CHARACTERIZATIONS OF ∗-LIE DERIVABLE MAPPINGS ON PRIME ∗-RINGS

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Abstract. Let $R$ be a ∗-ring containing a nontrivial self-adjoint idempotent. In this paper it is shown that under some mild conditions on $R$, if a mapping $d: R \to R$ satisfies
$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)]$$
for all $U, V \in R$, then there exists $Z_{U, V} \in Z(R)$ (depending on $U$ and $V$), where $Z(R)$ is the center of $R$, such that $d(U + V) = d(U) + d(V) + Z_{U, V}$. Moreover, if $R$ is a 2-torsion free prime ∗-ring additionally, then $d = \psi + \xi$, where $\psi$ is an additive ∗-derivation of $R$ into its central closure $T$ and $\xi$ is a mapping from $R$ into its extended centroid $C$ such that $\xi(U + V) = \xi(U) + \xi(V) + Z_{U, V}$ and $\xi([U, V]) = 0$ for all $U, V \in R$. Finally, the above ring theoretic results have been applied to some special classes of algebras such as nest algebras and von Neumann algebras.

1. Introduction

Throughout this paper $R$ will denote an associative ring with the center $Z(R)$. Recall that a ring $R$ is said to be $n$-torsion free, where $n > 1$ is an integer, if $nU = 0$ implies $U = 0$ for all $U \in R$. A ring $R$ is said to be prime if for any $U, V \in R$, $URV = \{0\}$ implies $U = 0$ or $V = 0$. An additive mapping $x \mapsto x^*$ on a ring $R$ is called involution in case $(UV)^* = V^*U^*$ and $(U^*)^* = U$ hold for all $U, V \in R$. A ring equipped with an involution is called a ring with involution or ∗-ring (see [7]). An additive mapping $d: R \to R$ is said to be a derivation on $R$ if $d(UV) = d(U)V + Ud(V)$ for all $U, V \in R$. In particular, derivation $d$ is called an inner derivation if there exists some $X \in R$ such that $d(U) = UX - XU$ for all $U \in R$. An additive mapping $d: R \to R$ is called a Lie derivation if $d([U, V]) = [d(U), V] + [U, d(V)]$ holds for all $U, V \in R$, where $[U, V] = UV - VU$ is the usual Lie product. If the condition of additivity is dropped from the above definition, then the corresponding Lie derivation is

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called a Lie derivable map. Obviously, every derivation is a Lie derivation. However, the converse statements are not true in general.

Let $R$ be a $\ast$-ring. An additive mapping $d : R \to R$ is said to be an additive $\ast$-derivation on $R$ if $d(UV) = d(U)V + Ud(V)$ and $d(U\ast) = d(U)\ast$ for all $U, V \in R$. More generally, a mapping $d : R \to R$ is said to be an additive $\ast$-Lie derivable mapping if $d([U\ast,V]) = [d(U)\ast,V] + [U\ast,d(V)]$. Indeed, if $d(U\ast) = d(U)\ast$ for all $U \in R$, then $d$ is a Lie derivable mapping if and only if $d$ is a $\ast$-Lie derivable mapping. An additive $\ast$-Lie derivable mapping is said to be a $\ast$-Lie derivation. It is not difficult to observe that any $\ast$-derivation is a $\ast$-Lie derivation but the converse is not true in general.

There has been a great interest in the study of characterizations of Lie derivations and $\ast$-Lie derivations for many years. The first quite surprising result is due to Martindale III who proved that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive (see [14]). Miers [16] initially established that every Lie derivation $d$ on a von Neumann algebra $A$ can be uniquely written as the sum $d = \psi + \xi$ where $\psi$ is an inner derivation of $A$ and $\xi$ is a linear mapping from $A$ into its center $Z(A)$ vanishing on each commutator. Yu and Zhang [18] proved that every Lie derivable mapping of a triangular algebra is the sum of an additive derivation and a mapping from triangular algebra into its center sending commutators to zero. Mathieu and Villena [15] gave the characterizations of Lie derivations on $C^\ast$-algebras. W. Jing and F. Lu [8] showed that every Lie derivable mapping on a 2-torsion free prime ring $R$ can be expressed as $d = \psi + \xi$, where $\psi : R \to T$ is an additive derivation and $\xi : R \to C$ is nearly additive i.e. $\xi(U + V) = \xi(U) + \xi(V) + Z_{U,V}$ where $Z_{U,V} \in Z(R)$ (depending on $U$ and $V$ in $R$) and vanishes on each commutator. Yu and Zhang [19] proved that every $\ast$-Lie derivable mapping from a factor von Neumann algebra into itself is an additive $\ast$-derivation. Also, Li, Chen and Wang [9] obtained the same result for $\ast$-Lie derivable mappings on von Neumann algebras and proved that every $\ast$-Lie derivable mapping on a von Neumann algebra with no central abelian projections can be expressed as the sum of an additive $\ast$-derivation and a mapping with image in the centre vanishing on commutators. In addition, the characterization of Lie derivations and $\ast$-Lie derivations on various algebras are considered in [1], [2], [5], [4],[6], [8], [12],[13], [17], [20].

Motivated by the results due to W. Jing & F. Lu [8] and C. Li et al. [9], in Section 2, we investigate the additivity of $\ast$-Lie derivable mappings on $\ast$-rings and show that every $\ast$-Lie derivable mapping on $R$ is almost additive in the sense that for any $U, V \in R$ there exists $Z_{U,V} \in Z(R)$ (depending on $U$ and $V$) such that $d(U + V) = d(U) + d(V) + Z_{U,V}$. In Section 3, we study the characterization of $\ast$-Lie derivable mappings on prime $\ast$-rings. Under some mild conditions on $R$, we prove that, if $d$ is an additive Lie derivable mapping on $R$, then $d = \psi + \xi$, where $\psi$ is an additive $\ast$-derivation of $R$ into its central
If a mapping $d$ is idempotent, a proof of the above theorem is given in a series of the following Lemmas.

For all $V$, let $A \subseteq R$ be a $R$-algebra such as nest algebras and von Neumann algebras. The above ring theoretic results have been applied to some special class of algebras such as nest algebras and von Neumann algebras.

2. ADDITIVITY OF $*$-LIE DERIVABLE MAPPINGS ON $*$-RINGS

In this section, we examine the additivity of $*$-Lie derivable mappings on rings. Let $\mathcal{R}$ be a $*$-ring with a nontrivial self-adjoint idempotent $P$. We write $Q = I - P$. It is to be noted that $\mathcal{R}$ may be without identity element. It is obvious that $PQ = QP = 0$. By the Peirce decomposition of $\mathcal{R}$, we have $\mathcal{R} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$, where $\mathcal{A}_{11} = P\mathcal{R}P$, $\mathcal{A}_{12} = \mathcal{P}\mathcal{R}Q$, $\mathcal{A}_{21} = Q\mathcal{R}P$ and $\mathcal{A}_{22} = Q\mathcal{R}Q$. Throughout this paper, $U_{ij}$ will denote an arbitrary element of $\mathcal{A}_{ij}$ and any element $U \in \mathcal{R}$ can be expressed as $U = U_{11} + U_{12} + U_{21} + U_{22}$.

The main result of this section starts as follows.

**Theorem 2.1.** Let $\mathcal{R}$ be a $*$-ring containing a nontrivial self-adjoint idempotent $P$ and satisfying the following conditions:

1. If $U_{ii}V_{ij} = V_{ij}U_{jj}$ for all $V_{ij} \in \mathcal{A}_{ij}$ and $1 \leq i \neq j \leq 2$, then $U_{ii} + U_{jj} \in Z(\mathcal{R})$.
2. If $U_{ij}V_{jk} = 0$ for all $V_{jk} \in \mathcal{A}_{jk}$ and $1 \leq i, j, k \leq 2$, then $U_{ij} = 0$.

If a mapping $d : \mathcal{R} \to \mathcal{R}$ satisfies

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)],$$

for all $U, V \in \mathcal{R}$, then there exists $Z_{U, V} \in Z(\mathcal{R})$ such that $d(U + V) = d(U) + d(V) + Z_{U, V}$.

Throughout assume that $\mathcal{R}$ satisfies the hypothesis of Theorem 2.1. The proof of the above theorem is given in a series of the following Lemmas.

**Lemma 2.2.** $d(0) = 0$.

**Proof.** $d(0) = d([0^*, 0]) = [d(0)^*, 0] + [0^*, d(0)] = 0$. \[\square\]

**Lemma 2.3.** For any $U_{ii} \in \mathcal{A}_{ii}$, $V_{ij} \in \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$, there exists $Z_{U_{ii}, V_{ij}} \in Z(\mathcal{R})$ such that

(i) $d(U_{ii} + V_{ij}) = d(U_{ii}) + d(V_{ij}) + Z_{U_{ii}, V_{ij}}$,

(ii) $d(U_{ii} + V_{ij}) = d(U_{ii}) + d(V_{ij}) + Z_{U_{ii}, V_{ij}}$.

**Proof.** (i) Let $A = d(U_{ii} + V_{ij}) - d(U_{ii}) - d(V_{ij})$. For any $U_{ii} \in \mathcal{A}_{ii}$, $V_{ij} \in \mathcal{A}_{ij}$, we have

$$d(V_{ij}) = d([P^*, U_{ii} + V_{ij}])$$

$$= [d(P)^*, U_{ii} + V_{ij}] + [P^*, d(U_{ii} + V_{ij})].$$
On the other hand by Lemma 2.2, we have
\[
d(V_{ij}) = d([P^*, U_{ii}]) + d([P^*, V_{ij}]) = [d(P)^*, U_{ii} + V_{ij}] + [P^*, d(U_{ii}) + d(V_{ij})].
\]
Comparing the above two identities, we get \([P, A] = 0\). Hence \(A_{ij} = A_{ji} = 0\).

For any \(W_{ji} \in \mathfrak{A}_{ii}\), we compute
\[
d(-U_{ii}W_{ji}) = d([W_{ji}^*, U_{ii} + V_{ij}]) = [d(W_{ji})^*, U_{ii} + V_{ij}] + [W_{ji}^*, d(U_{ii}) + d(V_{ij})].
\]
Using Lemma 2.2, \(d(-U_{ii}W_{ji}^*)\) can also be expressed as
\[
d(-U_{ii}W_{ji}^*) = d([W_{ji}^*, U_{ii}]) + d([W_{ji}^*, V_{ij}]) = [d(W_{ji})^*, U_{ii} + V_{ij}] + [W_{ji}^*, d(U_{ii}) + d(V_{ij})].
\]
From the above two equations it follows that \([W_{ji}^*, A] = 0\). In other words \(W_{ji}^*A = AW_{ji}^*\) for all \(W_{ji} \in \mathfrak{A}_{ii}\). By the condition \((G_1)\), we see that \(A_{ii} + A_{jj} \in Z(\mathfrak{R})\). Hence \(d(U_{ii} + V_{ij}) = d(U_{ii}) + d(V_{ij}) + Z_{U_{ii},V_{ij}}\) for some \(Z_{U_{ii},V_{ij}} \in Z(\mathfrak{R})\). Similarly, one can get \((ii)\).

**Lemma 2.4.** For any \(U_{ij}, V_{ij} \in \mathfrak{A}_{ii}, 1 \leq i \neq j \leq 2\), we have
\[
d(U_{ij} + V_{ij}) = d(U_{ij}) + d(V_{ij}).
\]

**Proof.** By Lemma 2.3, we see that
\[
d(U_{ij} + V_{ij}) = d([U_{ij}^* + P]^*, V_{ij} + Q]) = [d(U_{ij}^* + P)^*, V_{ij} + Q] + [d(U_{ij}^* + P)^*, d(V_{ij} + Q)] = [d(U_{ij}^* + p)^*, V_{ij} + Q] + [d(U_{ij}^* + P), d(V_{ij}) + d(Q)] = [d(U_{ij})^*, V_{ij}] + [d(U_{ij})^*, V_{ij}] + [d(P)^*, V_{ij}] + [d(P)^*, Q] + [U_{ij}, d(V_{ij})] + [V_{ij}, d(Q)] + [P, d(V_{ij})] + [P, d(Q)] = d([U_{ij}^*]^*, V_{ij}) + d([U_{ij}^*]^*, Q) + d([P^*, V_{ij}]) + d([P^*, Q]) = d(U_{ij}) + d(V_{ij}) .
\]

**Lemma 2.5.** For any \(U_{ii}, V_{ii} \in \mathfrak{A}_{ii}, i = 1, 2\), there exists \(Z_{U_{ii},V_{ii}} \in Z(\mathfrak{R})\) such that
\[
d(U_{ii} + V_{ii}) = d(U_{ii}) + d(V_{ii}) + Z_{U_{ii},V_{ii}}.
\]

**Proof.** Let \(A = d(U_{11} + V_{11}) - d(U_{11}) - d(V_{11})\). For any \(U_{11}, V_{11} \in \mathfrak{A}_{11}\), we have
\[
0 = d([Q^*, U_{11} + V_{11}]) = [d(Q)^*, U_{11} + V_{11}] + [Q^*, d(U_{11} + V_{11})].
\]
On the other hand, we have
\[ 0 = d([Q^*, U_{11}]) + d([Q^*, V_{11}]) \]
\[ = [d(Q^*), U_{11} + V_{11}] + [Q^*, d(U_{11}) + d(V_{11})]. \]
Comparing the above two identities, we get [Q, A] = 0. Hence \( A_{12} = A_{21} = 0 \).

For any \( W_{12} \in \mathfrak{A}_{12} \), we compute
\[ d(W_{12}^*(U_{11} + V_{11})) = d([W_{12}^*, U_{11} + V_{11}]) \]
\[ = [d(W_{12}^*), U_{11} + V_{11}] + [W_{12}^*, d(U_{11} + V_{11})]. \]
Comparing the above two equations, we have \( W_{12}^*, A = 0 \). Thus \( W_{12}^*, A_{11} = A_{22}W_{12}^* \) for all \( W_{12} \in \mathfrak{A}_{12} \). By using the condition \( G_1 \), we see that \( A_{11} + A_{22} \in Z(\mathcal{R}) \). Therefore \( d(U_{11} + V_{11}) = d(U_{11}) + d(V_{11}) + Z_{U_{11}, V_{11}} \) for all \( U_{11}, V_{11} \in \mathfrak{A}_{11} \) and for some \( Z_{U_{11}, V_{11}} \in Z(\mathcal{R}) \).

Similarly, the result is true for the case when \( i = 2 \).

**Lemma 2.6.** For any \( U_{12} \in \mathfrak{A}_{12} \) and \( V_{21} \in \mathfrak{A}_{21} \), we have
\[ d(U_{12} + V_{21}) = d(U_{12}) + d(V_{21}). \]

**Proof.** Suppose \( A = d(U_{12} + V_{21}) - d(U_{12}) - d(V_{21}) \). For any \( U_{12} \in \mathfrak{A}_{12} \) and \( V_{21} \in \mathfrak{A}_{21} \), we compute
\[ d(U_{12} + V_{21}) = d([P^*, U_{12} - V_{21}]) \]
\[ = [d(P^*), U_{12} - V_{21}] + [P^*, d(U_{12} - V_{21})] \]
\[ = [d(P^*), U_{12}] - [P, d(U_{12})] + d([P^*, -V_{21}]) - [P^*, d(-V_{21})] \]
\[ + [P^*, d(U_{12} - V_{21})] \]
\[ = d(U_{12}) + d(V_{21}) + [P^*, d(U_{12} - V_{21}) - d(U_{12}) - d(-V_{21})]. \]
Consequently \( A = P(d(U_{12} - V_{21}) - d(U_{12}) - d(-V_{21})) - (d(U_{12} - V_{21}) - d(U_{12}) - d(-V_{21}))P \). Hence we see that \( A_{11} = A_{22} = 0 \).

For any \( W_{12} \in \mathfrak{A}_{12} \), we have
\[ d([W_{12}^*, U_{12}]) = d([W_{12}^*, U_{12} + V_{21}]) \]
\[ = [d(W_{12}^*), U_{12} + V_{21}] + [W_{12}^*, d(U_{12} + V_{21})]. \]
On the other hand, by Lemma 2.2 we have
\[ d([W_{12}^*, U_{12}]) = d([W_{12}^*, U_{12}]) + d([W_{12}^*, V_{21}]) \]
\[ = [d(W_{12}^*), U_{12} + V_{21}] + [W_{12}^*, d(U_{12}) + d(V_{21})]. \]
Comparing the above two identities, we get \([W_{12}^*, A] = 0\). This gives that 
\(A_{12}W_{12}^* = 0\) for all \(W_{12} \in \mathfrak{A}_{12}\). By the condition \((G_2)\), we see that \(A_{12} = 0\). 
Similarly, we obtain that \(A_{21} = 0\). Thus we are done. 

**Lemma 2.7.** For any \(U_{11} \in \mathfrak{A}_{11}, V_{12} \in \mathfrak{A}_{12}\) and \(W_{22} \in \mathfrak{A}_{22}\), we have 
\[d(U_{11} + V_{12} + W_{22}) = d(U_{11}) + d(V_{12}) + d(W_{22}) + Z_{U_{11}, V_{12}, W_{22}}.\]

**Proof.** Suppose \(A = d(U_{11} + V_{12} + W_{22}) - d(U_{11}) - d(V_{12}) - d(W_{22})\).*

For any \(U_{11} \in \mathfrak{A}_{11}, V_{12} \in \mathfrak{A}_{12}\) and \(W_{22} \in \mathfrak{A}_{22}\), we compute 
\[d(V_{12}) = d([P^*, U_{11} + V_{12} + W_{22}]) = [d(P^*), U_{11} + V_{12} + W_{22}] + [P^*, d(U_{11} + V_{12} + W_{22})].\]

On the other hand, by Lemma 2.2 we have 
\[d(V_{12}) = d([P^*, U_{11}]) + d([P^*, V_{12}]) + d([P^*, W_{22}]) = [d(P^*), U_{11} + V_{12} + W_{22}] + [P^*, d(U_{11}) + d(V_{12}) + d(W_{22})].\]

Comparing the above two identities, we get \([P^*, A] = 0\). This gives that \(A_{12} = A_{21} = 0\).

Now for any \(S_{21} \in \mathfrak{A}_{21}\), we see that 
\[d([S_{21}^*, U_{11} + V_{12} + W_{22}]) = [d(S_{21}^*), U_{11} + V_{12} + W_{22}] + [S_{21}^*, d(U_{11} + V_{12} + W_{22})].\]

On the other hand, by Lemmas 2.2 & 2.4 we have 
\[d([S_{21}^*, U_{11} + V_{12} + W_{22}]) = d([S_{21}^*, U_{11}]) + d([S_{21}^*, V_{12}]) = d(S_{21}^* W_{22} - U_{11} S_{21}^*) + d(S_{21}^* W_{22}) = d(S_{21}^* W_{22}) + d(U_{11} S_{21}^*) + d([S_{21}^*, V_{12}]) = [d(S_{21}^*), U_{11} + V_{12} + W_{22}] + [S_{21}^*, d(U_{11}) + d(V_{12}) + d(W_{22})].\]

Comparing the above two identities, we get \([S_{21}^*, A] = 0\). This gives that 
\(S_{21}^* A_{22} = A_{11} S_{21}^*\) for all \(S_{21} \in \mathfrak{A}_{21}\). By the condition \((G_1)\), we get \(A_{11} + A_{22} \in Z(\mathcal{R})\). Thus we have obtained that 
\[d(U_{11} + V_{12} + W_{22}) = d(U_{11}) + d(V_{12}) + d(W_{22}) + Z_{U_{11}, V_{12}, W_{22}},\]

for some \(Z_{U_{11}, V_{12}, W_{22}} \in Z(\mathcal{R})\). 

**Lemma 2.8.** For any \(U_{11} \in \mathfrak{A}_{11}, V_{12} \in \mathfrak{A}_{12}, W_{21} \in \mathfrak{A}_{21}\) and \(X_{22} \in \mathfrak{A}_{22}\), we have 
\[d(U_{11} + V_{12} + W_{21} + X_{22}) = d(U_{11}) + d(V_{12}) + d(W_{21}) + d(X_{22}) + Z_{U_{11}, V_{12}, W_{21}, X_{22}}.\]
Proof. Assume $A = d(U_{11} + V_{12} + W_{21} + X_{22}) - d(U_{11}) - d(V_{12}) - d(W_{21}) - d(X_{22})$. For any $U_{11} \in \mathfrak{A}_{11}$, $V_{12} \in \mathfrak{A}_{12}$, $W_{21} \in \mathfrak{A}_{21}$ and $X_{22} \in \mathfrak{A}_{22}$, we see that

$$d(V_{12} - W_{21}) = d([P^*, U_{11} + V_{12} + W_{21} + X_{22}])$$
$$= [d(P)^*, U_{11} + V_{12} + W_{21} + X_{22}]$$
$$= [P^*, d(U_{11} + V_{12} + W_{21} + X_{22})].$$

On the other hand, by using Lemmas 2.2 & 2.6, we have

$$d(V_{12} - W_{21}) = d([P^*, U_{11}]) + d([P^*, V_{12}]) + d([P^*, W_{21}]) + d([P^*, X_{22}])$$
$$= [d(P)^*, U_{11} + V_{12} + W_{21} + X_{22}]$$
$$= [P^*, d(U_{11} + V_{12} + W_{21} + X_{22})].$$

Comparing the above two equations, we have $[P, A] = 0$. This gives that $A_{12} = A_{21} = 0$.

Now for any $S_{12} \in \mathfrak{A}_{12}$, we compute

$$d([S_{12}^*, U_{11} + V_{12} + W_{21} + X_{22}])$$
$$= [d(S_{12}^*)^*, U_{11} + V_{12} + W_{21} + X_{22}] + [S_{12}^*, d(U_{11} + V_{12} + W_{21} + X_{22})].$$

On the other hand, by using Lemma 2.7, we have

$$d([S_{12}^*, U_{11} + V_{12} + W_{21} + X_{22}])$$
$$= d([S_{12}^*, U_{11} + V_{12} + X_{22}]) + d([S_{12}^*, W_{21}])$$
$$= [d(S_{12}^*)^*, U_{11} + V_{12} + X_{22}] + [S_{12}^*, d(U_{11} + V_{12} + X_{22})]$$
$$+ [d(S_{12}^*)^*, W_{21}] + [S_{12}^*, d(W_{21})]$$
$$= [d(S_{12}^*)^*, U_{11} + V_{12} + W_{21} + X_{22}]$$
$$+ [S_{12}^*, d(U_{11} + V_{12} + W_{21} + X_{22})].$$

Comparing the above two identities, we get $[S_{12}^*, A] = 0$. This gives that $S_{12}^*A_{11} = A_{22}S_{12}^*$ for all $S_{12} \in \mathfrak{A}_{12}$. By using condition $(G_1)$, we see that $A_{11} + A_{22} \in \mathcal{Z}(\mathcal{R})$. Thus we have obtained that $d(U_{11} + V_{12} + W_{21} + X_{22}) = d(U_{11}) + d(V_{12}) + d(W_{21}) + d(X_{22}) + Z_{U_{11}, V_{12}, W_{21}, X_{22}}$ for some $Z_{U_{11}, V_{12}, W_{21}, X_{22}} \in \mathcal{Z}(\mathcal{R})$. \qed
Proof of Theorem 2.1. Now take \( U = U_{11} + U_{12} + U_{21} + U_{22} \) and \( V = V_{11} + V_{12} + V_{21} + V_{22} \). By using Lemmas 2.4, 2.5 & 2.8, we see that
\[
\begin{align*}
d(U + V) &= d(U_{11} + U_{12} + U_{21} + U_{22} + V_{11} + V_{12} + V_{21} + V_{22}) \\
&= d(U_{11} + V_{11}) + (U_{12} + V_{12}) + (U_{21} + V_{21}) + (U_{22} + V_{22}) \\
&= d(U_{11} + V_{11}) + d(U_{12} + V_{12}) + d(U_{21} + V_{21}) \\
&\quad + d(U_{22} + V_{22}) + Z_1 \\
&= d(U_{11}) + d(V_{11}) + Z_2 + d(U_{12}) + d(V_{12}) + d(U_{21}) \\
&\quad + d(V_{21}) + d(U_{22}) + d(V_{22}) + Z_3 + Z_1 \\
&= (d(U_{11}) + d(U_{12}) + d(U_{21}) + d(U_{22})) + (d(V_{11}) \\
&\quad + d(V_{12}) + d(V_{21}) + d(V_{22})) + Z_1 + Z_2 + Z_3 \\
&= d(U_{11} + U_{12} + U_{21} + U_{22}) - Z_4 + d(V_{11} + V_{12} + V_{21} + V_{22}) \\
&\quad - Z_5 + Z_1 + Z_2 + Z_3 \\
&= d(U) + d(V) + (Z_1 + Z_2 + Z_3 - Z_4 - Z_5).
\end{align*}
\]
Take \( Z_{U,V} = Z_1 + Z_2 + Z_3 - Z_4 - Z_5 \). Thus we see that \( d(U + V) = d(U) + d(V) + Z_{U,V} \) for some \( Z_{U,V} \in Z(R) \). This completes the proof of our main theorem. \( \square \)

Now we apply Theorem 2.1 to prime \(*\)-rings and nest algebras. We begin with the following important lemma.

**Lemma 2.9.** Let \( R \) be a prime \(*\)-ring containing a nontrivial self-adjoint idempotent \( P \) with centre \( Z(R) \).

(i) If \( U_{ij} V_{jk} = 0 \) for all \( V_{jk} \in A_{ik} \) and \( 1 \leq i, j, k \neq 2 \) then \( U_{ij} = 0 \).

(ii) If \( U_{11} V_{12} = V_{12} U_{22} \) for all \( V_{12} \in A_{12} \), then \( U_{11} + U_{22} \in Z(R) \).

**Proof.** (i) is the direct consequence of the primeness of \( R \).

(ii) For any \( V_{11} \in A_{11} \) and \( V_{22} \in A_{22} \), we get \( U_{11} V_{11} V_{12} = V_{11} V_{12} U_{22} = V_{11} U_{11} V_{12} \) for all \( V_{12} \in A_{12} \). As \( R \) is prime, we have \( U_{11} V_{11} = V_{11} U_{11} \).

For any \( V_{12} \in A_{12} \) and \( V_{22} \in A_{22} \), we get \( V_{12} V_{22} U_{22} = U_{11} V_{12} V_{22} = V_{12} U_{22} V_{22} \) for all \( V_{12} \in A_{12} \). It follows by the primeness of \( R \) that \( V_{22} U_{22} = U_{22} V_{22} \).

For any \( V_{12} \in A_{12} \) and \( V_{21} \in A_{21} \), we get \( U_{22} V_{21} V_{12} = V_{21} V_{12} U_{22} = V_{21} U_{11} V_{12} \) for all \( V_{12} \in A_{12} \). It follows that \( U_{22} V_{21} = V_{21} U_{11} \).

For any \( V \in R \), we have
\[
(U_{11} + U_{22}) V = (U_{11} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22}) \\
= U_{11} V_{11} + U_{11} V_{12} + U_{22} V_{21} + U_{22} V_{22} \\
= V_{11} U_{11} + V_{12} U_{22} + V_{21} U_{11} + V_{22} U_{22} \\
= (V_{11} + V_{12} + V_{21} + V_{22})(U_{11} + U_{22}) \\
= V(U_{11} + U_{22}).
\]
Hence it follows that $U_{11} + U_{22} \in \mathcal{Z}(R)$. 

It follows from Lemma 2.9 that every prime $*$-ring with nontrivial self-adjoint idempotent satisfies the conditions $(G_1)$ and $(G_2)$ of Theorem 2.1. So we have the following immediate corollary.

**Corollary 2.10.** Let $R$ be a prime $*$-ring containing a nontrivial self-adjoint idempotent $P$. If a mapping $d : R \rightarrow R$ satisfies

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)],$$

for all $U, V \in R$, then there exists $Z_{U,V} \in \mathcal{Z}(R)$ such that $d(U + V) = d(U) + d(V) + Z_{U,V}$. 

Let $\mathcal{H}$ be a complex Hilbert space. Recall that a nest $\mathcal{N}$ of projections on $\mathcal{H}$ is a chain of orthogonal projections on $\mathcal{H}$ containing zero operator $0$ and the identity operator $I$ and is closed in the strong operator topology. By $B(\mathcal{H})$, we mean the algebra of all bounded linear operators on $\mathcal{H}$. The nest algebra $T(\mathcal{N})$ corresponding to the nest $\mathcal{N}$ is the set of all operators $U$ in $B(\mathcal{H})$ such that $UP = PUP$ for all $P \in \mathcal{N}$. It is to be noted that $T(\mathcal{N})$ is a weak $*$-closed operator algebra. A nest is said to be nontrivial if it contains at least one nontrivial projection. The centre of the nest algebra $T(\mathcal{N})$ is $\mathbb{C}I$, where $\mathbb{C}$ is the complex field. It is to be noted that by every nest algebra $T(\mathcal{N})$ with non trivial projection $P$ satisfies the conditions $(G_1)$ and $(G_2)$ of Theorem 2.1 (see [10, Lemma 2.6]). Thus we have the following immediate corollary.

**Corollary 2.11.** Let $\mathcal{N}$ be a nontrivial nest on a complex Hilbert space $\mathcal{H}$ and $T(\mathcal{N})$ be the associated nest algebra. If a mapping $d : T(\mathcal{N}) \rightarrow T(\mathcal{N})$ satisfies

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)],$$

for all $U, V \in T(\mathcal{N})$, then there exists $\lambda_{U,V} \in \mathbb{C}$ such that $d(U + V) = d(U) + d(V) + \lambda_{U,V}I$.

**3. Characterization of $*$-Lie derivable mappings on Prime $*$-rings**

In this section, we list some notations and results which will be used frequently to prove our results. Let $\mathcal{R}$ be a prime $*$-ring containing a nontrivial self-adjoint idempotent $P$ with the centre $\mathcal{Z}(\mathcal{R})$. The maximal right ring of quotients is denoted by $Q_{mr}(\mathcal{R})$ and the two-sided right ring of quotients of $\mathcal{R}$ by $Q_r(\mathcal{R})$. The centre of $Q_r(\mathcal{R})$ is called the extended centroid of $\mathcal{R}$ and is denoted by $\mathcal{C}$. It is to be noted that $\mathcal{C}$ of any prime ring is a field. The subring $\mathcal{R}C$ of $Q_{mr}(\mathcal{R})$ is called the central closure of $\mathcal{R}$ which is also prime for any prime ring. We denote the central closure of $\mathcal{R}$ by $\mathcal{T}$.

We facilitate our discussion with the following known results.
LEMMA 3.1 ([3, Theorem 2.3.4]). If $R$ is a prime ring and $U, V \in \mathcal{Q}_{m,r}(R)$ such that $UXV = VUX$ for all $X \in R$, then $U = CV$ for some $C \in C$. In other words $U$ and $V$ are $C$-dependent.

LEMMA 3.2 ([11, Lemma 2 (ii)]). For $U = U_{11} + U_{12} + U_{21} + U_{22} \in R$. If $U_{ij}V_{jk} = 0$ for every $U_{ij} \in \mathfrak{A}_{ij}$, $1 \leq i, j, k \leq 2$, then $V_{jk} = 0$. Dually, if $V_{ki}U_{ij} = 0$ for every $U_{ij} \in \mathfrak{A}_{ij}$, $1 \leq i, j, k \leq 2$, then $V_{ki} = 0$.

THEOREM 3.3. Let $R$ be a 2-torsion free prime $*$-ring containing a non-trivial self-adjoint idempotent $P$. If a mapping $d : R \to R$ satisfies

\begin{equation}
\label{3.1}
d([U, V]) = [d(U), V] + [U, d(V)],
\end{equation}

for all $U, V \in R$, then there exists $Z_{U,V} \in Z(R)$ such that $d(U + V) = d(U) + d(V) + Z_{U,V}$ and $d = \psi + \xi$, where $\psi$ is an additive $*$-derivation from $R$ into its central closure $Z$ and $\xi$ is a mapping from $R$ into its extended centroid $C$ such that $\xi(U + V) = \xi(U) + \xi(V) + Z_{U,V}$ and $\xi([U, V]) = 0$ for all $U, V \in R$.

Now we shall use the hypothesis of Theorem 3.3 freely without any specific mention in proving the following lemmas.

LEMMA 3.4. For any non-trivial self-adjoint idempotents $P$ and $Q = I - P$, we have

(i) $Pd(P)P + Qd(P)Q \in Z(R)$,

(ii) $Pd(P)Q = Pd(P)^*Q$, $Qd(P)P = Qd(P)^*P$.

PROOF.

(i) For any $U_{12} \in \mathfrak{A}_{12}$, we have

\[
d(U_{12}) = d([P^*, U_{12}])
= [d(P^*), U_{12}] + [P^*, d(U_{12})]
= d(P)^*U_{12} - U_{12}d(P)^* + P^*d(U_{12}) - d(U_{12})P^*.
\]

Multiplying the above identity from the left by $P$ and from the right by $Q$, we arrive at

\[Pd(P)^*PU_{12} = U_{12}Qd(P)^*Q.\]

By using Lemma 2.9, it follows that $Pd(P)P + Qd(P)Q \in Z(R)$.

(ii) We compute

\[
0 = d([P^*, P])
= [d(P^*), P] + [P^*, d(P)]
= d(P)^*P - Pd(P)^* + Pd(P) - d(P)P.
\]

Multiplying the above identity from the left by $P$ and from the right by $Q$, we arrive at $Pd(P)Q = Pd(P)^*Q$. Similarly, we can also obtain $Qd(P)P = Qd(P)^*P$. \qed
In the sequel, we define $\phi : \mathcal{R} \to \mathcal{R}$ by

$$
\phi(U) = d(U) + [S, U] \text{ for all } U \in \mathcal{R}
$$

where $S = Pd(P)Q - Qd(P)P$. It is to be noted that by Lemma 3.4, we have $S^* = -S$.

**Lemma 3.5.**

1. $\phi([U^*, V]) = [\phi(U^*), V] + [U^*, \phi(V)]$,
2. $\phi(P) \in Z(\mathcal{R})$,
3. $\phi(Q) \in Z(\mathcal{R})$,
4. $\phi(U + V) = \phi(U) + \phi(V) + Z_U, \ Z_U \in Z(\mathcal{R})$,
5. $\phi$ is additive on $\mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$.

**Proof.** Since (i), (iv) and (v) are easy to verify, we prove only (ii) and (iii).

1. By the definition of $\phi$, we see that

$$
\phi(P) = d(P) + [S, P] = d(P) - Qd(P)P - Pd(P)Q = d(P)P + d(P)Q - Qd(P)P - Pd(P)Q \ {\text{since } P + Q = I} = Pd(P)P + Qd(P)Q \in Z(\mathcal{R}).
$$

1. In order to prove that $\phi(Q) \in Z(\mathcal{R})$, we first show that $\phi(PUQ + QUP) = P\phi(U)Q + Q\phi(U)P$ for all $U \in \mathcal{R}$. Since $[P^*, [P^*, U]] = PU - 2PU + U = PU + QUP$, it follows, applying (i) twice,

$$
\phi(PUQ + QUP) = \phi([P^*, [P^*, U]]) = [P^*, [P^*, \phi(U)]] = P\phi(U)Q + Q\phi(U)P.
$$

By Lemma 3.4(i), $Pd(Q)P + Qd(Q)Q \in Z(\mathcal{R})$. By the definition of $\phi$, we see that

$$
\phi(Q) = d(Q) + [S, Q] = d(Q) + Pd(P)Q + Qd(P).P.
$$

The above equation gives us that $Pd(Q)P = P\phi(Q)P$ and $Qd(Q)Q = Q\phi(Q)Q$ and hence $Pd(Q)P + Qd(Q)Q = P\phi(Q)P + Q\phi(Q)Q$.

Now we know that $\phi(Q) = P\phi(Q)P + Q\phi(Q)Q + Q\phi(Q)P + Q\phi(Q)Q$, by (3.2), we have

$$
P\phi(Q)Q + Q\phi(Q)P = \phi(PQQ + QQP) = 0.
$$

Consequently, we get $\phi(Q) = P\phi(Q)P + Q\phi(Q)Q \in Z(\mathcal{R})$.

**Lemma 3.6.** $\phi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$.

**Proof.** For $U_{12} \in \mathcal{A}_{12}$, we have $U_{12} = [P^*, U_{12}]$. Compute

$$
\phi(U_{12}) = \phi([P^*, U_{12}]) = [P, \phi(U_{12})] = P\phi(U_{12}) - \phi(U_{12})P,
$$
and hence we see that $P\phi(U_{12})P = Q\phi(U_{12})P = Q\phi(U_{12})Q = 0$. This implies that $\phi(\mathfrak{A}_{12}) \subseteq \mathfrak{A}_{12}$. Similarly, $\phi(U_{21}) = Q\phi(U_{21})P \in \mathfrak{A}_{21}$ for each $U_{21} \in \mathfrak{A}_{21}$ and therefore $\phi(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$.

**Lemma 3.7.** There is a functional $f_i : \mathfrak{A}_{ii} \to \mathcal{C}$ such that $\phi(U_{ii}) - f_i(U_{ii}) \in \mathcal{T}_{ii}$ for all $U_{ii} \in \mathfrak{A}_{ii}$, $i = 1, 2$.

**Proof.** For $U_{11} \in \mathfrak{A}_{11}$, by Lemma 3.5(ii), we have

$$0 = \phi([P^*, U_{11}]) = [P^*, \phi(U_{11})] = P\phi(U_{11}) - \phi(U_{11})P,$$

and hence we see that $P\phi(U_{11})Q = Q\phi(U_{11})P = 0$. Thus, it can be assumed that $\phi(U_{11}) = A_{11} + A_{22}$ and similarly, $\phi(U_{22}) = B_{11} + B_{22}$, here $A_{ii}, B_{ii} \in \mathfrak{A}_{ii}, i = 1, 2$. Since $[U_{11}^*, U_{22}] = 0$, a simple calculation gives $[A_{22}^*, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$; $[U_{11}^*, B_{11}] = 0$ for all $U_{11} \in \mathfrak{A}_{11}$. Since $[A_{22}^*, U_{22}] = 0$ for all $U_{22} \in \mathfrak{A}_{22}$, we see that $A_{22}XQ = QX A_{22}$ for any $X \in \mathcal{R}$. As both $A_{22}, Q \in Q_{mr}(\mathcal{R})$, by Lemma 3.1, $A_{22} = QC$ for some $C \in \mathcal{C}$. A simple calculation gives us that $\phi(U_{11}) \in \mathcal{T}_{11} + \mathcal{C}$. Similarly one can see that $\phi(\mathfrak{A}_{22}) \in \mathcal{T}_{22} + \mathcal{C}$. Therefore, there exist scalars $f_1(U_{11})$ and $f_2(U_{22})$ such that $A_{22} = f_1(U_{11})Q$ and $B_{11} = f_2(U_{22})P$. Hence $\phi(U_{11}) - f_1(U_{11})I \in \mathcal{T}_{11}$ and $\phi(U_{22}) - f_2(U_{22})I \in \mathcal{T}_{22}$.

Now for any $U \in \mathcal{R}$, we define a mapping $\Delta : \mathcal{R} \to \mathcal{T}$ by $\Delta(U) = \phi(PUP) + \phi(PUQ) + \phi(QUP) + \phi(QUQ) - (f_1(PUP) + f_2(QUQ))I$. Further, by the definitions of $\phi(U)$ and $\Delta(U)$ and by Corollary 2.10, it is clear that the difference $\phi(U) - \Delta(U) \in \mathcal{C}$. So, define a mapping $\xi : \mathcal{R} \to \mathcal{C}$ by $\xi(U) = \phi(U) - \Delta(U)$ for all $U \in \mathcal{R}$. By Lemmas 3.6 and 3.7, $\Delta$ has the following properties.

**Lemma 3.8.** Let $U_{ij} \in \mathfrak{A}_{ij}, 1 \leq i, j \leq 2$. Then

(i) $\Delta(U_{ij}) \in \mathcal{T}_{ij}, 1 \leq i \neq j \leq 2$,

(ii) $\Delta(U_{12}) = \phi(U_{12})$ and $\Delta(U_{21}) = \phi(U_{21})$,

(iii) $\Delta(U_{ii}) \in \mathfrak{A}_{ii}, i = 1, 2$,

(iv) $\Delta(U_{11} + U_{12} + U_{21} + U_{22}) = \Delta(U_{11}) + \Delta(U_{12}) + \Delta(U_{21}) + \Delta(U_{22})$.

Now, we shall show that $\Delta$ is an additive $*$-derivation. First, we shall prove the additivity of $\Delta$.

By Lemma 2.4 and Lemma 3.8(ii), we get the following result.

**Lemma 3.9.** $\Delta$ is additive on $\mathfrak{A}_{12}$ and $\mathfrak{A}_{21}$.

**Lemma 3.10.** Let $U_{ii} \in \mathfrak{A}_{ii}, U_{ij} \in \mathfrak{A}_{ij}, 1 \leq i \neq j \leq 2$. Then

(i) $\Delta(U_{ij}^*) = \Delta(U_{ij})^*$,

(ii) $\Delta(U_{ii}V_{ij}) = \Delta(U_{ii})V_{ij} + U_{ii}\Delta(V_{ij})$,

(iii) $\Delta(V_{ij}U_{jj}) = \Delta(V_{ij})U_{jj} + V_{ij}\Delta(U_{jj})$,

(iv) $\Delta(P) = \Delta(Q) = 0$.

**Proof.**
(i) By Lemmas 3.5 & 3.8, for any \( V_{21} \in \mathfrak{A}_{21} \), we compute

\[
\Delta(V_{21}^*) = \Delta([V_{21}, Q]) \\
= [\phi(V_{21}), Q] + [V_{21}, \phi(Q)] \\
= \Delta(V_{21})^*.
\]

Similarly, it is easy to prove the other case.

(ii) Since \([V_{21}, U_{11}] = -U_{11}V_{21}',\) by Lemmas 3.7 & 3.8, we have

\[
-\Delta(U_{11}V_{21}') = -\phi(U_{11}V_{21}') = \phi([V_{21}', U_{11}]) \\
= [\phi(V_{21})^*, U_{11}] + [V_{21}', \phi(U_{11})] \\
= [\Delta(V_{21})^*, U_{11}] + [V_{21}, \Delta(U_{11})] \\
= -\Delta(U_{11})V_{21}' - U_{11}\Delta(V_{21})^*.
\]

Thus, we have \( \Delta(U_{11}V_{21}') = \Delta(U_{11})V_{21}' + U_{11}\Delta(V_{21})^* \). Hence \( \Delta(U_{11}V_{12}) = \Delta(U_{11})V_{12}' + U_{11}\Delta(V_{12})^* = \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12}) \). Similarly, it is easy to prove the other identity.

(iii) Proof is same as that of part (ii).

(iv) Since \( \Delta(V_{12}) = \Delta(PV_{12}) = \Delta(P)V_{12} + P\Delta(V_{12}) \), multiplying above expression by \( P \) from the left we have \( P\Delta(P)PV_{12} = 0 \), which implies \( P\Delta(P)P = 0 \) because \( \mathcal{R} \) is prime. By Lemma 3.8, \( \Delta(P) \in \mathfrak{A}_{11} \), hence \( \Delta(P) = P\Delta(P)P = 0 \). Similarly, \( \Delta(Q) = 0 \). \( \blacksquare \)

**Lemma 3.11.** \( \Delta \) is additive on \( \mathfrak{A}_{11} \) and \( \mathfrak{A}_{22} \).

**Proof.** Let \( U_{11}, V_{11} \in \mathfrak{A}_{11} \). For any \( W_{12} \in \mathfrak{A}_{22} \), by Lemma 3.10, we have

\[
\Delta((U_{11} + V_{11})W_{12}) = \Delta(U_{11} + V_{11})W_{12} + (U_{11} + V_{11})\Delta(W_{12}).
\]

On the other hand, by Lemmas 3.9 & 3.10, we have

\[
\Delta((U_{11} + V_{11})W_{12}) \\
= \Delta(U_{11}W_{12} + V_{11}W_{12}) = \Delta(U_{11}W_{12}) + \Delta(V_{11}W_{12}) \\
= \Delta(U_{11})W_{12} + U_{11}\Delta(W_{12}) + \Delta(V_{11})W_{12} + V_{11}\Delta(W_{12}).
\]

Comparing the above two identities, we get \( (\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))W_{12} = 0 \). In other words \( (\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))PQ = 0 \). Since \( \mathcal{R} \) is prime, it follows that \( (\Delta(U_{11} + V_{11}) - \Delta(U_{11}) - \Delta(V_{11}))P = 0 \). Hence, \( \Delta(U_{11} + V_{11}) = \Delta(U_{11}) + \Delta(V_{11}) \) as \( \mathfrak{A}_{11} \subseteq \mathfrak{A}_{11} \). Similarly, \( \Delta \) is additive on \( \mathfrak{A}_{22} \). \( \blacksquare \)

**Lemma 3.12.** \( \Delta \) is additive.
Proof. Let $U = \sum_{i,j=1}^{2} U_{ij}$, $V = \sum_{i,j=1}^{2} V_{ij}$ be in $R$. By Lemmas 3.8, 3.9 & 3.11, we have

$$\Delta(U + V) = \Delta \left( \sum_{i,j=1}^{2} (U_{ij} + V_{ij}) \right) = \sum_{i,j=1}^{2} \Delta(U_{ij} + V_{ij}) = \sum_{i,j=1}^{2} (\Delta(U_{ij}) + \Delta(V_{ij})) = \Delta \left( \sum_{i,j=1}^{2} U_{ij} \right) + \Delta \left( \sum_{i,j=1}^{2} V_{ij} \right) = \Delta(U) + \Delta(V).$$

In the sequel, we shall prove that $\Delta$ is a derivation.

Lemma 3.13. Let $U_{ii}, V_{ii} \in \mathfrak{A}_{ii}, i = 1, 2$. Then $\Delta(U_{ii}V_{ii}) = \Delta(U_{ii})V_{ii} + U_{ii}\Delta(V_{ii})$ and $\Delta(U_{ii}^*) = \Delta(U_{ii})*$.

Proof. For any $U_{11}, V_{11} \in \mathfrak{A}_{11}$ and $W_{12} \in \mathfrak{A}_{12}$, we have by Lemma 3.10 that

$$\Delta(U_{11}V_{11}^*W_{12}) = \Delta(U_{11}V_{11}^*)W_{12} + U_{11}V_{11}^*\Delta(W_{12}).$$

On the other hand by Lemmas 3.5, 3.7, 3.8 & 3.10 we have,

$$\begin{align*}
\Delta(U_{11}V_{11}^*W_{12})
&= \Delta(U_{11})V_{11}^*W_{12} + U_{11}\Delta(V_{11}^*)W_{12} \\
&= \Delta(U_{11})V_{11}^*W_{12} + U_{11}\phi([V_{11}^*, W_{12}]) \\
&= \Delta(U_{11})V_{11}^*W_{12} + U_{11}([\Delta(V_{11}^*)^*, W_{12}]) + U_{11}([V_{11}^*, \Delta(W_{12})]) \\
&= \Delta(U_{11})V_{11}^*W_{12} + U_{11}\Delta(V_{11}^*)^*W_{12} + U_{11}V_{11}^*\Delta(W_{12}).
\end{align*}$$

Comparing the above two identities, we get $(\Delta(U_{11}V_{11}^*) - \Delta(U_{11})V_{11}^* - U_{11}\Delta(V_{11}^*))W_{12} = 0$. In other words

$$(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11}))PRQ = 0.$$ 

Since $R$ is prime, it follows that $(\Delta(U_{11}V_{11}^*) - \Delta(U_{11})V_{11}^* - U_{11}\Delta(V_{11}^*))P = 0$. Hence, $\Delta(U_{11}V_{11}^*) = \Delta(U_{11})V_{11}^* + U_{11}\Delta(V_{11})^*$ as $\Delta(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Since $U_{11}^* = PU_{11}^*$, we see that $\Delta(U_{11}^*) = \Delta(PU_{11}^*) = \Delta(U_{11})^*$. Thus $\Delta(U_{11}V_{11}) = \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11})$. Similarly, $\Delta(U_{22}V_{22}) = \Delta(U_{22})V_{22} + U_{22}\Delta(V_{22})$. \(\Box\)

Lemma 3.14. Let $U_{12} \in \mathfrak{A}_{12}$ and $W_{21} \in \mathfrak{A}_{21}$. Then $\Delta(U_{12}W_{21}) = \Delta(U_{12})W_{21} + U_{12}\Delta(W_{21})$ and $\Delta(U_{21}W_{12}) = \Delta(U_{21})W_{12} + U_{21}\Delta(W_{12})$. \(\Box\)
PROOF. For any $W_{21} \in \mathfrak{A}_{21}$, by Lemmas 3.8 & 3.10, we compute
\[
\phi([[V_{12}^*, U_{12}], W_{12}^*]) = \phi(V_{12}^* U_{12} W_{12}^* + W_{12}^* U_{12} V_{12}^*)
\]
\[
= \Delta(V_{12}^* U_{12} W_{12}^* + W_{12}^* U_{12} V_{12}^*)
\]
\[
= \Delta(V_{12}^* U_{12} W_{12}^*) + \Delta(W_{12}^* U_{12} V_{12}^*)
\]
\[
= \Delta(V_{12}^*) U_{12} W_{12}^* + V_{12}^* \Delta(U_{12} W_{12}^*)
\]
\[
+ \Delta(W_{12}^* U_{12}) V_{12}^* + W_{12}^* U_{12} \Delta(V_{12})^* .
\]

On the other hand, by Lemmas 3.5 & 3.8 we have
\[
\phi([[V_{12}^*, U_{12}], W_{12}^*])
\]
\[
= [[\phi(V_{12})^*, U_{12}], W_{12}^*] + [[V_{12}, \phi(U_{12})], W_{12}^*] + [[V_{12}^*, U_{12}], \phi(W_{12})^*]
\]
\[
= [[\Delta(V_{12})^*, U_{12}], W_{12}^*] + [[V_{12}, \Delta(U_{12})], W_{12}^*] + [[V_{12}^*, U_{12}], \Delta(W_{12})^*]
\]
\[
= \Delta(V_{12})^* U_{12} W_{12}^* + W_{12}^* U_{12} \Delta(V_{12})^* + V_{12}^* \Delta(U_{12}) W_{12}^*
\]
\[
+ W_{12}^* \Delta(U_{12}) V_{12}^* + V_{12}^* U_{12} \Delta(W_{12})^* + \Delta(W_{12})^* U_{12} V_{12}^* .
\]

Comparing the above two identities, we arrive at
\[
V_{12}^* (\Delta(U_{12} W_{12})^* - \Delta(U_{12}) W_{12}^* - U_{12} \Delta(W_{12})^*)
\]
\[
= ( - \Delta(W_{12}^* U_{12}) + \Delta(W_{12})^* U_{12} + W_{12}^* \Delta(U_{12}) ) V_{12}^* .
\]

By using Lemma 2.9, we see that
\[
\Delta(U_{12} W_{12}^*) - \Delta(U_{12}) W_{12}^* - U_{12} \Delta(W_{12})^* - \Delta(W_{12}^* U_{12})
\]
\[
+ \Delta(W_{12})^* U_{12} + W_{12}^* \Delta(U_{12}) = C \in \mathcal{C} .
\]

From the later relation we obtain the two identities
\[
\Delta(U_{12} W_{12}^*) - \Delta(U_{12}) W_{12}^* - U_{12} \Delta(W_{12})^* = PC
\]
and
\[
\Delta(W_{12}^* U_{12}) - \Delta(W_{12})^* U_{12} - W_{12}^* \Delta(U_{12}) = - QC .
\]

Since $\Delta(W_{21}^*) = \Delta(W_{21})^*$, we have
\[
\Delta(U_{12} W_{21}^*) - \Delta(U_{12}) W_{21} - U_{12} \Delta(W_{21}^*) = \Delta(U_{12} W_{21}^*) - \Delta(U_{12}) W_{21}^* - U_{12} \Delta(W_{21})^* = PC .
\]

Similarly, we obtain the other identity as
\[
\Delta(W_{21} U_{12}) - \Delta(W_{21}) U_{12} - W_{21} \Delta(U_{12}) = - QC .
\]

Now it is sufficient to show that $C = 0$. Assume $C \neq 0$. Then by using equations (3.3) and (3.4) together with Lemma 3.10, we have
\[
\Delta(U_{12} W_{21} U_{12})
\]
\[
= \Delta(U_{12}) W_{21} U_{12} + U_{12} \Delta(W_{21} U_{12})
\]
\[
= \Delta(U_{12}) W_{21} U_{12} + U_{12} \Delta(W_{21}) U_{12} + U_{12} W_{21} \Delta(U_{12}) - CU_{12} .
\]
and
\[
\Delta(U_{12}W_{21}U_{12}) = \Delta(U_{12}W_{21})U_{12} + U_{12}W_{21}\Delta(U_{12}) \\
= \Delta(U_{12})W_{21}U_{12} + U_{12}\Delta(W_{21})U_{12} + U_{12}W_{21}\Delta(U_{12}) + CU_{12}.
\]
Comparing the above two identities, we obtain \(CU_{12} = 0\). Since \(C\) is a field, we have \(U_{12} = 0\), a contradiction. Consequently, \(\Delta(U_{12}W_{21}) = \Delta(U_{12})W_{21} + U_{12}\Delta(W_{21}) + \Delta(U_{21}W_{12}) = \Delta(U_{21})W_{12} + U_{21}\Delta(W_{12})\).

**Proof of Theorem 3.3.** Let \(U, V \in \mathcal{R}\). Assume that \(U = U_{11} + U_{12} + U_{21} + U_{22}\) and \(V = V_{11} + V_{12} + V_{21} + V_{22}\). By Lemmas 3.8-3.14, we see that
\[
\Delta(UV) = \Delta((U_{11} + U_{12} + U_{21} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22})) \\
= \Delta(U_{11}V_{11} + U_{11}V_{12} + U_{12}V_{21} + U_{12}V_{22} + U_{21}V_{11} + U_{21}V_{12} + U_{22}V_{21} + U_{22}V_{22}) \\
= \Delta(U_{11}V_{11}) + \Delta(U_{12}V_{12}) + \Delta(U_{11}V_{12} + U_{12}V_{21}) + \Delta(U_{21}V_{11} + U_{22}V_{21}) + \Delta(U_{21}V_{12} + U_{22}V_{22}) \\
= \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11}) + \Delta(U_{12})V_{21} + U_{12}\Delta(V_{21}) + \Delta(U_{21})V_{11} + U_{21}\Delta(V_{11}) + \Delta(U_{22})V_{21} + U_{22}\Delta(V_{21}) \\
+ \Delta(U_{11}V_{12} + U_{12}V_{21}) + \Delta(U_{21}V_{12} + U_{22}V_{22}) \\
= \Delta(U_{11} + U_{12} + U_{21} + U_{22})\Delta(V_{11} + V_{12} + V_{21} + V_{22}) + \Delta(U_{11} + U_{12} + U_{21} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22}) \\
= U\Delta(V) + \Delta(U)V.
\]
It is easy to show that \(\Delta(U^*) = \Delta(U)^*\). Hence, \(\Delta\) is an additive \(*\)-derivation.

Now using the definition of \(\xi\), we see that
\[
\xi(U + V) = \phi(U + V) - \Delta(U + V) \\
= \phi(U) + \phi(V) + Z_{U,V} - \Delta(U) - \Delta(V) \\
= \xi(U) + \xi(V) + Z_{U,V}.
\]
and
\[
\xi([U, V]) = \phi([U, V]) - \Delta([U, V]) \\
= [\phi(U^*), V] + [U, \phi(V)] - \Delta([U, V]) \\
= [\Delta(U), V] + [U, \Delta(V)] - \Delta([U, V]) = 0.
\]
Finally, let us define \(\psi(U) = \Delta(U) - (SU - US)\) for all \(U \in \mathcal{R}\), where \(S = Pd(P)Q - Qd(P)P\). It is easy to check that \(\psi\) is an additive \(*\)-derivation.
on $R$. By the definitions of $\Delta$ and $\phi$, $\psi$ is an additive $*$-derivation and $d(U) = \psi(U) + \xi(U)$ for all $U \in R$.

We conclude this section by the following result. Recall that a von Neumann algebra $M$ is called a factor if its centre is $CI$. It is to be noted that every factor von Neumann algebra is prime. So we have the following immediate corollary.

**Corollary 3.15.** Let $M$ be a factor von Neumann algebra. Suppose that a mapping $d : M \to M$ satisfies $d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)]$ for all $U, V \in M$. Then there exists $\lambda_{U,V} \in \mathbb{C}$ such that $d(U + V) = d(U) + d(V) + \lambda_{U,V}$ and $d = \psi + \xi$, where $\psi$ is an additive $*$-derivation on $M$ and $\xi$ is a mapping from $M$ into $\mathbb{C}$ such that $\xi(U + V) = \xi(U) + \xi(V) + \lambda_{U,V}$ and $\xi([U, V]) = 0$ for all $U, V \in M$.

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**References**


**Karakterizacije $*$-Liejevih derivabilnih preslikavanja na prostim $*$-prstenima**

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**Sažetak.** Neka je $R$ $*$-prsten koji sadrži netrivijalni samoadjunirajući idempotentni element. U ovom članku se pokazuje da uz izvjesne pretpostavke na $R$, ako preslikavanje $d : R \to R$ zadovoljava

$$d([U^*, V]) = [d(U)^*, V] + [U^*, d(V)]$$

za sve $U, V \in R$, tada postoji $Z_{U,V} \in Z(R)$ (koji ovisi o $U$ i $V$), gdje je $Z(R)$ u centru od $R$, tako da vrijedi $d(U + V) = d(U) + d(V) + Z_{U,V}$. Štoviše, ako je $R$ slobodan od 2-torzie prosti $*$-prsten, tada je $d = \psi + \xi$, gdje je $\psi$ aditivna $*$-derivacija od $R$ u njegov centralni zatvarač $T$ i $\xi$ je preslikavanje $s R$ u njegov prošireni centroid $C$ tako da $\xi(U + V) = \xi(U) + \xi(V) + Z_{U,V}$ i $\xi([U,V]) = 0$ za sve $U, V \in R$. Naposljetku, gornji rezultati iz teorije prstena primijenjeni su na neke specijalne klase algebri kao što su ugniježđene algebre i von Neumannove algebre.
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