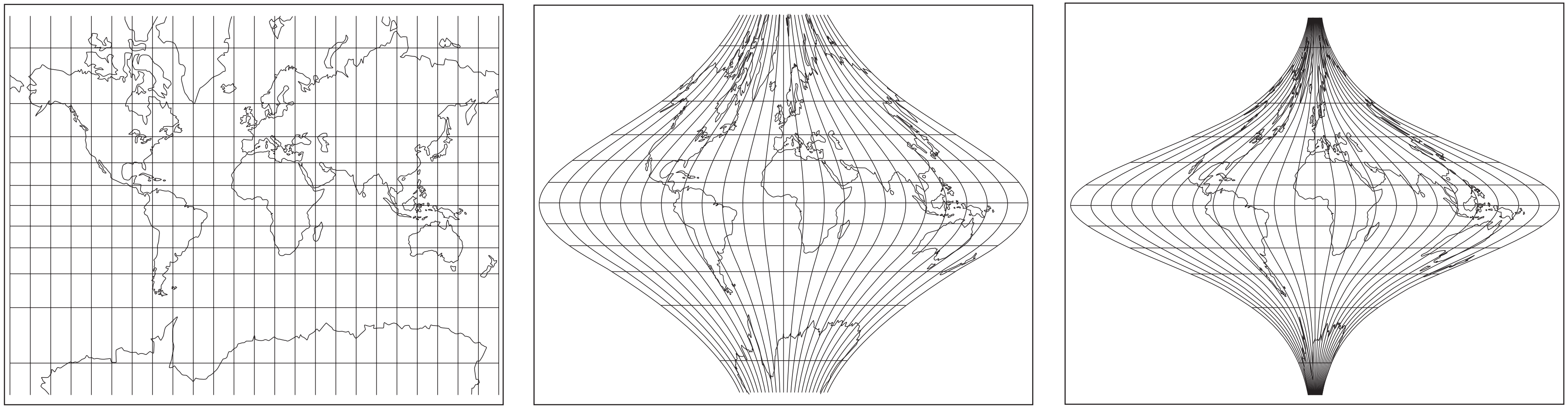
$$x = \lambda \qquad (17)$$
$$y = y(\varphi),$$

where function $y = y(\varphi)$ can be defined in different ways. We will not linger here, because conventional cylindrical projections are amply covered in many textbooks about map projections. We will only say that a visual comparison of the normal Mercator (conformal) projection with an equidistant or equal-area cylindrical projection leads to the conclusion that the Mercator projection significantly deforms both distances and area (Fig. 3a, b, c). What is new is the name ‘companion’, which will be explained in the next section.

COMPANIONS ALONG PARALLELS OF THE MERCATOR PROJECTION

The consequence of Euler’s proof that there can be no mapping without distortion is that there can be no map projection that would at the same time be conformal and equal-area. Since both are equally desirable characteristics of map projections, some people have tried to combine the incompatible. For example, W. Tobler (1978) proposed an equal-area projection of the entire world on the basis of the Mercator projection. The result was a map of the world that is not continuous, where each territory is separated from its neighbour. In the same paper, Tobler gave equations for continuous equal-area projection with parallel distances equal to those of the Mercator projection. Tobler briefly reported on this projection in a short article published in 2018, in which he named the projection a ‘companion for Mercator’.

In order to obtain an equal-area projection with the same distribution of latitudes as in the Mercator projection (Fig. 4c), it was necessary to change



(a) (b) (c)

Slika 4. *(a) Mercatorova konformna projekcija; (b) Lapaine–Tobler–Mercatorova projekcija ekvidistantna uzduž paralela s Mercatorovim razmještajem paralela, Lapaineova ekvidistantna partnerica Mercatora; (c) Tobler–Mercatorova ekvivalentna projekcija s Mercatorovim razmještajem paralela, Toblerova ekvivalentna partnerica Mercatora*

Figure 4 *(a) The Mercator conformal projection; (b) The Lapaine–Tobler–Mercator projection equidistant along parallels with Mercator's distribution of parallels. Lapaine's equidistant companion for Mercator; (c) The Tobler–Mercator equal-area projection with Mercator's distribution of parallels. Tobler's equal-area companion for Mercator* nadžbu koja definira razmještaj geografskih dužina. Tobler (2018.) je dobio jednadžbe te projekcije u obliku

$$x = \lambda \cos^2 \varphi$$
$$y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \quad (18)$$

U ovome članku želimo izvesti jednadžbe projekcije koja će biti ekvidistantna uzduž paralela, a u isto vrijeme imati razmještaj paralela jednak onome u Mercatorovoj projekciji. Tu projekciju nazvat ćemo ekvidistantnom partnericom Mercatorove projekcije uzduž paralela. Osim toga, izvest ćemo relacije koje opisuju distorzije Mercatorove projekcije i njezinih partnerica.

Sve partnerice Mercatorove projekcije uzduž paralela imaju jednadžbe

$$x = x(\varphi, \lambda)$$
$$y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right), \quad (19)$$

gdje se funkcija $x = x(\varphi, \lambda)$ može definirati na različite načine. Iz jednadžbi (19) vidimo da takvih projekcija ima beskonačno mnogo i da će one pripadati skupini pseudocilindričnih projekcija jer će slike paralela biti ravne, međusobno paralelne crte. Da bismo dobili kartografske projekcije različitih svojstava, ali s jednadžbama oblika (19), potrebni su koeficijenti prve diferencijalne forme preslikavanja:

$$E = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \frac{1}{\cos^2 \varphi}$$
$$F = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} \quad (20)$$
$$G = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2.$$

Faktori lokalnih linearnih mjerila uzduž meridijana h i uzduž paralela k su:

the equation defining the longitude distribution. Tobler (2018) put the equations of that projection in the form

$$x = \lambda \cos^2 \varphi$$
$$y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \quad (18)$$

In this paper, we want to derive equations of projection that will be equidistant along parallels, while at the same time having a parallel arrangement equal to that in the Mercator projection. We will call this projection an equidistant companion along the parallels for Mercator. Additionally, we will derive the relationships that describe the distortions of the Mercator projection and its companions.

All Mercator projection companions along parallels have the equations

$$x = x(\varphi, \lambda)$$
$$y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right), \quad (19)$$

where function $x = x(\varphi, \lambda)$ can be defined in different ways. From the equations (19) we can see that there is an infinite number of such projections. All belong to a group of pseudocylindrical projections, because the images of parallels will be straight, mutually parallel lines. In order to obtain map projections of different properties but with the equations in (19), the first differential form coefficients are needed:

$$E = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \frac{1}{\cos^2 \varphi}$$
$$F = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} \quad (20)$$
$$G = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2.$$

The local linear scale factors along meridians h and parallels k are:

$$h = \sqrt{E} = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \frac{1}{\cos^2 \varphi}}$$
$$k = \frac{\sqrt{G}}{\cos \varphi} = \frac{1}{\cos \varphi} \frac{\partial x}{\partial \varphi} . \quad (21)$$

Da bi projekcija kojoj odgovaraju jednadžbe (19) bila konformna, potrebno je da bude $F = 0$ i $h = k$. Iz (20) slijedi da mora biti

$$\frac{\partial x}{\partial \varphi} = 0$$

ili (22)

$$\frac{\partial x}{\partial \lambda} = 0.$$

Kad bi bilo

$$\frac{\partial x}{\partial \lambda} = 0,$$

onda u jednadžbama (19) ne bi bilo geografske dužine, to jest ne bi bile jednadžbe neke kartografske projekcije. Prema tome mora biti

$$\frac{\partial x}{\partial \varphi} = 0.$$

Sada iz (5) slijedi da je

$$h = \frac{1}{\cos \varphi}$$

a zbog $h = k$ mora biti

$$\frac{\partial x}{\partial \lambda} = \frac{dx}{\partial \lambda} = 1$$

i konačno

$$x = \lambda + C$$

gdje je C proizvoljna konstanta. Uz uvjet $x = 0$ za $\lambda = 0$ (srednji meridijan preslikava se na os y), slijedi da je $C = 0$, pa imamo jednadžbe (19), to jest poznatu uspravnu Mercatorovu projekciju. Dokazali smo da je jedina pseudocilindrična konformna kartografska projekcija s jednadžbama oblika (19) upravo cilindrična Mercatorova projekcija.

Da bi projekcija kojoj odgovaraju jednadžbe

$$h = \sqrt{E} = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \frac{1}{\cos^2 \varphi}}$$
$$k = \frac{\sqrt{G}}{\cos \varphi} = \frac{1}{\cos \varphi} \frac{\partial x}{\partial \varphi} . \quad (21)$$

For the projection corresponding to equations (19) to be conformal, there should be $F = 0$ and $h = k$. From (20) follows that there must be

$$\frac{\partial x}{\partial \varphi} = 0$$

or (22)

$$\frac{\partial x}{\partial \lambda} = 0.$$

If there were

$$\frac{\partial x}{\partial \lambda} = 0,$$

then the equations (19) would not include longitude, which means that (19) would not represent the equations of a map projection. Therefore there must be

$$\frac{\partial x}{\partial \varphi} = 0.$$

From (5) it follows

$$h = \frac{1}{\cos \varphi}$$

and due to $h = k$ there must be

$$\frac{\partial x}{\partial \lambda} = \frac{dx}{\partial \lambda} = 1$$

and finally

$$x = \lambda + C$$

where C is a constant. From the condition $x = 0$ for $\lambda = 0$ (the central meridian is mapped on the y axis), it follows that $C = 0$, and we have the equations (19), that is, the well-known Mercator projection in normal aspect. We have just proved that the only pseudocylindrical conformal map projection with the equations in the form (19) is precisely the Mercator cylindrical projection.

For a projection with equations (19) to be

(19) bila ekvivalentna, treba vrijediti

$$hk \sin \beta = 1 \tag{23}$$

gdje je β kut između slika meridijana i paralela u ravnini projekcije. Taj je kut općenito određen relacijama

$$\cos \beta = \frac{F}{\sqrt{EG}}$$
$$\sin \beta = \sqrt{\frac{EG - F^2}{EG}} \tag{24}$$

Uzevši u obzir (21), relacija (23) se lako prevede u

$$\frac{\partial x}{\partial \lambda} = \cos^2 \varphi \tag{25}$$

odakle je

$$x = \lambda \cos^2 \varphi + f(\varphi), \tag{26}$$

gdje je $f(\varphi)$ bilo koja funkcija geografske širine φ. Uz uvjet $x = 0$ za $\lambda = 0$ (srednji meridijan se preslikava na os y), slijedi da je $f(\varphi) = 0$, pa konačno imamo jednadžbe (18) koje daju Toblerovu ekvivalentnu partnericu Mercatorovoj projekciji uzduž paralela (Sl. 2c).

Uvjet za ekvidistantnost uzduž paralela glasi $k = 1$, što je u skladu s (20) ekvivalentno s

$$\frac{\partial x}{\partial \lambda} = \cos \varphi \tag{27}$$

Opće rješenje parcijalne diferencijalne jednadžbe (27) je

$$x = \lambda \cos \varphi + f(\varphi), \tag{28}$$

gdje se funkcija $f = f(\varphi)$ može odabrati proizvoljno. Ako ponovno odaberemo $x = 0$ i $\lambda = 0$, dobit ćemo $f(\varphi) = 0$ za svaku geografsku širinu φ.

Na taj način dobili smo konačne jednadžbe za projekciju koja je ekvidistantna uzduž paralela, a ima jednak razmještaj paralela kao u Mercatoro-

equal-area,

$$hk \sin \beta = 1 \tag{23}$$

where β is the angle between the images of meridians and parallels in the plane of projection. This angle is generally determined by the relations

$$\cos \beta = \frac{F}{\sqrt{EG}}$$
$$\sin \beta = \sqrt{\frac{EG - F^2}{EG}} \tag{24}$$

Taking into account (21), relation (23) can be easily transformed into

$$\frac{\partial x}{\partial \lambda} = \cos^2 \varphi \tag{25}$$

from where

$$x = \lambda \cos^2 \varphi + f(\varphi), \tag{26}$$

where $f(\varphi)$ is any function of latitude φ. From the condition $x = 0$ for $\lambda = 0$ (the central meridian is mapped on the y axis), it follows that $f(\varphi) = 0$, and finally we have the equations (18) for Tobler's equal-area companion along the parallels for Mercator (Fig. 4c).

The condition of equidistance along parallels reads $k = 1$, which is according to (20) equivalent to

$$\frac{\partial x}{\partial \lambda} = \cos \varphi \tag{27}$$

The general solution of the partial differential equation (27) is

$$x = \lambda \cos \varphi + f(\varphi), \tag{28}$$

where function $f = f(\varphi)$ can be chosen arbitrarily. If we again take $x = 0$ and $\lambda = 0$, we will get $f(\varphi) = 0$ for any latitude φ. Thus, we have the final equations for the projection that is equidistant along the parallels and has the same parallel arrangement as the Mercator projection:

voj projekciji:

$$x = \lambda \cos \varphi$$
$$y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \quad (29)$$

Na Slici 4b je karta svijeta u toj projekciji, uz srednji grinički meridijan.

Uvjet za ekvidistantnost uzduž meridijana glasi $h = 1$ što je ekvivalentno s

$$\left(\frac{\partial x}{\partial \varphi}\right)^2 + \frac{1}{\cos^2 \varphi} = 1, \quad (30)$$

odnosno

$$\left(\frac{\partial x}{\partial \varphi}\right)^2 = -\tan^2 \varphi \quad (31)$$

Očito je da posljednja diferencijalna jednadžba nema realnih rješenja. Drugim riječima, ne postoji partnerica Mercatorove projekcije uzduž paralela koja bi bila ekvidistantna uzduž meridijana.

DISTORZIJE PARTNERICA MERCATOROVE PROJEKCIJE UZDUŽ PARALELA

Najprije uočimo da se jednadžbe Mercatorove projekcije i njezinih partnerica uzduž paralela, ekvivalentne i ekvidistantne, mogu napisati u obliku

$$x = \lambda \cos^t \varphi$$
$$y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \quad (32)$$

gdje je $t = 0$ za Mercatorovu projekciju, $t = 1$ za njezinu ekvidistantnu partnericu, i $t = 2$ za ekvivalentnu partnericu.

Prema (20), za koeficijente prve diferencijalne forme potrebne su parcijalne derivacije

$$\frac{\partial x}{\partial \varphi} = -\lambda t \cos^{t-1} \varphi \sin \varphi \quad (33)$$

$$\frac{\partial x}{\partial \lambda} = \cos^{t-1} \varphi. \quad (34)$$

$$x = \lambda \cos \varphi$$
$$y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \quad (29)$$

Figure 4b is a world map in this projection, with the Greenwich meridian as the central meridian.

The condition of equidistance along the meridians reads $h = 1$ which is equivalent to

$$\left(\frac{\partial x}{\partial \varphi}\right)^2 + \frac{1}{\cos^2 \varphi} = 1, \quad (30)$$

or

$$\left(\frac{\partial x}{\partial \varphi}\right)^2 = -\tan^2 \varphi \quad (31)$$

Obviously, the last differential equation has no real solutions. In other words, there is no companion for Mercator along parallels which would be equidistant along the meridians.

DISTORTIONS OF COMPANIONS FOR MERCATOR ALONG PARALLELS

First of all, let us note that the equations of the Mercator projection and its companions along the parallels can be written as

$$x = \lambda \cos^t \varphi$$
$$y = \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \quad (32)$$

where $t = 0$ for the Mercator projection, $t = 1$ for its equidistant companion and $t = 2$ for its equal-area companion.

According to (20), we need partial derivatives for the coefficients of the first differential form

$$\frac{\partial x}{\partial \varphi} = -\lambda t \cos^{t-1} \varphi \sin \varphi \quad (33)$$

$$\frac{\partial x}{\partial \lambda} = \cos^{t-1} \varphi. \quad (34)$$

Tablica 1. *Vrijednosti lokalnih faktora mjerila za Mercatorovu projekciju (t = 0), i njezine partnerice u smjeru paralela – ekvidistantnu (t = 1) i ekvivalentnu (t = 2)*
Table 1 *Values of local scale factors for the Mercator projection (t = 0), and its companions along the parallels – equidistant (t = 1) and equal-area (t = 2)*

	t	*h*	*k*	*p*
Mercatorova projekcija *Mercator projection*	0	$\frac{1}{\cos\varphi}$	$\frac{1}{\cos\varphi}$	$\frac{1}{\cos^2\varphi}$
Ekvidistantna partnerica uzduž paralela *Equidistant companion along the parallels*	1	$\frac{1}{\cos\varphi}\sqrt{\lambda^2\cos^2\varphi\sin^2\varphi+1}$	1	$\frac{1}{\cos\varphi}$
Ekvivalentna partnerica uzduž paralela *Equal-area companion along the parallels*	2	$\frac{1}{\cos\varphi}\sqrt{4\lambda^2\cos^4\varphi\sin^2\varphi+1}$	$\cos\varphi$	1

Sada, prema (21), imamo faktore lokalnih linearnih mjerila

$$h=\frac{1}{\cos\varphi}\sqrt{\lambda^2 t^2\cos^{2t}\varphi\sin^2\varphi+1}$$
$$k=\cos^{t-1}\varphi. \tag{35}$$

Faktor lokalnog mjerila površina p je

$$p=hk\sin\beta=\frac{\sqrt{EG-F^2}}{\cos\varphi}=\cos^{t-2}\varphi. \tag{36}$$

Vrijednosti lokalnih faktora mjerila za Mercatorovu projekciju (t = 0), i njezine partnerice u smjeru paralela − ekvidistantnu (t = 1) i ekvivalentnu (t = 2) − prikazane su u Tablici 1.

Iz Tablice 1. može se vidjeti da su vrijednosti svih triju lokalnih faktora mjerila ekvidistantne partnerice Mercatorove projekcije (t = 1) između vrijednosti Mercatorove projekcije i njezine ekvivalentne partnerice gledano u smjeru paralela.

ZAKLJUČAK

Izvorna Mercatorova kartografska projekcija je konformna, to jest kutovi se lokalno prikazuju

Now, according to (21) we have the local liner scale factors

$$h=\frac{1}{\cos\varphi}\sqrt{\lambda^2 t^2\cos^{2t}\varphi\sin^2\varphi+1}$$
$$k=\cos^{t-1}\varphi. \tag{35}$$

Local area scale factor p reads

$$p=hk\sin\beta=\frac{\sqrt{EG-F^2}}{\cos\varphi}=\cos^{t-2}\varphi. \tag{36}$$

The values of local scale factors for the Mercator projection (t = 0), and its companions along the parallels − equidistant (t = 1) and equal-area (t = 2) − are shown in Table 1.

In Table 1, one can see that the values of all three local scale factors of the equidistant companion for Mercator (t = 1) are between the values of the Mercator projection and the value of its equal-area companion along parallels.

CONCLUSION

The original Mercator map projection was conformal, that is, the angles were faithfully displayed vjerno, dok su površine deformirane. Tobler-Mercatorova projekcija to obrće, ali su sad maksimalne promjene kutova ekstremno velike. Ekvivalentna Tobler-Mercatorova projekcija ostavlja snažan dojam o distorzijama površina u Mercatorovoj projekciji. Ekvidistanta projekcija uzduž paralela, predložena u ovome članku, po svojstvima deformacija kutova i površina nalazi se između Mercatorove i Tobler-Mercatorove projekcije. Ona je još jedna partnerica Mercatorove projekcije.

Partnerice Mercatorove projekcije prikazane u ovome radu nisu samo neobični reprezentanti ekvivalentnih i ekvidistantnih projekcija. Njima se želi dodatno osvijestiti činjenicu da karta svijeta izrađena u Mercatorovoj projekciji znatno deformira udaljenosti i površine. U tom bi smislu ovaj članak mogao imati didaktičku komponentu.

locally, while the areas were distorted. The Tobler-Mercator projection changes this, but now the maximum angular changes are extremely large. The equal-area Tobler-Mercator projection gives a strong impression of area distortions in the Mercator projection. The equidistant projection proposed in this article is somewhere between the Mercator and the Tobler-Mercator projections in terms of angular and area distortions. It is another companion for Mercator.

The Mercator companions presented in this paper are not merely unusual representations of equal-area and equidistant projections. Through these projections, we would like to highlight the fact that the map of the world made in the Mercator projection significantly distorts distances and areas. So, this article may have a didactic component.

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