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1D and 2D parameter estimation of a rotary drum filter based on RLS and IV methods

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ABSTRACT

This work deals with a physical one- and two-dimensional (1D and 2D) parameters estimation of a filtration process of slurry, the second stage of phosphoric acid manufacture. This study focuses on recursive least square and instrumental variable techniques applied to the (1D) and (2D) models. The model of the rotary drum filter is based on different physical laws involved in the filtration phase in order to get a simulator of the filtration process. Besides, many physical parameters rise in the system model and effect enormously the efficiency which should be modelled with precision such as permeability, porosity and viscosity. We use a constructive realization procedure for (2D) systems which may lead to a Fornasini–Marchesini local state-space model to describe the dynamic of the system states.

1. Introduction

In recent years, membrane filtration technology and particularly low pressure membrane technology have been widely applied in liquid aspiration fields more and more [1].

For some time, it is well known that membrane separation processes are considered very beneficial, advanced and efficient technologies to use in factories. These methods are used to separate suspended particles. The filtration technique is classified according to the size of the pores. So, there are four types of filtration: macrofiltration, microfiltration, ultrafiltration and nanofiltration [2]. Filtration processes can be classified into two categories:

- Discontinuous: The press filter and the Nutshe filter are most common.
- Continuous: The rotary drum filter and the pass band filter are most common.

In our case, we are interested in the rotary drum filter which ensures the separation between the phosphoric acid and the phosphogypsum. Few works are dealt with the modelling of the filtration phases due to its complexity and the number of parameters interfering in the filtration process. The problem is therefore to model the system to be studied so that the model is as faithful as possible to the actual behaviour of the system.

Modelling filter is essentially based on the Darcy's law treated by [3] that applies to an homogeneous and isotropic porous environment traversed by a flow at low speed in order to calculate the rate of phosphoric acid produced. Darcy's law is processed also by [4], a similar strategy has been discussed by [5]. Then a relationship frequently quoted was proposed by [6] and modified by [7].

In this research, we describe the dynamic behaviour of a rotary drum filter by an homogeneous two-dimensional (2D) model. Fornasini and Marchesini [8] focus on the importance of (2D) systems. (2D) systems have attracted attention because of their wide range of practical projects and significant significance. (2D) system theory can be used efficiently in many fields such as digital filtering and batch processes [9, 10]. For instance, investigations on (2D) systems have been related to state-space model realization and stability [11], estimation and observers [12–14] and filtering [15–17].

More recently, many other interesting contexts where (2D) systems prove to be the appropriate setting for carrying on a thorough and successful analysis have been illuminated [18, 19]. Indeed, in the (1D) context, there has been a long stream of research, which originated in the 70s and flourished in the 80s [20, 21], but still represents a very important research topic. During the last years, research focused on the (2D) system theory and it have been interested in most of the classical topics already investigated in the (1D) setting [22–25]. Generally, the main interesting point of the (2D) systems is its dependence on two variables which are the time and the space. However, there are many (2D) systems such that none of their variables is time or spatial

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[26–29]. Moreover, one of the fundamental issues in the (2D) systems theory is the realization of a given transfer function or transfer matrix by a certain kind of the (2D) local state-space model, typically by Roesser model or Fornasini–Marchesini second (FM-II) model.

Our contribution is mainly based on the modelling of a rotary drum filter using physical laws and particularly concerned with the realization problem for a given (2D) MIMO system by the (FM-II) model. As well as a (2D) least squares identification is applied and it has a great importance in a huge range of applications [30]. However, the special structure of the (2D) data set gives rise to the development of cost-effective algorithms for the sequential determination of parameters [31, 32].

This paper is organized as follows: Section 2 provides the characteristics of the rotary drum filter subject of steady with its different running modes. Section 3 describes the (1D) nonlinear modelling of the filter using several laws. Section 4 presents the linearization of the (1D) system. Section 5 describes the implementation of (FM-II) model on our system and the algorithms used. Section 6 represents parameter estimation with the recursive least square (RLS) method in the case (1D) and (2D). Finally, concluding remarks are made in Section 7.

2. The rotary drum filter

There are mainly two vacuum-operated devices that are rotary drum filters and band filters. They have the same applications but the band filters deal thicker slurries (50% solids). Among the continuous filters which are considered, we find the rotary drum filter. The rotary drum filter is constituted by two coaxial cylindrical drum and a outer drum carries a filter canvas.

2.1. Characteristics

Our filter has a huge toric surface which has a value close to 262 m^2. This filter (Figure 1) is composed by 36 flat cells covered by 36 membranes with 36 tubes. Each tube is connected to a cell, a vacuum box and a driving motor which is responsible for the transition of the slurry by five sectors which are: the pre-sector, the strong acid sector, the acid medium sector, the weak acid sector and the sector of wash canvases. Figure 2 shows a sketch [33] of the filter. The outputs of the filter are the strong acid which goes to storage, the gypsum which goes to outer area and the medium acid which returns to the reactor [34]. Tables 1 and 2 gathers the different characteristics of the filter.

The cake (slurry) reaches a certain thickness and by the end of the cycle, the gypsum is removed from the filter by scraping system. A cycle of wash and spin is often assistant. These filters provide greater investment but they have a lower cost of operation: they are therefore suitable for large productions.

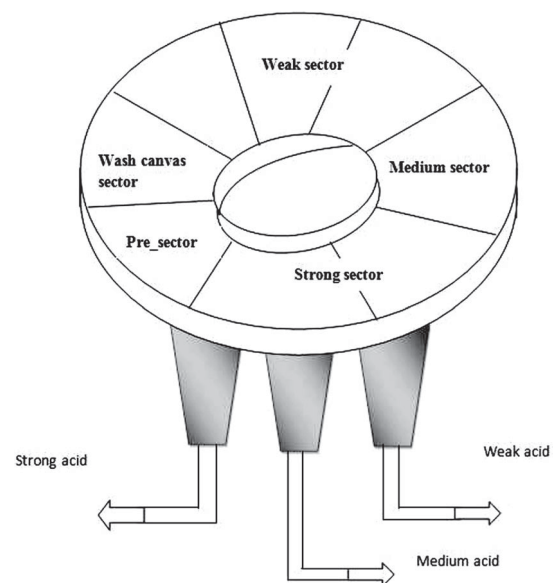**Figure 1.** Photo of the rotary drum filter.

Figure 2. Functional block of the rotary drum filter.

Table 1. Nomenclature of parameters.

Symbol	Designation	Value (IS)
S_O	Peripheral filter surface	262.314 m^2
S_I	Internal filter surface	42.754 m^2
D_O	Overall filter diameter	18.280 m
D_I	Internal filter diameter	7.380 m
S_n	Surface of one nacelle	6.098 m^2
S_F	Surface of the fort compartment	24, 395 m^2
S_M	Surface of the medium compartment	48, 784 m^2
S_{FW}	Surface of the fabric wash compartment	24.395 m^2
S_W	Surface of the weak compartment	60.98 m^2
S_G	Surface of the gypsum compartment	60.98 m^2

In continuous filtration, a cake stand (or filter media) moves slowly to the suspension. When the cake is thick enough (or clogging of the mass is large enough), we shall remove the cake with, where appropriate, an intermediate steps of washing and dewatering of the cake.

2.2. Running modes

The membrane filtration can be used in two main operations: frontal filtration (dead end) and tangential flow
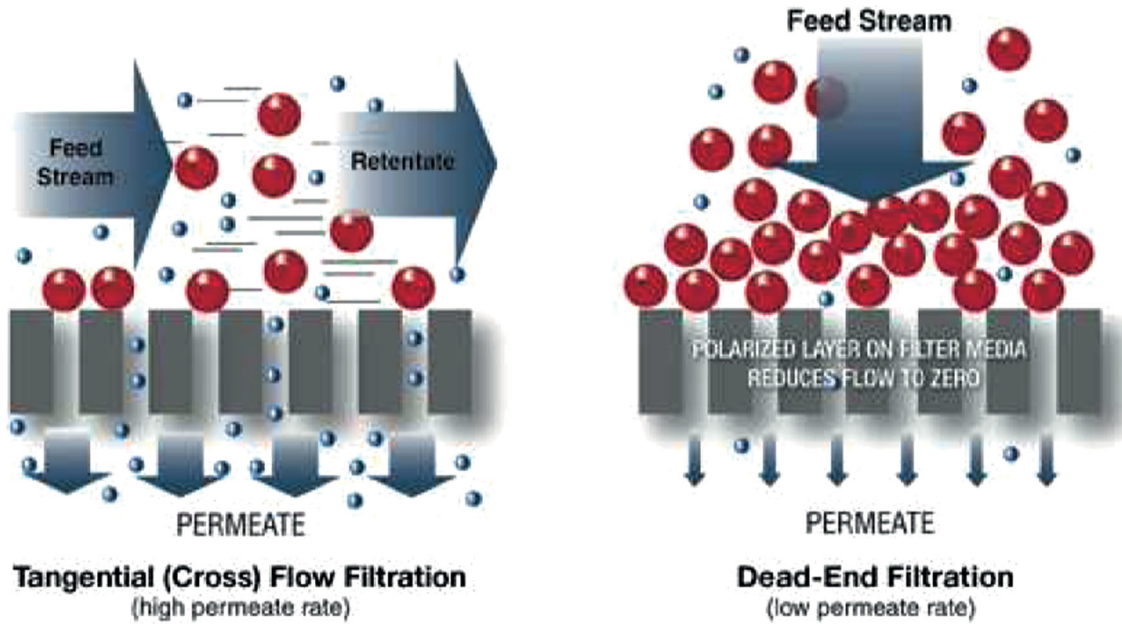
Figure 3. Operating principle of frontal and tangential filtration.

Table 2. Nomenclature of parameters.

Symbol	Designation	Unit
Q_e	Flow of strong acid	$m^3 \cdot s^{-1}$
z	vertical displacement along the height of the cake	m
P	Loss of pressure	Pa
P_s	Solid pressure	Pa
P_l	Liquid pressure	Pa
w_f	filter Speed	RPMn
S	Surface of the strong acid compartment	m^2
V	Volume flow filtration	m^3
K	Permeability	m^2
h	Height of the cake	m
μ_l	Liquid viscosity	Pas $\cdots$ s
μ	Viscosity	Pas $\cdot$ s
ϑ	filtration Speed	$m \cdot s^{-1}$
V_b	Volume of slurry	m^3
t	Time	s
L	Length of the portion of the cake	m
r	Radial displacement	m
q_b	Flow rate of slurry	$m^3 \cdot s^{-1}$
ϵ	Porosity of particles of phosphate	Dimensionless
θ	Opening angle of the nacelle of the filter	R
θ_0	Initial opening angle of the nacelle of the filter	R
E	ratio of lateral and normal stress	Dimensionless

filtration (cross flow) (see figure 3). These two modes are very important and correspond to two technologies and two completely different approaches to filtration.

In our case, we have a frontal filtration because the slurry is injected in a perpendicular way in the filter. For this mode of running, the fluid flows vertical to the membrane by causing a shear which allows the accumulation of material. All the material retained accumulates on the membrane. Nevertheless, this type of filtration increases a little the occurrence of clogging phenomenon. The frontal filtration is the best solution but it requires a washing step to avoid the phenomenon of clogging. In general, the filtration can be operated at constant pressure or constant flow. In our case, we have a constant pressure. The pressure gradient may be generated by the operation of a pump that circulates the liquid above the membrane. When the pressure of the vacuum pump decreases, the thickness of the cake increases along the axis of the membrane [35]. Due to the coriolis effect, the thickness of the cake has tendency to move to the outer edge which decreases the height of the edge. This phenomenon is neglected in our case because the speed of the rotation of the filter is so low.

3. (1D) nonlinear modelling of the filter

The literature distinguishes two types of approach to model filtration: macroscopic models based on the material conservation equations and a filtration rate determined empirically as well as a microscopic models describing the grain transportation process. Concerning macroscopic models, these types are not based on a physical modelling of particle deposition mechanisms. The parameters of these models do not necessarily have a clear physical meaning but its results are usually close to the experimental results.

Developing a model requires first to write mathematical equations that relate state variables descriptive of physical process. Compared to other procedures, fixed cultures, the filtration system has the complementary particularity to include an additional model of physical filtration, which increases the difficulty of resolution and integration of system equations. For activated sludge processes like the filter, the problem of the complexity of the models make them control possibilities in practice very limited cases [36].

The model we are looking for is a model for a control objective. So we will look for the physical equations that contain the state variables X and control variables U.
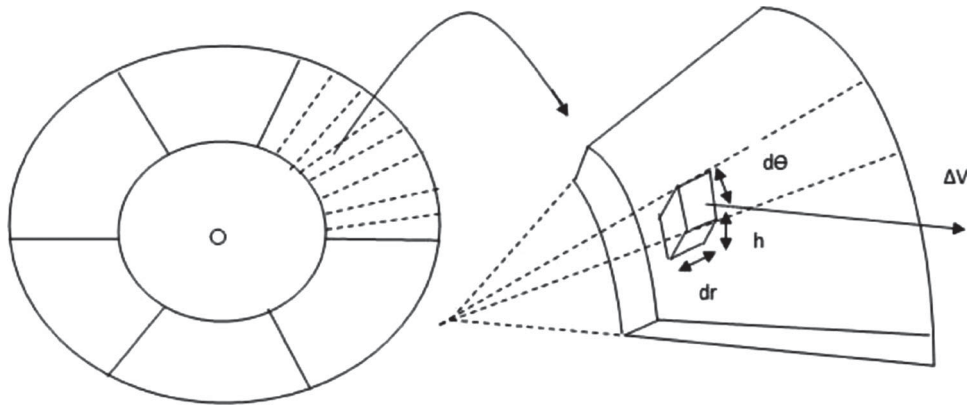
Figure 4. Decomposition of a filter cell.

The state variables are the height of the slurry in the filter which denoted h and the volume of the filtrate acid denoted V. The two actuators involved in the process are the pump providing slurry which is regulated to a prescribed volumetric flow q_b, and the motor for rotating the filter which is regulated to the described angular speed w_f.

3.1. The state variable calculation

3.1.1. The height of the cake h
According to the mass balance, the quantity of the slurry in the filter is equal to the difference between the quantity of slurry injected in the filter and the quantity of acid gotten from the slurry:

$$M_b = M_{\text{filter}} + M_A, \qquad (1)$$

where M_b is the mass of the slurry injected into the filter, M_{filter} is the mass of the slurry after filtration and M_A is the mass of the phosphoric acid filtrated.

The mass of the slurry M_b injected from the reactor to the filter is modelled by the volume of the slurry V_b injected into the filter:

$$M_b = \rho_b V_b, \qquad (2)$$

where ρ_b is the volumetric density of the slurry and V_b is the volume of slurry filled the nacelle.

The mass of the slurry M_{filter} after filtration is modelled by the height of the cake h and the surface S of the nacelle:

$$M_{\text{filter}} = \rho_b S h. \qquad (3)$$

The mass of the filtrated acid M_A is modelled by the volume of this acid V:

$$M_A = \rho_A V, \qquad (4)$$

where ρ_A is the volumetric density of the phosphoric acid P_2O_5.

Replacing (2), (3) and (4) into (1), we get

$$\rho_b V_b = \rho_b S h + \rho_A V. \qquad (5)$$

If we divide (5) by ρ_b and we suppose that $\rho_A / \rho_b = m_\rho$, we find the following formula:

$$V_b = S h + m_\rho V. \qquad (6)$$

Then the filter has the shape of a ring with external radius R_1 and internal radius R_2 as shown in Figure 4. So the surface S of the nacelle is modelled by the angle θ:

$$S = \frac{1}{2}(R_1^2 - R_2^2)\theta. \qquad (7)$$

If we introduce (7) into (6), we obtain

$$V_b = \frac{1}{2}(R_1^2 - R_2^2)\theta h + m_\rho V. \qquad (8)$$

If we derivate (8) with respect to time, we obtain

$$\frac{dV_b}{dt} = \frac{1}{2}(R_1^2 - R_2^2)\theta \frac{dh}{dt} + \frac{1}{2}(R_1^2 - R_2^2)h \frac{d\theta}{dt} + m_\rho \frac{dV}{dt}. \qquad (9)$$

In fact, the angular deviation with respect to time $d\theta/dt$ is equivalent to the angular speed of the filter w_f, and the slurry volume deviation with respect to time dV_b/dt is equivalent to the volumetric flow of the slurry q_b, so we obtain from (9) the following equation:

$$q_b = \frac{1}{2}(R_1^2 - R_2^2)\theta \frac{dh}{dt} + \frac{1}{2}(R_1^2 - R_2^2)h w_f + m_\rho \frac{dV}{dt}. \qquad (10)$$

We take $dV/dt = q_{\text{AF}}$. From (10), we can get the height deviation with respect to time dh/dt:

$$\frac{dh}{dt} = \frac{2}{(R_1^2 - R_2^2)\theta} q_b - \frac{h}{\theta} w_f - \frac{2m_\rho}{(R_1^2 - R_2^2)\theta} q_{\text{AF}}. \qquad (11)$$

Finally, we get

$$\frac{dh}{dt} = f_1(X(t), U(t)), \qquad (12)$$

where $X = [h\ q_{\text{AF}}]'$ and $U = [q_b\ w_f]'$.

3.1.2. Volume flow rate of the phosphoric acid q$_{AF}$

To describe the flow of a fluid through a porous medium, we use the Navier Stokes law and the Darcy's law. The Darcy's law is a relation between the flow of a fluid and the pressure. Permeability is a physical characteristic that represents the ease of a material to allow the transfer of fluid through a connected network. The Darcy's law makes it possible to connect a flow rate to a pressure gradient applied to the fluid thanks to a characteristic parameter of the crossed place: the permeability k. The mass conservation equation coupled to the fluid motion equation yields the incompressible Navier Stokes equations and the main unknown parameters in this equation are the mass density, the velocity, the pressure and the temperature, but this list could be longer depending on the case being studied. The purpose of this part is to describe the corresponding fundamental solution for the equations that model an incompressible fluid in the exterior of a rigid body that is rotating at a constant velocity and rotation [37]. The Navier Stokes equation is:

$$\rho \frac{\partial \vartheta}{\partial t} + \rho(\vartheta \cdot \Delta)\vartheta = -\nabla P + \mu \cdot \Delta \vartheta \qquad (13)$$

$$\nabla \cdot \vartheta = 0 \qquad (14)$$

with ϑ is the velocity field, P is the pressure, ρ is the fluid density, μ is the kinematic viscosity, t is the time and Δ is Laplace operator. $\Delta = \sum_{j=1}^{1} \partial^2/\partial z_j^2$ stands for the Laplace operator.

If we multiply (13) by S, we obtain

$$\rho \frac{\partial \vartheta \cdot S}{\partial t} + \rho \cdot \vartheta \cdot \frac{\partial \vartheta \cdot S}{\partial z} = -S \cdot \frac{\partial P}{\partial z} + \mu \cdot \frac{\partial^2 \vartheta \cdot S}{\partial z^2}. \qquad (15)$$

If we take $\vartheta \cdot S = q_{AF}$, so we find

$$\frac{\partial q_{AF}}{\partial t} = -\vartheta \cdot \frac{\partial q_{AF}}{\partial z} - \frac{S}{\rho} \cdot \frac{\partial P}{\partial z} + \frac{\mu}{\rho} \cdot \frac{\partial^2 q_{AF}}{\partial z^2}. \qquad (16)$$

Now we should calculate the following two terms: $\partial q_{AF}/\partial z, \partial^2 q_{AF}/\partial z^2$.

To find these two terms, Darcy's law is used which has the following formula:

$$\vec{\vartheta} = \frac{K \Delta P S}{\mu z}.$$

The strong acid flow (28%P_2O_5) is characterized by a $\vec{\vartheta}$, according to the Darcy's law, is proportional to the pressure ΔP and the permeability K and inversely proportional to the dynamic viscosity μ of the fluid.

First, we will express the pressure drop across the filter. The term is derived from the Darcy's law.

$$\begin{aligned} q_{AF}(z) &= \frac{K(P_0 - P(z))S}{\mu \cdot z} \\ &= \frac{KP_0 \cdot S}{\mu \cdot z} - \frac{KS}{\mu} \frac{P(z)}{z} \\ &= \frac{\beta_1}{z} - \beta_2 \frac{P(z)}{z} \end{aligned} \qquad (17)$$

with $\beta_1 = (K \cdot P_0 \cdot S)/\mu$ and $\beta_2 = K \cdot S/\mu$ are two constants.

If we derivate the previous equation, we obtain

$$\frac{\partial q_{AF}}{\partial z} = \frac{\beta_1}{z^2} - \beta_2 \frac{\frac{\partial P(z)}{\partial z}}{z} - \frac{P(z)}{z^2}. \qquad (18)$$

If we derivate (18), we get (19)

$$\begin{aligned} \frac{\partial^2 q_{AF}}{\partial z^2} &= \frac{2\beta_1}{z^3} + \beta_2 \frac{\frac{\partial P(z)}{\partial z}}{z} - 2\frac{P(z)}{z^2} \\ &\quad - \beta_2 \left(\frac{\partial^2 P(z) z}{\partial z^2} - \frac{\partial P(z)}{\partial z} \right). \end{aligned} \qquad (19)$$

Replacing (18) and (19) in (16), we find

$$\begin{aligned} \frac{\partial q_{AF}}{\partial t} &= \frac{\partial P(z)}{\partial z} \left(\frac{\vartheta \cdot \beta_2}{z} - \frac{S}{\rho} + \frac{\mu \cdot \beta_2}{\rho \cdot z} + \frac{\mu \cdot \beta_2}{\rho} \right) \\ &\quad - \upsilon \left(\frac{\beta_1}{z^2} + \beta_2 \frac{P(z)}{z^2} \right) + \frac{\mu}{\rho} \left(2\frac{\beta_1}{z^3} - 2\frac{P(z)}{z^2} \right). \end{aligned} \qquad (20)$$

The overall pressure P in the slurry mixture is the sum of the pressure of the solid particles P_s and the pressure of the fluid P_l such as

$$P = P_s + P_l.$$

The differential of the pressure with respect to the z axis:

$$\frac{\partial P}{\partial z} = \frac{\partial P_s}{\partial z} + \frac{\partial P_l}{\partial z}. \qquad (21)$$

The expressions of P_s and P_l are given by the following relations:

$$\begin{aligned} \frac{\partial P_s}{\partial z} &= \frac{Q_e}{2\pi K z^2}(1 - E)\frac{P_s}{z} + (\rho_s - \rho_l)w_f^2 z, \\ \frac{\partial P_l}{\partial z} &= -\frac{\mu_l Q_e}{K} + \mu_l w_f^2 z, \end{aligned}$$

where Q_e is the speed of strong acid (ms^{-1}); ρ_s is the solid density (dimensionless); ρ_l is the liquid density (dimensionless); μ_l is the liquid viscosity (Pas.s); E is the ratio of lateral and normal stress (dimensionless).

So, the overall pressure is the sum of the liquid and solid pressure:

$$\frac{\partial P}{\partial z} = \frac{Q_e}{2\pi K z^2}(1-E)\frac{P_s}{z} + (\rho_s - \rho_l)w_f^2 z$$
$$- \frac{\mu_l Q_e}{K} + \mu_l w_f^2 z. \tag{22}$$

If we replace $\partial P(z)/\partial z$ by its expression, we obtain

$$\lim_{z \to h}\frac{\partial q_{AF}}{\partial t} = \frac{1}{h^4}\left(\frac{Q_e \cdot P_s \cdot S(1-E)}{2\pi \cdot K}\right.$$
$$+ \frac{Q_e \cdot P_s \cdot \mu \cdot \beta_2 (1-E)}{2\pi \cdot K \cdot \rho}\right)$$
$$+ \frac{1}{h^3}\left(\frac{-Q_e \cdot P_s \cdot \vartheta \cdot \beta_2 (1-E)}{2\pi \cdot K \cdot \rho}\right.$$
$$+ \frac{Q_e \cdot P_s \cdot \beta_2 \cdot \mu (1-E)}{2\pi \cdot K \cdot \rho} + \frac{2 \cdot \mu \cdot \beta_2}{\rho}\right)$$
$$+ \frac{1}{h^2}(\vartheta \cdot \beta_2 - \vartheta \cdot \beta_2 \cdot P - 2\frac{\mu \cdot P}{\rho})$$
$$+ \frac{1}{h}\left(\frac{-Q_e \cdot \mu_l \cdot \vartheta \cdot \beta_2}{K} - \frac{-Q_e \cdot \mu_l \cdot \mu \cdot \beta_2}{K \cdot \rho}\right)$$
$$+ w_f^2(\rho_s - \rho_l + \mu_l)\beta_2(\vartheta + \frac{\mu}{\rho})$$
$$+ w_f^2 \cdot h(\frac{\rho_s - \rho_l + \mu_l}{\rho})(\mu \cdot \beta_2 - S) + \frac{\mu \cdot Q_e \cdot S}{K \cdot \rho}, \tag{23}$$

finally, we get

$$\frac{dq_{AF}}{dt} = f_2(X(t), U(t)). \tag{24}$$

3.2. The (1D) nonlinear model

The nonlinear model obtained from (12) and (24) is as follows:

$$\frac{dh}{dt} = f_1(X(t), U(t)),$$
$$\frac{dq_{AF}}{dt} = f_2(X(t), U(t)). \tag{25}$$

For controlling such systems, it is quite necessary to take into account the nonlinear phenomena. The simulation of the nonlinear system gives the two following outputs:

- The height of the cake, h (cm).
- The volume flow rate of the strong acid, q_{AF} (m$^3 \cdot$ h^{-1}).

We note that the thickness of the cake increases over time until it reaches 6 cm. The volume flow of the filtrate increases to a value close to $74 \, \text{m}^3 \cdot \text{h}^{-1}$. These results are close to the real system.

4. Linearization of the (1D) model

In this section, we make linearization around an operating point $(\bar{X}, \bar{U})$.

We will linearize the terms: $1/h$, $1/h^2$, $1/h^3$ and $1/h^4$.

We make for h a linear approximation with the first-order Taylor's formula:

$$\frac{1}{h} \to \frac{h}{h_0^2}; \frac{1}{h^2} \to \frac{h}{h_0^3}; \frac{1}{h^3} \to \frac{h}{h_0^4}; \frac{1}{h^4} \to \frac{h}{h_0^5}$$

with h_0 is a small variation of the height of the cake. So, we obtain

$$\frac{\partial q_{AF}}{\partial t} = \left(\left(\left(\frac{Q_e \cdot P_s \cdot S(1-E)}{2\pi \cdot K.h_0^5} + \frac{Q_e \cdot P_s \cdot \mu \cdot \beta_2 (1-E)}{2\pi \cdot K \cdot \rho \cdot h_0^5}\right)\right.\right.$$
$$+ \left(\frac{-Q_e \cdot P_s \cdot \vartheta \cdot \beta_2 (1-E)}{2\pi \cdot K \cdot \rho \cdot h_0^4} + \frac{Q_e \cdot P_s \cdot \beta_2 \cdot \mu (1-E)}{2\pi \cdot K \cdot \rho \cdot h_0^4}\right.$$
$$+ \frac{2 \cdot \mu \cdot \beta_2}{\rho \cdot h_0^4}\right) + \left(\frac{\vartheta \cdot \beta_2}{h_0^3} - \frac{\vartheta \cdot \beta_2 \cdot P}{h_0^3} - 2\frac{\mu \cdot P}{\rho \cdot h_0^3}\right)$$
$$+ \left.\left(\frac{-Q_e \cdot \mu_l \cdot \vartheta \cdot \beta_2}{K.h_0^2} - \frac{-Q_e \cdot \mu_l \cdot \mu \cdot \beta_2}{K \cdot \rho h_0^2}\right)\right) h$$
$$+ (\rho_s - \rho_l + \mu_l)\beta_2 \left(\vartheta + \frac{\mu}{\rho}\right) w_f^2$$
$$+ \left(\frac{\rho_s - \rho_l + \mu_l}{\rho}\right)(\mu \cdot \beta_2 - S)w_f^2 \cdot h + \frac{\mu \cdot Q_e \cdot S}{K \cdot \rho} \tag{26}$$

We suppose that:

$$\alpha = \left(\left(\frac{Q_e \cdot P_s \cdot S(1-E)}{2\pi \cdot K.h_0^5} + \frac{Q_e \cdot P_s \cdot \mu \cdot \beta_2 (1-E)}{2\pi \cdot K \cdot \rho \cdot h_0^5}\right)\right.$$
$$+ \left(\frac{-Q_e \cdot P_s \cdot \vartheta \cdot \beta_2 (1-E)}{2\pi \cdot K \cdot \rho \cdot h_0^4} + \frac{Q_e \cdot P_s \cdot \beta_2 \cdot \mu (1-E)}{2\pi \cdot K \cdot \rho \cdot h_0^4}\right.$$
$$+ \frac{2 \cdot \mu \cdot \beta_2}{\rho \cdot h_0^4}\right) + \left(\frac{\vartheta \cdot \beta_2}{h_0^3} - \frac{\vartheta \cdot \beta_2 \cdot P}{h_0^3} - 2\frac{\mu \cdot P}{\rho \cdot h_0^3}\right)$$
$$+ \left(\frac{-Q_e \cdot \mu_l \cdot \vartheta \cdot \beta_2}{K.h_0^2} - \frac{-Q_e \cdot \mu_l \cdot \mu \cdot \beta_2}{K \cdot \rho h_0^2}\right)\right). \tag{27}$$

So the linear model of the rotary drum filter is given by the two following equations:

$$f_1(X(t), U(t)) = \frac{dh}{dt}$$
$$= \frac{2 \cdot \rho_g}{(R_1^2 - R_2^2)\theta}q_b - \frac{h}{\theta}w_f + \frac{2}{(R_1^2 - R_2^2)\theta} \cdot q_{AF} \tag{28}$$

$$f_2(X(t), U(t)) = \frac{dq_{AF}}{dt}$$

$$= \alpha \cdot h + (\rho_s - \rho_l + \mu_l)\beta_2\left(\vartheta + \frac{\mu}{\rho}\right)w_f^2$$

$$+ \left(\frac{\rho_s - \rho_l + \mu_l}{\rho}\right)$$

$$\times (\mu \cdot \beta_2 - S)w_f^2 \cdot h + \frac{\mu \cdot Q_e \cdot S}{K \cdot \rho}. \qquad (29)$$

The equations of the model contains the product of a state variable and a control variable. In order to linearize the model, we must use the Jacobian linearization. We will select an operating point at steady-state $P(\bar{X}, \bar{U})$ such as $f(\bar{X}(t), \bar{U}(t)) = 0$. Moreover, if we suppose that $\bar{U} = 0$, so the operating point is:

$$\left(\begin{bmatrix} \dfrac{\mu_l Q_e S}{K\rho\alpha} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

The model is as follows:

$$\begin{aligned} \dot{X} &= F_x X + B_u U \\ Y &= G_x X, \end{aligned} \qquad (30)$$

where F_x, B_u are the Jacobian matrices of the partial derivatives of $f(X, U)$ with respect to X and U, the state and the input variable of the system, respectively, which measured at point $P(\bar{X}, \bar{U})$. The two matrices F_x and B_u change according to the operating point.

At steady-state, we have

$$\bar{X} = \begin{pmatrix} \bar{h} + \delta h \\ \bar{q}_{AF} + \delta q_{AF} \end{pmatrix}, \quad \bar{U} = \begin{pmatrix} \bar{q}_b + \delta q_b \\ \bar{w}_f + \delta w_f \end{pmatrix}$$

$$F_x = \begin{pmatrix} \dfrac{\partial f_1}{\partial h} & \dfrac{\partial f_1}{\partial q_{AF}} \\ \dfrac{\partial f_2}{\partial h} & \dfrac{\partial f_2}{\partial q_{AF}} \end{pmatrix} = \begin{pmatrix} F_{x11} & F_{x12} \\ F_{x21} & F_{x22} \end{pmatrix}$$

$$B_u = \begin{pmatrix} \dfrac{\partial f_1}{\partial q_b} & \dfrac{\partial f_1}{\partial w_f} \\ \dfrac{\partial f_2}{\partial q_b} & \dfrac{\partial f_2}{\partial w_f} \end{pmatrix} = \begin{pmatrix} B_{u11} & B_{u12} \\ B_{u21} & B_{u22} \end{pmatrix}$$

$$G_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $F_{x11} = \partial f_1/\partial h = -\bar{w}_f/\theta$; $F_{x12} = \partial f_1/\partial q_{AF} = -2m_\rho/((R_1^2 - R_2^2) \cdot \theta)$; $F_{x21} = \partial f_2/\partial h = \alpha + ((\rho_s - \rho_l + \mu_l)/\rho)(\mu \cdot \beta_2 - S)\bar{w}_f^2$; $F_{x22} = \partial f_2/\partial q_{AF} = 0$; $B_{u11} = \partial f_1/\partial q_b = 2 \cdot \rho_g/((R_1^2 - R_2^2)\theta)$; $B_{u12} = \partial f_1/\partial w_f = -\bar{h}/\theta$; $B_{u21} = \partial f_2/\partial q_b = 0$; $B_{u22} = 2(\rho_s - \rho_l + \mu_l)\beta_2(\vartheta + \mu/\rho)\bar{w}_f + 2((\rho_s - \rho_l + \mu_l)/\rho)(\mu \cdot \beta_2 - S)\bar{w}_f \cdot h$
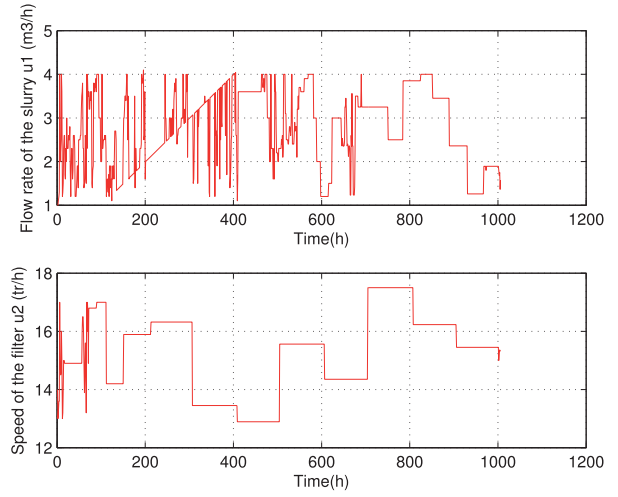

Figure 5. Evolution of the inputs of the filter.

So, the final Jacobian matrices of the state-space are

$$F_x = \begin{pmatrix} F_{x11} & F_{x12} \\ F_{x21} & F_{x22} \end{pmatrix}, \quad B_u = \begin{pmatrix} B_{u11} & B_{u12} \\ B_{u21} & B_{u22} \end{pmatrix}$$

$$G_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Validation of the model is the process of determining the degree to which a simulation model and its associated data are an accurate representation of the real world from the perspective of the intended uses of the model.

In order to investigate the filter and its changing states, our system was modelled and simulated by "MATLAB" which is among the most developed software for the simulation of production systems. We suppose that the rotary filter on the TCG factory is a multi-input multi-output nonlinear system (See Figure 5), which has as inputs $U1$ flow rate of the slurry and $U2$ speed of the filter, and outputs $Y1$ height of the cake and $Y2$ the volume of the filtrate.

Also we have

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}.$$

The Figure 6 has the output $Y_1 = Y_r$, which corresponds to the thickness of the cake as well as the output of the model Y_m are almost constant. The relative error between the output of the real system Y_r and the output of the model Y_m belongs to $\pm 20\%$ (see Figure 8)

The mean squared error is $MSE = 0.4\%$.

The Figure 7 describes the output $Y_2 = Y_r$, which corresponds to the flow rate of the filtrate and the output of the model Y_m are almost constant. The relative error between the output of the real system Y_r and the model Y_m belongs to $\pm 0,47\%$, see Figure 9. The mean squared error is $MSE = 0.313\%$. We note that there is a gap between the real output and the output of the model, this difference is due to the variation of the
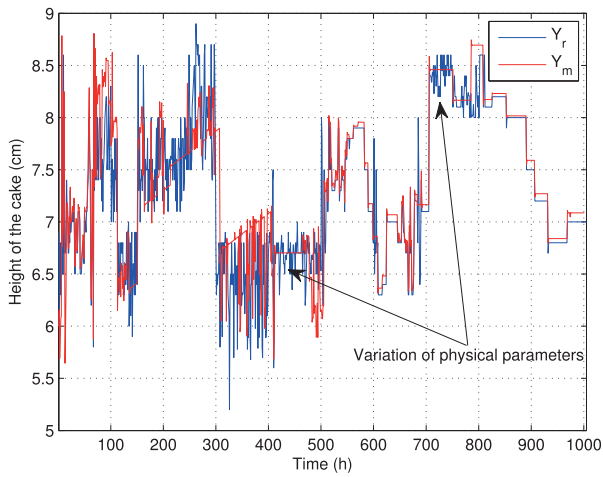
Figure 6. Calculated and estimated output of the height of the cake.
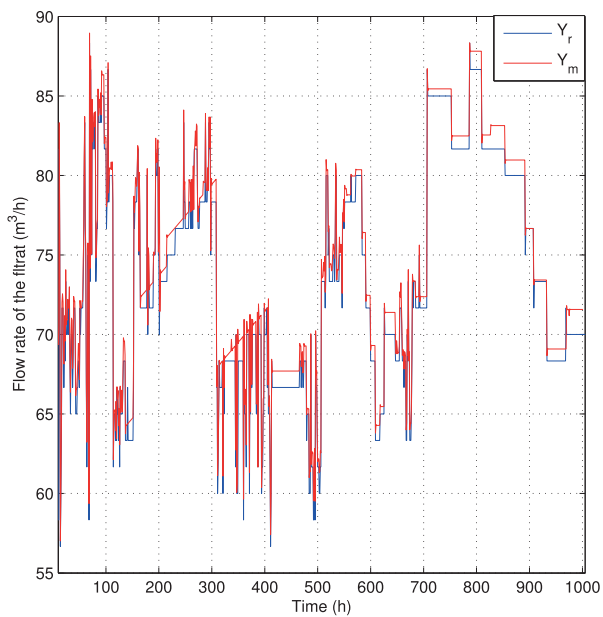
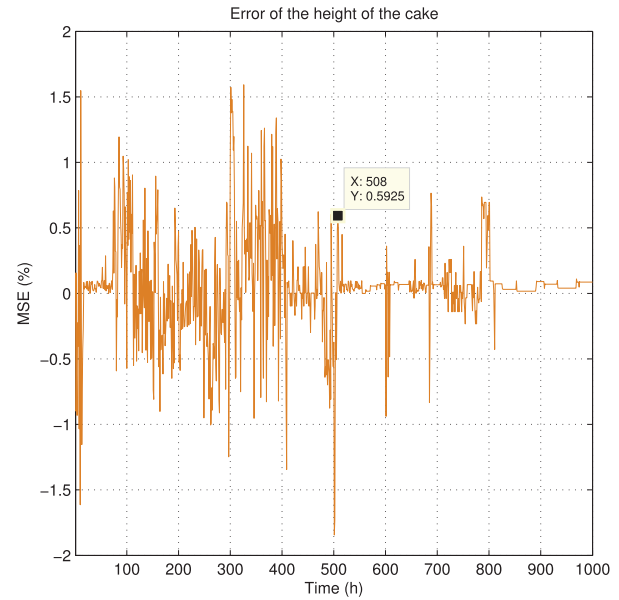**Figure 7.** Calculated and estimated output of the flow rate of the filtrate.

physical parameters of the system during the filtration phase.

## 5. 2D modelling of the rotary drum filter based on (FM-II) method

Consider the linear 2D system described by the following (FM-II) model:

$$X(d+1, k+1) = A_1 X(d, k+1) + A_2 X(h+1, k)$$
$$+ B_1 U(d, k+1) + B_2 U(d+1, k)$$
$$Y(d, k) = CX(d, k), \tag{31}$$

where $X(d, k) \in R^n$, $U(d, k) \in R^l$ and $Y(d, k) \in R^m$ are, respectively, the local state, input and output vectors, d is the time t and k is the angle $1/\theta$ and A_1, A_2, B_1, B_2 and C are real system matrices of suitable sizes. The system is also conventionally denoted by (A_1, A_2, B_1, B_2, C).

Figure 8. The relative error between the calculated and estimated values of the height of the cake.
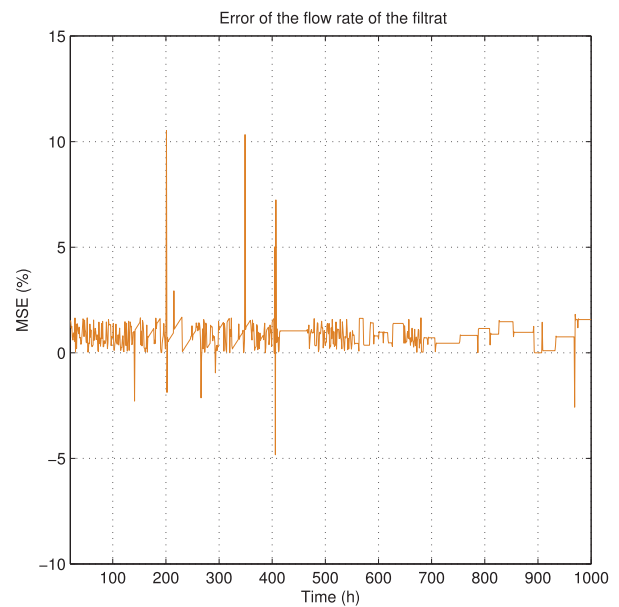
Figure 9. The relative error between the calculated and estimated values of the flow rate of the filtrate.

Definition 5.1: The (2D) linear system (31) is said to be asymptotically stable if:

$$\lim_{i \to \infty} \|x(i)\| = 0$$

under

$$\sup_k \|x(0)\| < \infty$$

where $\quad x(i) = \{X(d, k), d + k = i\} \quad$ and $\quad \|x(i)\| = \sup_{X \in x(i)} \|x\|$.

The following lemma gives a sufficient condition for the asymptotic stability of the (2D) system in term of an LMI.

Lemma 5.2 ([38]): *The (2D) system is asymptotically linear if there exist matrices $P \succ 0$ and $Q \succ 0$. Thus, we introduce the equation of "Bézout" in matrices $P(z_1, z_2)$ and $Q(z_1, z_2)$ such that:*

$$I_X = [Q(z_1, z_2)\, P(z_1, z_2)] \begin{bmatrix} I_X - A_1 z_1 - A_2 z_2 \\ C \end{bmatrix} \quad (32)$$

where

$$\prod = \begin{pmatrix} P - Q - 2R & R \\ R^{\mathrm{T}} & Q \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \quad \text{and}$$

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^{\mathrm{T}}.$$

Lemma 5.3 ([39]): *The (2D) system is asymptotically stable if there exist matrices $P \succ 0$, $Q \succ 0$, $R = R^{\mathrm{T}} \succeq 0$, $\prod \succ 0$, $F \in R^{2n \cdot n}$ and $G \in R^{n \cdot n}$ such that:*

$$\begin{pmatrix} -\prod + FA + A^{\mathrm{T}}F^{\mathrm{T}} & -F + A^{\mathrm{T}}G^{\mathrm{T}} \\ * & P - G - G^{\mathrm{T}} \end{pmatrix} \succ 0 \quad (33)$$

where

$$\prod = \begin{pmatrix} P - Q - 2R & R \\ R^{\mathrm{T}} & Q \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

and $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^{\mathrm{T}}$ The different stages of the (2D) modelling are:

(1) *The transfer matrix of the previous model is:*

$$\psi = \mathrm{diag}\{\psi_1, \ldots, \psi_l\}$$

where ψ_i is the column vector defined as:

$$\psi_i = \left[z_2^{k_i}, z_2^{k_i-1}, z_1 \cdots z_2 z_1^{k_i-1}, z_1^{k_i}, \ldots \right.$$
$$\left. z_2 z_1, z_1^2, z_2, z_1 \right]$$

and diag is a diagonal matrix.

(2) *Write $D(z_1, z_2)$ and $N(z_1, z_2)$ in the form of:*

$$D(z_1, z_2) = D_{HT} \cdot \psi, \quad (34)$$

$$N(z_1, z_2) = N_{HT} \cdot \psi, \quad (35)$$

where:

$$D_{HT} = \begin{pmatrix} D_{11} & \cdots & D_{1l} \\ \vdots & \ddots & \vdots \\ D_{l1} & \cdots & D_{ll} \end{pmatrix},$$

$$N_{HT} = \begin{pmatrix} N_{11} & \cdots & N_{1l} \\ \vdots & \ddots & \vdots \\ N_{m1} & \cdots & N_{ml} \end{pmatrix},$$

We have $F(z_1, z_2) = \begin{pmatrix} N(z_1, z_2) \\ D(z_1, z_2) \end{pmatrix}$ where D_{HT} and N_{HT} are real matrices with sizes conformable to ψ.

(3) *Construct the matrices $A_{i0}, B_i; i = 1, 2$ such that:*

$$\psi = (I - A_{10}z_1 - A_{20}z_2)^{-1}(B_1 z_1 + B_2 z_2) \quad (36)$$

(4) *The realization is finally obtained as:*

$$A_i = A_{i0} + B_i D_{HT}, \quad B_i, \ i = 1, 2 \quad \text{and}$$
$$C = N_{HT}.$$

By thoroughly investigating the structural properties of (FM-II) model, using the relations (34), (35) and (36), a realization can be obtained.

ψ has to satisfy the following conditions:

(1) Condition 1: ψ_i contains all the power products occurring in the polynomial entries of the i_{th} column of $F(z_1, z_2)$.
(2) Condition 2: ψ_i contains z_1 or z_2, or both z_1 and z_2.
(3) Condition 3: Let $n_i (i = 1, \ldots, l)$ be the dimension of ψ_i. For every entry $\psi_i(j)$ $(j \in 1, \ldots, n_i)$ except for the entry that is either z_1 or z_2, there exists another entry $\psi_i(d)$ $(d \in 1, \ldots, n_i)$ such that $\psi_i(j) = z_1 \psi_i(d)$ or $\psi_i(j) = z_2 \psi_i(d)$.

The above realization procedure produces a state-space description (A_1, A_2, B_1, B_2, C) which have the following properties:

- $[I - A_1 z_1 - A_2 z_2 B_1 z_1 + B_2 z_2]$ is full rank in C^2
- $\det(I - A_1 z_1 - A_2 z_2) = \det D(z_1, z_2)$

Starting from the initial ψ_i $(i = 1, \ldots, l)$, the following algorithm is given below for constructing the final ψ_i $(i = 1, \ldots, l)$ which has to satisfy the conditions of (2) and (3).

Once $\psi_1, \ldots, \psi_l$ are constructed by Algorithm 1, $\psi = \mathrm{diag}\{\psi \cdots \psi\}$, can be readily obtained. ψ is now of size $n \times l$, where $n = \sum_{i=1}^{l} n_i$. Note that n will be smaller than n' if some power products are absent in the polynomial entries of $F(z_1, z_2)$. This will be illustrated by a non-trivial example in the next section. Next, express $D(z_1, z_2)$ and $N(z_1, z_2)$ in the form $D(z_1, z_2) = D_{HT}\psi$, $N(z_1, z_2) = N_{HT}\psi$, with $D_{i_j} \in R^{1n_j}$ and $D_{i_j} \in R^{1n_j}$ are row vectors whose entries are the coefficients of the $(i, j)-$ indexed polynomial in $D(z_1, z_2)$ and $N(z_1, z_2)$, respectively. The system matrices A_1, A_2, B_1, B_2 and C can then be constructed by the Algorithm 2.

Calculation of the transfer matrix M

The state vector of the rotary drum filter has the following form:

$$F_x = \begin{pmatrix} F_{x11} & F_{x12} \\ F_{x21} & F_{x22} \end{pmatrix} \quad (37)$$

Algorithm 1 Calculation of the matrix: ψ

1: $i = 0$
2: $i = i + 1; j = 0$. If $i \succ l$, exit. Otherwise, proceed to Step 3.
3: $j = j + 1$, $r = 1$. Check whether there exists an entry $\psi(h)$ ($j < h \leq n_i$) such that condition (3) is satisfied. If yes, repeat Step 3. Otherwise, proceed to Step 4.
4: : If $j = n(i - 1)$, check whether $\psi(j) = z_2$
5: If $j = n(i)$, check whether $\psi(j) = z_1$ or $\psi(j) = z_2$. If the answer is yes, return to Step 2, and if the answer is no, go to Step 7. If $j \neq n_i$, proceed to Step 6.
6: $r = r + 1$. Check whether there exists an entry $\psi(k)$ for $j < k \leq n_i$ that satisfies either $\psi_i(j) = z_1^r \psi_i(k)$ or $\psi_i(j) = z_2^r \psi_i(k)$. If the answer is yes, insert either $z_1^{r-s} \psi_i(k)$ or $z_2^{r-s} \psi_i(k)$ at an appropriate position according to the descending order of power products in ψ for $s = 1, ..., r - 1$. Let $n_i = n_i + (r - 1)$ and return to Step 3. In the case that the answer is no, check whether $r \prec deg\{\psi(j)_2\}$. If yes, repeat Step 6. Otherwise, proceed to Step 7.
7: Check whether steps 1 and 2 and that $deg_{z_1}\{\psi(j)\} \geq deg_{z_2}\{\psi(j)\}$. If yes, insert $z_1^{-1}\psi_i(k)$, while if no, insert $z_2^{-1}\psi_i(k)$, into ψ_i according to the descending order of power products. $n_i = n_{i+1}$.
8: **return** to Step 3.

with:

$$F_{x11} = \frac{\partial f_1}{\partial h} = -\frac{\bar{w}_f}{\theta};$$

$$F_{x12} = \frac{\partial f_1}{\partial q_{AF}} = -\frac{2m_\rho}{(R_1^2 - R_2^2) \cdot \theta};$$

$$F_{x21} = \frac{\partial f_2}{\partial h} = \alpha + \left(\frac{\rho_s - \rho_l + \mu_l}{\rho}\right)(\mu \cdot \beta_2 - S)\bar{w}_f^2;$$

$$F_{x22} = \frac{\partial f_2}{\partial q_{AF}} = 0.$$

The command vector of the rotary drum filter has the following form:

$$B_u = \begin{pmatrix} B_{u11} & B_{u12} \\ B_{u21} & B_{u22} \end{pmatrix} \tag{38}$$

with:

$$B_{u11} = \frac{\partial f_1}{\partial q_b} = \frac{2 \cdot \rho_g}{(R_1^2 - R_2^2)\theta};$$

$$B_{u12} = \frac{\partial f_1}{\partial w_f} = -\frac{\bar{h}}{\theta};$$

Algorithm 2 Calculation of parameters: A_i, b_i, c_i

1: Introduce $n_i \times n_i$ matrices $A_{10}^{(i)}$ and $A_{20}^{(i)}$, $i = 1, ..., l$ which are determined in the following way. Set initially all entries of $A_{10}^{(i)}$ and $A_{20}^{(i)}$ to zero. For $k = 1, ..., n_i$, let only $A_{20}^{(i)}(k, d_k) = 1$ if there exists certain h_k ($k \prec d_k \leq n_i$) such that $\psi_i(k) = z_1\psi(d_k)$, and let only $A_{20}^{(i)}(k, m_k) = 1$ if condition (1) step 7 does not hold and there exists certain m_k ($k \prec m_k \leq n_i$) such that $\psi_i(k) = z_2\psi_i(m_k)$. It is obvious that $A_{10}^{(i)}$ and $A_{20}^{(i)}$ are upper triangular matrices and have at most only one entry equal to 1 in the same row of $A_{10}^{(i)}$ and $A_{20}^{(i)}$.
2: For $k = 1, 2$, construct column vector $B_k^{(i)} \in R^{ni}$, $i = 1, ..., l$, by setting initially all the entries of $B_k^{(i)}$ to zero. If there exists some h such that $\psi_i(h) = z_k$, change the $h - th$ entry of $B_k^{(i)}$ to 1.
3: Construct the $n \times n$ matrices A_{10}, A_{20} and the $n \times l$ matrices B_1, B_2 as follows.

$$A_{k0} = diag\{A_{10}^{(1)}, A_{10}^{(2)}, ..., A_{10}^{(l)}\}$$

$$B_k = diag\{B_{10}^{(1)}, B_{10}^{(2)}, ..., B_{10}^{(l)}\}$$

with $k = 1, 2$.
4: To check the stability conditions of the (2D) system, it is easy to see that

$$(I - A_{10}z_1 - A_{20}z_2)\psi = (B_1z_1 + B_2z_2)$$

and it can be shown, in the same way of [40], that

$$\psi D_R^{-1}(z_1, z_2) = (I - (A_{10} + B_1 D_{HT})z_1 - (A_{20} + B_2 D_{HT})z_2)^{-1}(B_1z_1 + B_2z_2)$$

$$B_{u21} = \frac{\partial f_2}{\partial q_b} = 0;$$

$$B_{u22} = 2(\rho_s - \rho_l + \mu_l)\beta_2\left(\vartheta + \frac{\mu}{\rho}\right)\bar{w}_f$$

$$+ 2\left(\frac{\rho_s - \rho_l + \mu_l}{\rho}\right)$$

$$\times (\mu \cdot \beta_2 - S)\bar{w}_f \cdot h.$$

$$det(p \cdot I_2 - F_x) = p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2) \cdot \theta} \cdot F_{x21}$$

Throughout this part, we denote by p the Laplace operator.

At first, we calculate the determinant of F_x:

$$p \cdot I_2 - F_x = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} - \begin{pmatrix} F_{x11} & F_{x12} \\ F_{x21} & F_{x22} \end{pmatrix}$$

$$det(p \cdot I_2 - F_x) = p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2) \cdot \theta} \cdot F_{x21}.$$

Using the previous determinant, we calculate the adjugate matrix of F_x to obtain the final transfer matrix $M(t, 1/\theta)$.

$$G_X(\text{com}(p \cdot I_2 - F_x)^T) \cdot B_u$$

$$= \begin{pmatrix} p \cdot B_{u11} & p \cdot B_{u12} \\ F_{x21} \cdot B_{u11} & F_{x21} \cdot B_{u12} + (p - F_{x11})B_{u22} \end{pmatrix}$$

$$M\left(p, \frac{1}{\theta}\right) = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \tag{39}$$

with

$$M_1\left(p, \frac{1}{\theta}\right) = \left(p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2) \cdot \theta} \cdot F_{x21}\right), \tag{40}$$

$$M_3\left(p, \frac{1}{\theta}\right) = \frac{F_{x21} \cdot B_{u11}}{p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2) \cdot \theta} \cdot F_{x21}}, \tag{41}$$

$$M_2\left(p, \frac{1}{\theta}\right) = \frac{p \cdot B_{u12}}{p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2) \cdot \theta} \cdot F_{x21}}, \tag{42}$$

$$M_4\left(p, \frac{1}{\theta}\right) = \frac{F_{x21} \cdot B_{u12} + (p - F_{x11})B_{u22}}{p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2) \cdot \theta} \cdot F_{x21}}. \tag{43}$$

We have $p = z_1$ and $1/\theta = z_2$

so,

$$M(z_1, z_2) = \begin{pmatrix} \dfrac{b_1 z_1 z_2}{z_1^2 + a_1 z_1 z_2 + a_{11} z_2} & \dfrac{b_2 z_1 z_2}{z_1^2 + a_2 z_1 z_2 + a_{22} z_2} \\ \dfrac{b_3 z_2}{z_1^2 + a_3 z_1 z_2 + a_{33} z_2} & \dfrac{b_4 + b_{44} z_2 + b_{444} z_1}{z_1^2 + a_4 z_1 z_2 + a_{44} z_2} \end{pmatrix}$$

with $b_1 = B_{u11}$; $b_2 = B_{u12}$; $b_3 = F_{x21} \cdot B_{u11}$; $b_4 = F_{x21} \cdot B_{u12}$; $b_{44} = F_{x11} \cdot B_{u22}$; $b_{444} = B_{u22}$; $a_1 = a_2 = a_3 = a_4 = \bar{w}_f/\theta$. $a_{11} = a_{22} = a_{33} = a_{44} = (2m_\rho/((R_1^2 - R_2^2) \cdot \theta))F_{x21}$.

From the transfer matrix, we can formulate the following matrices:

$$D_R(z_1, z_2)$$
$$= \begin{pmatrix} z_1^2 + a_{11} z_1 z_2 + a_{12} z_2 & 0 \\ 0 & z_1^2 + a_{11} z_1 z_2 + a_{12} z_2 \end{pmatrix}$$

$$N_R(z_1, z_2) = \begin{pmatrix} b_{11} z_1 z_2 & b_{12} z_1 z_2 \\ b_{21} z_2 & b0 + b_{22} z_2 + b_{222} z_1 \end{pmatrix}$$

$$F(z_1, z_2) = \begin{pmatrix} N(z_1, z_2) \\ D(z_1, z_2) \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} z_1 z_2 & b_{21} z_2 \\ b_{12} z_1 z_2 & b0 + b_{22} z_2 + b_{222} z_1 \\ z_1^2 + a_{11} z_1 z_2 + a_{12} z_2 & 0 \\ 0 & z_1^2 + a_{11} z_1 z_2 + a_{12} z_2 \end{pmatrix}$$

Thus the column degrees of columns 1 and 2 in $F(z_1, z_2)$ are $k_1 = 1$ and $k_2 = 2$, and the power products occurring in the polynomial entries of the columns with non zero coefficients are $\{z_2, z_1\}$ and $\{z_2, z_1, z_1^2, z_2 z_1\}$, respectively. The initial column vectors ψ_1 and ψ_2 satisfying condition (1) are: $\psi_1 = (z_1 z_2 \ z_2 \ z_1)^T$; $\psi_2 = (z_1 z_2 \ z_1^2 \ z_2 \ z_1)^T$

Applying now the 1, ψ_1 and ψ_2 should satisfy conditions (2) and (3).

So, for ψ_1 a new term z_1^2 has to be inserted into ψ_1, so, $\psi_1 = (z_1 z_2 \ z_1^2 \ z_2 \ z_1)^T$. This new ψ_1 now satisfies conditions (2) and (3); therefore, we have

$$\psi = \text{diag}\{\psi_1, \psi_2\}$$
$$= \begin{pmatrix} z_1 z_2 & z_1^2 & z_2 & z_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1 z_2 & z_1^2 & z_2 & z_1 \end{pmatrix}$$

For ψ constructed above, it is ready to calculate that

$$D(z_1, z_2) = D_{HT}\psi; \quad N(z_1, z_2) = N_{HT}\psi$$

where

$$D_{HT} = \begin{pmatrix} 0 & 0 & a_{11} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{22} & 1 & 0 & 0 \end{pmatrix}$$

and

$$N_{HT} = \begin{pmatrix} 0 & 0 & b_{11} & 0 & 0 & 0 & b_{12} & 0 \\ 0 & 0 & b_{21} & 0 & 0 & 0 & b_{22} & b_{222} \end{pmatrix}$$

Since only $\psi_1(1) = z_1\psi_1(3)$ holds for ψ_1, its follows from Algorithm 2 that

$$A_{10}^{(1)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{20}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Calculation of A_i and B_i

On the other hand, for $\psi_2(1) = z_2\psi_2(4)$, Algorithm 2 produces

$$A_{10}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{20}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is easy to obtain

$$B_1^{(1)} = (0 \ 0 \ 1 \ 0)^T; \quad B_2^{(1)} = (0 \ 0 \ 0 \ 1)^T;$$

$$B_1^{(2)} = (0 \ 0 \ 1 \ 0)^T; \quad B_2^{(21)} = (0 \ 0 \ 0 \ 1)^T.$$

Since $\psi_1(3) = z_2, \psi_1(4) = z_1, \psi_2(3) = z_2, \psi_2(4) = z_1$

$$A_{10} = \{A_{10}^{(1)}, A_{10}^{(2)}\}$$

$$A_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{20} = 0$$

$$B_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}},$$

$$A_1 = A_{10} + B_1 D_{HT}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{22} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = A_{20} + B_2 D_{HT}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{22} & 1 & 0 & 0 \end{pmatrix}$$

$$C = N_{HT} = \begin{pmatrix} 0 & 0 & b_{11} & 0 & & 0 & 0 & b_{12} & 0 \\ 0 & 0 & b_{21} & 0 & & 0 & 0 & b_{22} & b_{222} \end{pmatrix}^{\mathrm{T}}$$

6. RLS identification: study case (1D) and (2D)

6.1. RLS for the (1D) system

RLS algorithms for parameter estimation are usually described in relation to the batch-type least squares algorithms, minimizing a quadratic error criterion. This error is the difference between the estimated and the observed values of the process output [41]. Consider the following linear process with slowly time-varying parameters, described by the following difference equation:

$$\sum_{i=0}^{N} a_i y_{k-i} = \sum_{j=0}^{M} b_i x_{k-j}, a_0 = 1. \tag{44}$$

The signal y_k is the process output and x_k is the input. For a notation most practical a parameter vector is defined: $\Theta^t = (b_0, b_1, \ldots, b_M, -a_1, -a_2, \ldots, -a_N)$ and a signal vector: $u^t = (x_k, \ldots, x_{k-M}, y_{k-1}, \ldots, y_{k-N})$ $u_k, \Theta \in R^p, p = M + N + 1$ (44) can be rewritten as follow:

$$y_k = u_k^t \Theta. \tag{45}$$

The parameter vector of the filter is estimated by an m-vector $\hat{\Theta}$. An error criterion is established:

$$J(\hat{\Theta}, k) = \eta^k(\hat{\Theta}_k - \Theta_0) + \sum_{i=1}^{k} \eta^{k-i}(y_i - u_i^t \hat{\Theta}_k)^2, \tag{46}$$

where η is an exponential forgetting factor $\eta \in]0, 1]$ and Θ_0 contains initial values of $\hat{\Theta}_0$. $Q \in R^{m \times m}$ is a positive definite matrix. Note that usually in the normal RLS algorithms the first term on the right-hand side of equation (51) is omitted By setting $\partial J / \partial \hat{\Theta} = 0$ it is readily calculated that

$$\hat{\Theta}_k = \left(\eta^k Q^{-1} + \sum_{i=1}^{k} \eta^{k-i} u_i u_i^t \right)^{-1}$$

$$\times \left(\eta^k Q^{-1} \Theta_0 + \sum_{i=1}^{k} \eta^{k-i} u_i y_i \right). \tag{47}$$
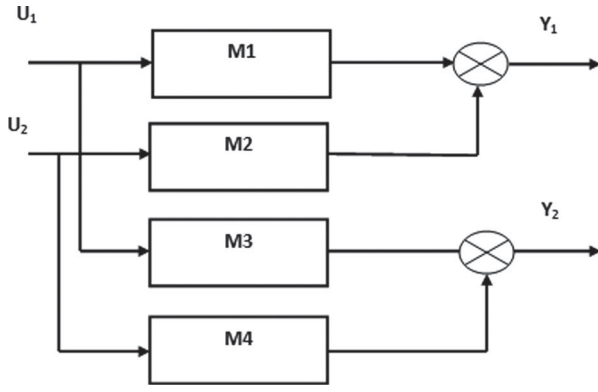
In this equation, a matrix P_k is introduced where

$$P_k^{-1} = \eta^k Q^{-1} + \sum_{i=1}^{k} \eta^{k-i} u_i u_i^t$$

$$= \eta \left(\eta^k Q^{-1} + \sum_{i=1}^{k-1} \eta^{k-1-i} u_i u_i^t \right) + u_k u_k^t$$

$$= \eta P_{k-1}^{-1} u_k u_k^t \tag{48}$$

which is a recursive algorithm; $P_0 = Q$

$$P_k = \frac{1}{\eta} \left(P_{k-1} - \frac{P_{k-1} u_k u_k^t P_{k-1}}{\eta + u_k^t P_k u_k} \right) \tag{49}$$

$P_0 = Q$ using P_k defined in (48), (50) can be written as follows:

$$\hat{\Theta}_k = P_k \left(\eta^k Q^{-1} \Theta_0 + \sum_{i=1}^{k} \eta^{k-i} u_i y_i \right)$$

$$= \hat{\Theta}_{k-1} + (\eta P_k - P_{k-1})$$

$$\times \left(\eta^k Q^{-1} \Theta_0 \sum_{i=1}^{k-1} \eta^{k-1-i} u_i y_i \right)$$

$$+ P_k u_k y_k \tag{50}$$

Figure 10. Transfer matrix of the model.

with $P_0 = Q$, it is seen that $\hat{\Theta}_0 = \Theta_0$. Substituting (50) yields

$$\hat{\Theta}_k = \hat{\Theta}_{k-1} - \frac{P_{k-1} u_k (u_k^t \hat{\Theta}_{k-1} - y_k)}{\eta + u_k^t P_{k-1} u_k} \quad (51)$$

$$.\hat{\Theta}_0 = \Theta_0$$

6.1.1. Simulation results for the (1D) linear system

At the end of the modelling, we obtained a transfer matrix consisting of four transfer functions M_1, M_2, M_3 and M_4. These transfer functions are arranged as in Figure 10.

From the filter-state representation, we got a transfer matrix which is as follows:

$$M(p) = \begin{pmatrix} M_1(p) & M_2(p) \\ M_3(p) & M_4(p) \end{pmatrix}. \quad (52)$$

with

$$M_1(p) = \left(\frac{p \cdot B_{u11}}{p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2)\cdot\theta} \cdot F_{x21}} \right) \quad (53)$$

$$M_2(p) = \left(\frac{p \cdot B_{u12}}{p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2)\cdot\theta} \cdot F_{x21}} \right) \quad (54)$$

$$M_3(p) = \left(\frac{F_{x21} \cdot B_{u11}}{p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2)\cdot\theta} \cdot F_{x21}} \right) \quad (55)$$

$$M_4(p) = \left(\frac{F_{x21} \cdot B_{u12} + (p - F_{x11})B_{u22}}{p^2 + \frac{\bar{w}_f}{\theta} \cdot p + \frac{2m_\rho}{(R_1^2 - R_2^2)\cdot\theta} \cdot F_{x21}} \right). \quad (56)$$

The performance of the *RLS* method was tested by computer simulation. Now, we want to look for the estimation vectors $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\theta}_4$ which correspond respectively to the transfer functions $M_1(p)$, $M_2(p)$,

Table 3. Estimation vector of $\hat{\theta}_1$.

$\hat{\theta}_1$	Calculated parameters	Estimated parameters
$\hat{a}_1$	−0.2501	−0.2331
$\hat{a}_{11}$	−0.4500	−0.4344
$\hat{b}_1$	−0.0133	−0.0034

Table 4. Estimation vector of $\hat{\theta}_2$.

$\hat{\theta}_2$	Calculated parameters	Estimated parameters
$\hat{a}_2$	−0.2501	−0.2090
$\hat{a}_{22}$	−0.4500	−0.3899
$\hat{b}_2$	−0.1535	−0.0015

Table 5. Estimation vector of $\hat{\theta}_3$.

$\hat{\theta}_2$	Calculated parameters	Estimated parameters
$\hat{a}_3$	−0.2501	−0.1771
$\hat{a}_{33}$	−0.4500	−0.7584
$\hat{b}_3$	−0.1155	−0.0187

Table 6. Estimation vector of $\hat{\theta}_4$.

$\hat{\theta}_2$	Calculated parameters	Estimated parameters
$\hat{a}_4$	−0.2501	−0.1425
$\hat{a}_{44}$	−0.4500	−0.7754
$\hat{b}_4$	−1.6004	−2.2350
$\hat{b}_{44}$	−0.1955	−0.0226

$M_3(p)$ and $M_4(p)$. The calculated parameters are found directly from the physical parameters of the rotary filter with a tolerance of $\pm10\%$. In the following, we will apply the RLS estimation method on the rotary drum filter. So we obtain four estimation vectors, which are cited in Tables 3, 4, 5 and 6:

Using these tables, we notice that $\hat{\theta}$ and θ are close, which validates the performance of the (1D) RLS identification method.

6.2. RLS for the (2D) system

This section is concerned with a basic spatial RLS algorithm for a second Fornasini–Marchesini (2D) model. Either, $x(d, k)$ is the input and $y(d, k)$ is the output of the system.

$$y(d, k) = - \sum_{(i,j) \in L^y | i| + |j| \neq 0} a_{i,j} y(d - i, k - j)$$

$$- \sum_{(i,j) \in L^x} b_{i,j} x(d - i, k - j). \quad (57)$$

L^y and L^x denote the support region for the ARX model. Given an input (2D) signal $x(d, k)$ and a desired response $\hat{x}(d, k)$, $(h, k) \in [0, 0].[N_1, N_2]$, the optimal least square (2D) filter is obtained by minimizing the

cost function.

$$z(d,k) = \sum_{t_1=0}^{h-1} \sum_{t_2=0}^{N_2} \eta^{(h-t_1)N_2+k-t_2} e^2(t_1,t_2)$$

$$+ \sum_{t_2=0}^{k} \eta^{(k-t_2)} e^2(d,t_2), \qquad (58)$$

where $e(t_1,t_2) = \hat{x}(t_1,t_2,k) - y(t_1,t_2)$ is the instantaneous error and $\eta \in [0,1]$ is the so called forgetting factor. In this section, (2D) support region of general shapes are considered for both L^y and L^x. More precisely, L consists of a union of intervals

$$L = \bigcup_{i=k_1}^{t_1} l(i).$$

Using the previous notation, equation (49) takes the form

$$l(i) = \{(i,j) : k_2(i) \le j \le s_2(i)\}$$

$$s_1 = \max\{i : (i,j) \in L\}$$

$$k_1 = \max\{j : (i,j) \in L\}$$

$$s_2 = \max\{j : (i,j) \in l(i)\}$$

$$k_2 = \max\{j : (i,j) \in l(i)\}$$

$$y(d,k) = - \sum_{i=k_1^y}^{s_1^y} \sum_{i=k_2^y(i)|i|+|j|\neq 0}^{s_2^y(i)} a_{i,j} y(d-i,k-j)$$

$$- \sum_{i=k_1^x}^{s_1^x} \sum_{i=k_2^x(i)|i|+|j|\neq 0}^{s_2^x(i)} b_{i,j} x(d-i,k-j). \quad (59)$$

Let us define the row-wise data vectors and the system parameters that are attached to the support regions L^y and L^x. We use the following notation $k_2^y = 1$

(1) Output data corresponding to L^y:

$$\forall i \in [k_1^y, s_1^y]$$

$$y_{l^y(i)}(d,k) = \left[y(d-i,k-j)\right]_{\substack{j=k_2^y(i)\cdots s_2^y(i) \\ k_2^y(0)=1}}$$

$$a_{l^y(i)}(i) = \left[a_{i,j}\right]_{\substack{j=k_2^y(i)\cdots s_2^y(i) \\ k_2^y(0)=1}}$$

(2) Input data corresponding to L^x:

$$\forall i \in [k_1^x, s_1^x]$$

$$x_{l^x(i)}(d,k) = \left[x(d-i,k-j)\right]_{\substack{j=k_2^x(i)\cdots s_2^x(i) \\ k_2^x(0)=1}}$$

$$b_{l^x(i)}(i) = \left[b_{i,j}\right]_{\substack{j=k_2^x(i)\cdots s_2^x(i) \\ k_2^x(0)=1}}$$

Then (59) can be expressed as a linear regression

$$y(d,k) = -\Phi^t(h,k)\Theta$$

The regressor vector $\Phi(h,k)$ is defined in terms of input–output data

$$\Phi^t(d,k) = \begin{bmatrix} Y_{L^y}(d,k) \\ X_{L^x}(d,k) \end{bmatrix}$$

where

$$Y_{L^y}(d,k) = \left[y_{l^y}(i)\right]_{i=k_1^y\cdots s_1^y}$$

$$X_{L^x}(d,k) = \left[x_{l^x}(i)\right]_{i=k_1^x\cdots s_1^x}$$

The parameter vector Θ carries the system coefficients

$$\Theta = \begin{bmatrix} A_{L^y} \\ B_{L^x} \end{bmatrix}$$

where $A_{L^y} = [a_{l^y}(i)]$ and $BA_{L^x} = [b_{l^x}(i)]$

The least square algorithm is as follows:

2D parameter update

$$\Theta(d,k+1) = \Theta(d,k) + w(d,k+1)\varepsilon(d,k+1),$$

$$\varepsilon(d,k+1) = e(d,k+1)/\beta(d,k+1),$$

$$e(d,k+1) = \hat{x}(d,k)\Phi^t(d,k)\Theta(d,k),$$

$$\beta(d,k+1) = 1 - \Phi(d,k+1)w(d,k+1).$$

2D Kalman gain update

$$w(d,k+1) = -\eta^{-1}P(d,k)\Phi(d,k+1),$$

$$P(d,k+1) = \eta^{-1}P(d,k) + \frac{w(d,k+1)w^t(d,k+1)}{\beta(d,k+1)}.$$

6.2.1. Simulation results for the (2D) linear system

We consider that $\hat{\theta}_1^{(2D)}$, $\hat{\theta}_2^{(2D)}$, $\hat{\theta}_3^{(2D)}$, $\hat{\theta}_4^{(2D)}$ are the estimated vectors, respectively, of the transfer functions M_1, M_2, M_3 and M_4. We apply the (2D) identification algorithm in order to obtain the following tables:

From the simulation results presented in Tables 7, 8, 9 and 10, we note that the estimated parameters tend to the calculated parameters with a tolerance of $\pm 10\%$. The parameters in (2D) model are closer to calculated values than those given by a (1D) model. Since the convergence of the parameters is done, we note that the (2D) (RLS) identification method is more efficient than the (1D) identification. For example for the (1D) model we have $a_1 = -0.2331$ whether for the (2D) model $a_1^{(2D)} = -0.2530$. So, we note that $a_1^{(2D)}$ is closer to the calculated value equal to -0.2501. So the convergence of parameters is performed and the (RLS) identification method is so effective in the (2D) case.

Table 7. Estimation vector of $\hat{\theta}_1^{(2D)}$.

$\hat{\theta}_1^{(2D)}$	Calculated parameters	Estimated parameters
$\hat{a}_1^{(2D)}$	-0.2501	-0.2530
$\hat{a}_{11}^{(2D)}$	-0.4500	-0.4344
$\hat{b}_1^{(2D)}$	-0.0133	-0.0134

Table 8. Estimation vector of $\hat{\theta}_2^{(2D)}$.

$\hat{\theta}_2^{(2D)}$	Calculated parameters	Estimated parameters
$\hat{a}_2^{(2D)}$	−0.2501	−0.2805
$\hat{a}_{22}^{(2D)}$	−0.4500	−0.4515
$\hat{b}_2^{(2D)}$	−0.1535	−0.1553

Table 9. Estimation vector of $\hat{\theta}_3^{(2D)}$.

$\hat{\theta}_3^{(2D)}$	Calculated parameters	Estimated parameters
$\hat{a}_3^{(2D)}$	−0.2501	−0.1574
$\hat{a}_{33}^{(2D)}$	−0.4500	−0.4504
$\hat{b}_3^{(2D)}$	−0.1155	−0.1187

Table 10. Estimation vector of $\hat{\theta}_4^{(2D)}$.

$\hat{\theta}_4^{(2D)}$	Calculated parameters	Estimated parameters
$\hat{a}_4^{(2D)}$	−0.2501	−0.2019
$\hat{a}_{44}^{(2D)}$	−0.4500	−0.4561
$\hat{b}_4^{(2D)}$	−1.6004	−1.6382
$\hat{b}_{44}^{(2D)}$	−0.1955	−0.1975

7. Conclusion

In this paper, (2D) model was proposed to describe the dynamic behaviour of a rotary filter, which is dedicated to the aspiration of various types of acid: strong, medium and weak.

Therefore, its study requires a deep research in order to know all the parameters that have an influence on the normal operation of the filter. Using the parameters that characterize the "AOUSTIN-UCEGO" filter, we get to find a two dimensions model for the filter. A constructive state-space realization procedure has been used for (2D) systems which may produce (FM-II) local state-space model. An *RLS* identification is made in both cases (1D) and (2D) and from which it is noticed that the results of simulation are near to the real values. The RLS identification for the (2D) system is more efficient since the parameters in this case depend on the temporal and spatial variables.

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Disclosure statement

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