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RESOLVENT OPERATOR OF SINGULAR DIRAC SYSTEM WITH TRANSMISSION CONDITIONS

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Abstract. This paper is concerned with the resolvent operator of one dimensional singular Dirac operator with transmission conditions. We study the Titchmarsh-Weyl function of this problem. Later, we construct a Green function and a spectral function for regular and singular problems. With the help of these functions, we obtain an expansion into a Fourier series of resolvent in regular case. Furthermore, we give integral representations in terms of the spectral function for the resolvent of this operator with transmission conditions in singular case. Finally, we obtain a formula for the Titchmarsh-Weyl function in terms of the spectral function of the singular Dirac system.

1. Introduction

Recently, much attention has been paid to the boundary value problems with transmission conditions, or discontinuous boundary value problems. It has been shown that they are closely related to various physically interesting models and theories, such as the heat and mass transfer theory (see [22]), radio science (see [23]), and geophysics (see [19]). For further references consider [1-16, 19, 21-28, 30, 32, 34-37].

On the other hand, the Dirac systems play an important role in the theory of relativistic quantum mechanics since basic physics of relativistic quantum mechanics was governed by the Dirac operators. For example, they predict the existence of a positron and elucidate the origin of spin 1/2 of an electron (see [29]). Dirac systems in the finite interval have been considered in [20], [33] whereas the Dirac system in the infinite interval were considered in [20]. Direct or inverse spectral problems for Dirac operators with transmission conditions were studied in [6-8, 13, 16, 24, 36]. In [14], Hıra and Altınışık investigated asymptotic behavior of eigenvalues and eigenfunctions of discontinuous Dirac system which includes an eigenvalue parameter in a

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transmission condition. In [30], Tharwat and Bhrawy computed the eigenvalues of a discontinuous regular Dirac systems with transmission conditions at the point of discontinuity numerically. In [15], Kablan and Özden studied a Dirac system with transmission conditions and eigenparameter in the boundary condition. They investigated the existence of the solution and some spectral properties of this problem. For the classical Sturm-Liouville equation, the integral representation of the resolvent was first proved by H. Weyl in 1910. Similar theorems were proved by E. C. Titchmarsh [31] and [20]. Levitan and Sargsjan obtained the integral representation of the resolvent for one dimensional Dirac operators ([20]).

The main result of the paper is obtaining an integral representations in terms of the spectral function for the resolvent of the one dimensional singular Dirac operator with transmission conditions.

This paper is organized as follows. In Section 2, we consider one dimensional singular Dirac operator with transmission (or impulsive) conditions. In Section 3, we study the Titchmarsh-Weyl function of one dimensional singular Dirac operator with transmission conditions. We will define limit-point and limit-circle singularities. In Section 4, we construct a Green function and a spectral function for regular problem. With the help of these functions, we obtain an expansion into a Fourier series of resolvent in regular case. In Section 5, we give the main result of this paper. We obtain integral representations in terms of the spectral function for the resolvent of this operator with transmission conditions in singular case. Finally, in Section 6, we obtain a formula for the Titchmarsh-Weyl function in terms of the spectral function of the singular Dirac system.

2. Preliminaries

We consider one dimensional Dirac expression

$$\tau(y) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{dy(x)}{dx} + Q(x)y(x),\ x \in \Omega := \Omega_1 \cup \Omega_2,$$

where

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\ Q(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix},$$

and $\Omega_1 := [a, c)$, $\Omega_2 := (c, b]$, $-\infty < a < c < b < +\infty$. We assume that the points a, b and c are regular for the differential expression τ. p and r are real-valued, Lebesgue measurable functions on Ω and $p, r \in L^1(\Omega_k)$, $(k = 1, 2)$. The point c is regular if $p, r \in L^1[c - \epsilon, c + \epsilon]$ for some $\epsilon > 0$.

Let us consider the Dirac system

$$\tau\,(y) = \lambda y,\ x \in \Omega, \tag{2.1}$$

with the boundary conditions

$$y_1(a)\cos\beta + y_2(a)\sin\beta = 0, \tag{2.2}$$

$$y_1(b)\cos\alpha + y_2(b)\sin\alpha = 0,\ \alpha,\beta\in\mathbb{R} := (-\infty,\infty), \tag{2.3}$$

and transmission (or impulsive) conditions

$$y(c+) = Cy(c-),\ C\in M_2(\mathbb{R}),\ \det C = \gamma > 0, \tag{2.4}$$

where λ is a complex spectral parameter and $M_2(\mathbb{R})$ denotes the the 2×2 matrices with entries from $\mathbb{R}$.

Now, we introduce the direct sum Hilbert space $H = L^2(\Omega_1; E)\dot{+} L^2(\Omega_2; E)$ (where $E := \mathbb{C}^2$) of vector-valued functions with values in $\mathbb{C}^2$ and with the inner product

$$\langle u, v\rangle_H := \int_a^c (u(x), v(x))_E\,dx + \gamma\int_c^b (u(x), v(x))_E\,dx,\ \delta = \frac{1}{\gamma},$$

where

$$u(x) = \begin{pmatrix} u_1(x)\\ u_2(x)\end{pmatrix},\ v(x) = \begin{pmatrix} v_1(x)\\ v_2(x)\end{pmatrix},$$

$$u_1(x) = \begin{cases} u_1^{(1)}(x), & x\in\Omega_1\\ u_1^{(2)}(x), & x\in\Omega_2\end{cases},\quad u_2(x) = \begin{cases} u_2^{(1)}(x), & x\in\Omega_1\\ u_2^{(2)}(x), & x\in\Omega_2\end{cases},$$

$$v_1(x) = \begin{cases} v_1^{(1)}(x), & x\in\Omega_1\\ v_1^{(2)}(x), & x\in\Omega_2\end{cases},\quad v_2(x) = \begin{cases} v_2^{(1)}(x), & x\in\Omega_1\\ v_2^{(2)}(x), & x\in\Omega_2\end{cases}.$$

Denote by D the linear set of all vector-valued functions $y\in H$ such that y_1, y_2 are locally absolutely continuous functions on Ω, one-sided limits $y_1(c\pm), y_2(c\pm)$ exist and are finite and $\tau(y)\in H$. The operator L defined by $Ly = \tau(y)$ $(y\in D)$ is called the maximal operator on H.

For two arbitrary vector-valued functions $y, z\in D$, we have Green's formula

$$\int_a^b \tau(y)\,\overline{z}dx - \int_a^b y\overline{\tau(z)}dx = [y,z]_{c-} - [y,z]_a + [y,z]_b - [y,z]_{c+}, \tag{2.5}$$

where $[y,z]_x := W_x(y,\overline{z}) := y_1(x)\overline{z_2(x)} - y_2(x)\overline{z_1(x)}$ $(x\in\Omega)$.

Now, we introduce the Hilbert space $\mathcal{H} := L^2(\Omega_1)\dot{+} L^2(\Omega_3)$, $(\Omega_1 := [a,c),\ \Omega_3 := (c,\infty))$ with the inner product

$$\langle u, v\rangle_{\mathcal{H}} := \int_a^c (u(x), v(x))_E\,dx + \delta\int_c^\infty (u(x), v(x))_E\,dx,\ \delta = \frac{1}{\gamma},$$

where

$$u(x)=\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix},\ v(x)=\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix},$$

$$u_1(x)=\begin{cases} u_1^{(1)}(x), & x\in\Omega_1 \\ u_1^{(2)}(x), & x\in\Omega_3 \end{cases},\ u_2(x)=\begin{cases} u_2^{(1)}(x), & x\in\Omega_1 \\ u_2^{(2)}(x), & x\in\Omega_3 \end{cases},$$

$$v_1(x)=\begin{cases} v_1^{(1)}(x), & x\in\Omega_1 \\ v_1^{(2)}(x), & x\in\Omega_3 \end{cases},\ v_2(x)=\begin{cases} v_2^{(1)}(x), & x\in\Omega_1 \\ v_2^{(2)}(x), & x\in\Omega_3 \end{cases}.$$

We will denote by

$$\varphi(x,\lambda)=\begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix},\ \varphi_1(x,\lambda)=\begin{cases} \phi_1^{(1)}(x,\lambda), & x\in\Omega_1 \\ \phi_1^{(2)}(x,\lambda), & x\in\Omega_3 \end{cases}, \tag{2.6}$$

$$\varphi_2(x,\lambda)=\begin{cases} \varphi_2^{(1)}(x,\lambda), & x\in\Omega_1 \\ \varphi_2^{(2)}(x,\lambda), & x\in\Omega_3 \end{cases},$$

and

$$\theta(x,\lambda)=\begin{pmatrix} \theta_1(x,\lambda) \\ \theta_2(x,\lambda) \end{pmatrix},\ \theta_1(x,\lambda)=\begin{cases} \theta_1^{(1)}(x,\lambda), & x\in\Omega_1 \\ \theta_1^{(2)}(x,\lambda), & x\in\Omega_3 \end{cases},$$

$$\theta_2(x,\lambda)=\begin{cases} \theta_2^{(1)}(x,\lambda), & x\in\Omega_1 \\ \theta_2^{(2)}(x,\lambda), & x\in\Omega_3 \end{cases},$$

the solution of the system $\tau(y)=\lambda y,\ x\in\Omega_1\cup\Omega_3$ which satisfy the initial conditions

$$\begin{gathered}\varphi_1^{(1)}(a,\lambda)=\sin\beta,\ \varphi_2^{(1)}(a,\lambda)=-\cos\beta,\\ \theta_1^{(1)}(a,\lambda)=\cos\beta,\ \theta_2^{(1)}(a,\lambda)=\sin\beta.\end{gathered} \tag{2.7}$$

and transmission conditions

$$\begin{gathered}\Phi(c+,\lambda)=C\Phi(c-,\lambda),\ \Phi(x,\lambda):=\begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix},\\ \Theta(c+,\lambda)=C\Theta(c-,\lambda),\ \Theta(x,\lambda):=\begin{pmatrix} \theta_1(x,\lambda) \\ \theta_2(x,\lambda) \end{pmatrix},\end{gathered} \tag{2.8}$$

$$C\in M_2(\mathbb{R}),\ \det C=\gamma>0.$$

3. The Titchmarsh-Weyl function

In this section, we will consider the Titchmarsh-Weyl function of one dimensional singular Dirac operator with transmission conditions. Later, we will define limit-point and limit-circle singularities.

We will denote by $\theta(x,\lambda) + m_b(\lambda)\varphi(x,\lambda)$ the solution of the equation (2.1) which satisfy the boundary condition

$$(\theta_1^{(2)}(b,\lambda) + m_b(\lambda)\varphi_1^{(2)}(b,\lambda))\cos\alpha$$
$$+ (\theta_2^{(2)}(b,\lambda) + m_b(\lambda)\varphi_2^{(2)}(b,\lambda))\sin\alpha = 0.$$

Then, $m_b(\lambda)$ satisfies the relation

$$m_b(\lambda) = -\frac{\theta_1^{(2)}(b,\lambda)\cot\alpha + \theta_2^{(2)}(b,\lambda)}{\varphi_1^{(2)}(b,\lambda)\cot\alpha + \varphi_2^{(2)}(b,\lambda)}.$$

It is clear that $m_b(\lambda)$ is a meromorphic function of λ, since $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ are entire functions of λ. Furthermore, since the eigenvalues of the regular problem are real, all poles of $m_b(\lambda)$ are real and simple. The function m_b is called the *Titchmarsh-Weyl function* of the regular problem (1)-(4). If $\cot\beta$ is replaced by a complex variable z, then we have

$$m_b(\lambda, z) = -\frac{\theta_1^{(2)}(b,\lambda)\, z + \theta_2^{(2)}(b,\lambda)}{\varphi_1^{(2)}(b,\lambda)\, z + \varphi_2^{(2)}(b,\lambda)}. \tag{3.1}$$

For every λ, the equality (3.1) is a one-to-one conformal mapping in z, which follows from the theory of Möbius transformations [17]. Hence, if $\operatorname{Im}\lambda \neq 0$, then $m_b(\lambda, z)$ varies on a circle $C_b(\lambda)$ with a finite radius in the m_b-plane as z varies over the real axis of the z-plane.

Using this notation we now state the result from [6].

THEOREM 3.1. *Let $\theta(x,\lambda), \varphi(x,\lambda)$ be two linearly independent solution of the system (2.1) satisfying the initial conditions (2.7) and transmission conditions (2.8). Then, the solution*

$$\omega(x,\lambda) = \theta(x,\lambda) + m_b(\lambda)\varphi(x,\lambda)$$

satisfies the boundary condition

$$(\theta_1^{(2)}(b,\lambda) + m_b(\lambda)\varphi_1^{(2)}(b,\lambda))\cos\alpha$$
$$+(\theta_2^{(2)}(b,\lambda) + m_b(\lambda)\varphi_2^{(2)}(b,\lambda))\sin\alpha = 0.$$

if and only if $m_b(\lambda)$ is on C_b with

$$\lim_{b\to\infty} W(\omega,\overline{\omega})(b,\lambda) = 0.$$

If $b \to \infty$, then C_b tends either to limit-circle C_∞ or to the limit-point m_∞. In the first case, all solutions of the system are in the space $\mathcal{H}$. In the second case, if $\operatorname{Im}\lambda \neq 0$, one linearly independent solution is in the space $\mathcal{H}$. In the limit-circle case, a point is on C_∞ if and only if

$$\lim_{b\to\infty} W(\omega,\overline{\omega})(b,\lambda) = 0.$$

The function $m(\lambda) := \lim_{b\to\infty} m_b(\lambda)$ is called the *Titchmars-Weyl* function, and $\chi(x,\lambda) := \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$ is called the *Weyl solution* of the singular system $\tau(y) = \lambda y$ $(x \in \Omega_1 \cup \Omega_3)$ satisfying (2.2), (2.4).

Let us define

$$\chi_b(x,\lambda) := \theta(x,\lambda) + m_b(\lambda)\varphi(x,\lambda),\ \ x \in \Omega. \tag{3.2}$$

Then, we have the following lemma.

Lemma 3.2. *For each nonreal* λ, *we have*

$$\chi_b(x,\lambda) \to \chi(x,\lambda),\ \ b \to \infty,$$

$$\begin{aligned}&\int_a^c \left\|\chi_b^{(1)}(x,\lambda)\right\|_E^2 dx + \delta \int_c^b \left\|\chi_b^{(2)}(x,\lambda)\right\|_E^2 dx\\ &\to \int_a^c \left\|\chi^{(1)}(x,\lambda)\right\|_E^2 dx + \delta \int_c^\infty \left\|\chi^{(2)}(x,\lambda)\right\|_E^2 dx,\ \ b \to \infty.\end{aligned}$$

Proof. It is clear that

$$\chi_b(x,\lambda) = \chi(x,\lambda) + \{m_b(\lambda) - m(\lambda)\}\varphi(x,\lambda),$$

where $\chi(x,\lambda) \in \mathcal{H}$ and $m_b(\lambda)$ is a point of the circle. According to [20], [6]

$$\begin{aligned}&|m_b(\lambda) - m(\lambda)| \le 2r_b(\lambda)\\ &\qquad = \left[|v|\left(\begin{array}{c}\int_a^c \left\|\varphi^{(1)}(x,\lambda)\right\|_E^2 dx\\ +\delta\int_c^b \left\|\varphi^{(2)}(x,\lambda)\right\|_E^2 dx\end{array}\right)\right]^{-1},\ \ \operatorname{Im}\lambda = v \ne 0.\end{aligned}$$

Since $r_b(\lambda) \to 0$, we get $\chi_b(x,\lambda) \to \chi(x,\lambda)$ $(b \to \infty)$. Furthermore, we have

$$\begin{aligned}&\int_a^c \left\|\{m_b(\lambda) - m(\lambda)\}\varphi^{(1)}(x,\lambda)\right\|_E^2 dx\\ &\quad + \delta\int_c^b \left\|\{m_b(\lambda) - m(\lambda)\}\varphi^{(2)}(x,\lambda)\right\|_E^2 dx\\ &= |m_b(\lambda) - m(\lambda)|^2 \int_a^c \left\|\varphi^{(1)}(x,\lambda)\right\|_E^2 dx\\ &\quad + \delta\,|m_b(\lambda) - m(\lambda)|^2 \int_c^b \left\|\varphi^{(2)}(x,\lambda)\right\|_E^2 dx\\ &\le \left(|v|^2\left[\int_a^c \left\|\varphi^{(1)}(x,\lambda)\right\|_E^2 dx + \delta\int_c^b \left\|\varphi^{(2)}(x,\lambda)\right\|_E^2 dx\right]\right)^{-1}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\int_a^c \left\|\chi_b^{(1)}(x,\lambda)\right\|_E^2 dx + \delta \int_c^b \left\|\chi_b^{(2)}(x,\lambda)\right\|_E^2 dx\\ &\to \int_a^c \left\|\chi^{(1)}(x,\lambda)\right\|_E^2 dx + \delta \int_c^\infty \left\|\chi^{(2)}(x,\lambda)\right\|_E^2 dx,\ \ b \to \infty.\end{aligned}$$

∎

4. An expansion of the resolvent in regular case

In this section, we will obtain an expansion into a Fourier series of resolvent in regular case. We will construct a Green function and a spectral function for this problem. With the help of these functions, we will obtain an expansion into a Fourier series of resolvent.

Let two functions $\chi_b(x,\lambda)$ and $\varphi(x,\lambda)$ be as in (3.2) and (2.6), respectively. Putting

$$G_b(x,t,\lambda) = \begin{cases} \chi_b(x,\lambda)\varphi^T(t,\lambda), & t \le x,\ x \ne c,\ t \ne c \\ \varphi(x,\lambda)\chi_b^T(t,\lambda), & t > x,\ x \ne c,\ t \ne c \end{cases}$$

$$= \begin{cases} \begin{pmatrix} \chi_{b1}(x,\lambda)\varphi_1(t,\lambda) & \chi_{b1}(x,\lambda)\varphi_2(t,\lambda) \\ \chi_{b2}(x,\lambda)\varphi_1(t,\lambda) & \chi_{b2}(x,\lambda)\varphi_2(t,\lambda) \end{pmatrix}, & t \le x,\ x \ne c,\ t \ne c \\ \begin{pmatrix} \varphi_1(x,\lambda)\chi_{b1}(t,\lambda) & \varphi_1(x,\lambda)\chi_{b2}(t,\lambda) \\ \varphi_2(x,\lambda)\chi_{b1}(t,\lambda) & \varphi_2(x,\lambda)\chi_{b2}(t,\lambda) \end{pmatrix}, & x < t,\ x \ne c,\ t \ne c, \end{cases} \tag{4.1}$$

$$(R_b f)(x,\lambda) = y(x,\lambda) = \int_a^b G_b(x,t,\lambda)\, f(t)\, dt,\ \lambda \in \mathbb{C}. \tag{4.2}$$

Hence, we have

$$G_b(x,t,\lambda)\, f(t)$$
$$= \begin{cases} \begin{pmatrix} \chi_{b1}(x,\lambda)\varphi_1(t,\lambda)f_1(t) \\ +\chi_{b1}(x,\lambda)\varphi_2(t,\lambda)f_2(t) \\ \chi_{b2}(x,\lambda)\varphi_1(t,\lambda)f_1(t) \\ +\chi_{b2}(x,\lambda)\varphi_2(t,\lambda)f_2(t) \end{pmatrix}, & t \le x,\ x \ne c,\ t \ne c \\ \begin{pmatrix} \varphi_1(x,\lambda)\chi_{b1}(t,\lambda)f_1(t) \\ +\varphi_1(x,\lambda)\chi_{b2}(t,\lambda)f_2(t) \\ \varphi_2(x,\lambda)\chi_{b1}(t,\lambda)f_1(t) \\ +\varphi_2(x,\lambda)\chi_{b2}(t,\lambda)f_2(t) \end{pmatrix}, & x < t,\ x \ne c,\ t \ne c. \end{cases}$$

Now, we shall show that (4.2) satisfies the equation $\tau(y) = \lambda y + f$ ($x \in \Omega$, $f \in H$) and the conditions (2.2)–(2.4). From (4.2), we have

$$y_1(x,\lambda) = \begin{cases} \chi_{b1}^{(1)}(x,\lambda) \int_a^x \left(\varphi_1^{(1)}(t,\lambda)f_1(t) + \varphi_2^{(1)}(t,\lambda)f_2(t)\right) dt \\ +\varphi_1^{(1)}(x,\lambda) \int_x^c \left(\chi_{b1}^{(1)}(t,\lambda)f_1(t) + \chi_{b2}^{(1)}(t,\lambda)f_2(t)\right) dt \\ +\varphi_1^{(1)}(x,\lambda)\delta \int_c^b \left(\chi_{b1}^{(2)}(t,\lambda)f_1(t) + \chi_{b2}^{(2)}(t,\lambda)f_2(t)\right) dt, x \in \Omega_1, \\ \\ \chi_{b1}^{(2)}(x,\lambda) \int_a^c \left(\varphi_1^{(1)}(t,\lambda)f_1(t) + \varphi_2^{(1)}(t,\lambda)f_2(t)\right) dt \\ +\chi_{b1}^{(2)}(x,\lambda)\delta \int_c^x \left(\varphi_1^{(2)}(t,\lambda)f_1(t) + \varphi_2^{(2)}(t,\lambda)f_2(t)\right) dt \\ +\varphi_1^{(2)}(x,\lambda)\delta \int_x^b \left(\chi_{b1}^{(2)}(t,\lambda)f_1(t) + \chi_{b2}^{(2)}(t,\lambda)f_2(t)\right) dt, x \in \Omega_2, \end{cases} \tag{4.3}$$

and
(4.4)

$$y_2(x,\lambda)=\begin{cases} \chi_{b2}^{(1)}(x,\lambda)\int_a^x\left(\varphi_1^{(1)}(t,\lambda)f_1(t)+\varphi_2^{(1)}(t,\lambda)f_2(t)\right)dt \\ +\varphi_2^{(1)}(x,\lambda)\int_x^c\left(\chi_{b1}^{(1)}(t,\lambda)f_1(t)+\chi_{b2}^{(1)}(t,\lambda)f_2(t)\right)dt \\ +\varphi_2^{(1)}(x,\lambda)\delta\int_c^b\left(\chi_{b1}^{(2)}(t,\lambda)f_1(t)+\chi_{b2}^{(2)}(t,\lambda)f_2(t)\right)dt, x\in\Omega_1, \\ \\ \chi_{b2}^{(2)}(x,\lambda)\int_a^c\left(\varphi_1^{(1)}(t,\lambda)f_1(t)+\varphi_2^{(1)}(t,\lambda)f_2(t)\right)dt \\ +\chi_{b2}^{(2)}(x,\lambda)\delta\int_c^x\left(\varphi_1^{(2)}(t,\lambda)f_1(t)+\varphi_2^{(2)}(t,\lambda)f_2(t)\right)dt \\ +\varphi_2^{(2)}(x,\lambda)\delta\int_x^b\left(\chi_{b1}^{(2)}(t,\lambda)f_1(t)+\chi_{b2}^{(2)}(t,\lambda)f_2(t)\right)dt, x\in\Omega_2. \end{cases}$$

From (4.3), it follows that

$$y_1'(x,\lambda)=\begin{cases} \chi_{b1}^{(1)\prime}(x,\lambda)\int_a^x\left(\varphi_1^{(1)}(t,\lambda)f_1(t)+\varphi_2^{(1)}(t,\lambda)f_2(t)\right)dt \\ +\varphi_1^{(1)\prime}(x,\lambda)\int_x^c\left(\chi_{b1}^{(1)}(t,\lambda)f_1(t)+\chi_{b2}^{(1)}(t,\lambda)f_2(t)\right)dt \\ +\varphi_1^{(1)\prime}(x,\lambda)\delta\int_c^b\left(\chi_{b1}^{(2)}(t,\lambda)f_1(t)+\chi_{b2}^{(2)}(t,\lambda)f_2(t)\right)dt \\ +W(\varphi,\chi_b)f_2(x), x\in\Omega_1, \\ \\ \chi_{b1}^{(2)\prime}(x,\lambda)\int_a^c\left(\varphi_1^{(1)}(t,\lambda)f_1(t)+\varphi_2^{(1)}(t,\lambda)f_2(t)\right)dt \\ +\chi_{b1}^{(2)\prime}(x,\lambda)\delta\int_c^x\left(\varphi_1^{(2)}(t,\lambda)f_1(t)+\varphi_2^{(2)}(t,\lambda)f_2(t)\right)dt \\ +\varphi_1^{(2)\prime}(x,\lambda)\delta\int_x^b\left(\chi_{b1}^{(2)}(t,\lambda)f_1(t)+\chi_{b2}^{(2)}(t,\lambda)f_2(t)\right)dt \\ +W(\varphi,\chi_b)f_2(x), x\in\Omega_2 \end{cases}$$

$$=\begin{cases} \{\lambda-r(x)\}\chi_{b2}^{(1)}(x,\lambda)\int_a^x\left(\varphi_1^{(1)}(t,\lambda)f_1(t)+\varphi_2^{(1)}(t,\lambda)f_2(t)\right)dt \\ \{\lambda-r(x)\}\varphi_2^{(1)}(x,\lambda)+\int_x^c\left(\chi_{b1}^{(1)}(t,\lambda)f_1(t)+\chi_{b2}^{(1)}(t,\lambda)f_2(t)\right)dt \\ \{\lambda-r(x)\}\varphi_2^{(1)}(x,\lambda)\delta\int_c^b\left(\chi_{b1}^{(2)}(t,\lambda)f_1(t)+\chi_{b2}^{(2)}(t,\lambda)f_2(t)\right)dt \\ +f_2(x), x\in\Omega_1, \\ \\ \{\lambda-r(x)\}\chi_{b2}^{(2)}(x,\lambda)\int_a^c\left(\varphi_1^{(1)}(t,\lambda)f_1(t)+\varphi_2^{(1)}(t,\lambda)f_2(t)\right)dt \\ \{\lambda-r(x)\}\varphi_2^{(2)}(x,\lambda)\delta\int_c^x\left(\chi_{b1}^{(2)}(t,\lambda)f_1(t)+\chi_{b2}^{(2)}(t,\lambda)f_2(t)\right)dt \\ \{\lambda-r(x)\}\varphi_2^{(2)}(x,\lambda)\delta\int_x^b\left(\chi_{b1}^{(2)}(t,\lambda)f_1(t)+\chi_{b2}^{(2)}(t,\lambda)f_2(t)\right)dt \\ +f_2(x), x\in\Omega_2, \end{cases}$$

$$=\{\lambda-r(x)\}y_2(x,\lambda)+f_2(x).$$

The validity of of the other equation in (2.1) is proved similarly. Hence the function $y(x,\lambda)=\begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}$ in (4.2) is the solution of the system (2.1). We check at once that (4.2) satisfies the conditions (2.2)-(2.4).

Let $\lambda_{m,b}$ ($m\in\mathbb{Z}:=\{0,\pm1,\pm2,...\}$) denote the eigenvalues of the self-adjoint problem (1)-(4) and by

$$\phi_{m,b}(x)=\begin{pmatrix} \phi_{m,b1}(x) \\ \phi_{m,b2}(x) \end{pmatrix},\ \phi_{m,b1}(x)=\begin{cases} \phi^{(1)}_{m,b1}(x), & x\in\Omega_1 \\ \phi^{(2)}_{m,b2}(x), & x\in\Omega_2 \end{cases},$$
$$\phi_{m,b2}(x)=\begin{cases} \phi^{(1)}_{m,b2}(x), & x\in\Omega_1 \\ \phi^{(2)}_{m,b2}(x), & x\in\Omega_2 \end{cases}$$

the corresponding eigenfunction which satisfy the conditions (2)-(4). If

$$f(x)=\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},\ f_1(x)=\begin{cases} f^{(1)}_1(x), & x\in\Omega_1 \\ f^{(2)}_1(x), & x\in\Omega_2 \end{cases},$$
$$f_2(x)=\begin{cases} f^{(1)}_2(x), & x\in\Omega_1 \\ f^{(2)}_2(x), & x\in\Omega_2 \end{cases},$$

$f\in H$, and

$$\alpha^2_{m,b}=\int_a^c\left(\left(\phi^{(1)}_{m,b1}(x)\right)^2+\left(\phi^{(1)}_{m,b2}(x)\right)^2\right)dx$$
$$+\,\delta\int_c^b\left(\left(\phi^{(2)}_{m,b1}(x)\right)^2+\left(\phi^{(2)}_{m,b2}(x)\right)^2\right)dx.$$

Then we have

$$\|f\|^2_H=\int_a^c\left(\left|f^{(1)}_1(x)\right|^2+\left|f^{(1)}_2(x)\right|^2\right)dx$$
$$+\,\delta\int_c^b\left(\left|f^{(2)}_1(x)\right|^2+\left|f^{(2)}_2(x)\right|^2\right)dx$$

$$(4.5)\quad =\sum_{m=-\infty}^{\infty}\frac{1}{\alpha^2_{m,b}}\left\{\begin{array}{c} \int_a^c\left(f^{(1)}_1(x)\phi^{(1)}_{m,b1}(x)+f^{(1)}_2(x)\phi^{(1)}_{m,b2}(x)\right)dx \\ +\delta\int_c^b\left(f^{(2)}_1(x)\phi^{(2)}_{m,b1}(x)+f^{(2)}_2(x)\phi^{(2)}_{m,b2}(x)\right)dx \end{array}\right\}^2.$$

which is called *the Parseval equality*.

Now, let us define the nondecreasing step function ϱ_b on $\mathbb{R}$ by

$$\varrho_b(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_{m,b}<0}\frac{1}{\alpha^2_{m,b}}, & \text{for } \lambda\le 0 \\ \sum_{0\le\lambda_{m,b}<\lambda}\frac{1}{\alpha^2_{m,b}}, & \text{for } \lambda\ge 0. \end{cases}$$

Then equalities (4.5) can be written as

$$\|f\|_H^2 = \int_{-\infty}^{\infty} |F(\lambda)|^2 \, d\varrho_b(\lambda), \tag{4.6}$$

where

$$F(\lambda) = \int_a^c \left(f_1^{(1)}(x)\phi_{m,b1}^{(1)}(x) + f_2^{(1)}(x)\phi_{m,b2}^{(1)}(x) \right) dx + \delta \int_c^b \left(f_1^{(2)}(x)\phi_{m,b1}^{(2)}(x) + f_2^{(2)}(x)\phi_{m,b2}^{(2)}(x) \right) dx.$$

By letting $b \to \infty$, we have the Parseval equality for the problem (2.1), (2.2), (2.4).

Now, we will give a definition.

A function f defined on an interval $[a,b]$ is said to be of *bounded variation* if there is a constant $C > 0$ such that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq C$$

for every partition

$$a = x_0 < x_1 < \ldots < x_n = b \tag{4.7}$$

of $[a,b]$ by points of subdivision $x_0, x_1, \ldots, x_n$.

Let f be a function of bounded variation. Then, by the *total variation* of f on $[a,b]$, denoted by $\bigvee_a^b (f)$, we mean the quantity

$$\bigvee_a^b (f) := \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|,$$

where the least upper bound is taken over all (finite) partitions (4.7) of the interval $[a,b]$ (see [18]).

Lemma 4.1. *For any positive N, there is a positive constant $\Upsilon = \Upsilon(N)$ not depending on b such that*

$$\bigvee_{-N}^{N} \{\varrho_b(\lambda)\} = \sum_{-N \leq \lambda_{m,b} < N} \frac{1}{\alpha_{m,b}^2} = \varrho_b(N) - \varrho_b(-N) < \Upsilon. \tag{4.8}$$

Proof. Let $\sin\beta \neq 0$. Since $\varphi_1^{(1)}(x,\lambda)$ is continuous on the region

$$\{(x,\lambda) : -N \leq \lambda \leq N, \ a \leq x \leq c\},$$

by condition $\varphi_1^{(1)}(a,\lambda) = \sin\beta$, there is a positive number k and near by a such that

$$\frac{1}{k^2}\left(\int_a^k \varphi_1^{(1)}(x,\lambda)dx\right)^2 > \frac{1}{2}\sin^2\beta. \tag{4.9}$$

Let us define $f_k(x) = \begin{pmatrix} f_{k1}(x) \\ f_{k2}(x) \end{pmatrix}$ by

$$f_{k1}(x) = 0,\ f_{k2}(x) = \begin{cases} \frac{1}{k}, & a \le x < k \\ 0, & x \ge k. \end{cases}$$

From (4.6), (4.8) and (4.9), we get

$$\begin{aligned} \int_a^k (f_{k1}^2(x) + f_{k2}^2(x))dx = \frac{k-a}{k^2} &= \int_{-\infty}^{\infty}\left(\frac{1}{k}\int_a^k \phi_1^{(1)}(x,\lambda)dx\right) d\varrho_b(\lambda) \\ &\ge \int_{-N}^{N}\left(\frac{1}{k}\int_a^k \phi_1^{(1)}(x,\lambda)dx\right)^2 d\varrho_b(\lambda) \\ &> \frac{1}{2}\sin^2\beta\left\{\varrho_b(N) - \varrho_b(-N)\right\}, \end{aligned}$$

which proves the inequality (4.8).

If $\sin\beta = 0$, then we define the function $f_k(x) = \begin{pmatrix} f_{k1}(x) \\ f_{k2}(x) \end{pmatrix}$ by the formula

$$f_{k1}(x) = \begin{cases} \frac{1}{k^2}, & a \le x < k \\ 0, & x \ge k \end{cases},\quad f_{k2}(x) = 0.$$

So, we obtain the inequality (4.8) by applying the Parseval equality. □

Now, we will obtain an expansion into a Fourier series of resolvent if one knows the expansion of the function $f(.)$.

By integration by parts, we find

$$\begin{aligned}
&\int_a^c \left(\tau(y^{(1)}), \varphi_{m,b}^{(1)}(x)\right)_E dx + \delta \int_c^b \left(\tau(y^{(2)}), \varphi_{m,b}^{(2)}(x)\right)_E dx \\
&= \int_a^c \left[-y_2^{(1)\prime}(x) + p(x)y_1^{(1)}\right] \varphi_{m,b1}^{(1)}(x)dx \\
&\quad + \delta \int_c^b \left[-y_2^{(2)\prime}(x) + p(x)y_1^{(2)}\right] \varphi_{m,b1}^{(2)}(x)dx \\
&\quad + \int_a^c \left[y_1^{(1)\prime}(x) + r(x)y_2^{(1)}\right] \varphi_{m,b2}^{(1)}(x)dx \\
&\quad + \delta \int_c^b \left[y_1^{(2)\prime}(x) + r(x)y_2^{(2)}\right] \varphi_{m,b2}^{(2)}(x)dx \\
&= \int_a^c \left[-\varphi_{m,b2}^{(1)\prime}(x) + p(x)\varphi_{m,b1}^{(1)}\right] y_2^{(1)} dx \\
&\quad + \delta \int_c^b \left[-\varphi_{m,b2}^{(2)\prime}(x) + p(x)\varphi_{m,b1}^{(2)}\right] y_2^{(2)} dx \\
&\quad + \int_a^c \left[\varphi_{m,b1}^{(1)\prime}(x) + r(x)\varphi_{m,b2}^{(1)}\right] y_1^{(1)} dx \\
&\quad + \delta \int_c^b \left[-\varphi_{m,b1}^{(2)\prime}(x) + p(x)\varphi_{m,b1}^{(2)}\right] y_1^{(2)} dx \\
&= \lambda_{m,b} \int_a^c \left(y^{(1)}(x,\lambda), \varphi_{m,b}^{(1)}(x)\right)_E dx \\
&\quad + \lambda_{m,b}\delta \int_c^b \left(y^{(2)}(x,\lambda)\varphi_{m,b}^{(2)}(x)\right)_E dx \\
&= \lambda_{\ m,b}\gamma_m(\lambda) \ \ (m \in \mathbb{Z}),
\end{aligned}$$

where

$$\gamma_m(\lambda) = \int_a^c \left(y^{(1)}(x,\lambda), \varphi_{m,b}^{(1)}(x)\right)_E dx + \delta \int_c^b \left(y^{(2)}(x,\lambda)\varphi_{m,b}^{(2)}(x)\right)_E dx \ (m \in \mathbb{Z}).$$

Set

$$y(x,\lambda) = \sum_{m=-\infty}^{\infty} \gamma_m(\lambda)\, \varphi_{m,b}(x),$$

$$c_m = \int_a^c \left(f^{(1)}(x), \varphi_{m,b}^{(1)}(x)\right)_E dx + \delta \int_c^b \left(f^{(2)}(x), \varphi_{m,b}^{(2)}(x)\right)_E dx \ (m \in \mathbb{Z}).$$

Since $y(x,\lambda)$ satisfies the system $\tau\left(y\right)=\lambda y+f\ (x\in\Omega,\ f\in H)$ and conditions (2.2)-(2.4), we get

$$\begin{aligned} c_m &= \int_a^c (f^{(1)}(x),\varphi^{(1)}_{m,b}(x))_E dx+\delta\int_c^b (f^{(2)}(x),\varphi^{(2)}_{m,b}(x))_E dx \\ &= \int_a^c \left(\tau\left(y^{(1)}\right)(x),\varphi^{(1)}_{m,b}(x)\right)_E dx+\delta\int_c^b \left(\tau\left(y^{(2)}(x)\right),\varphi^{(2)}_{m,b}(x)\right)_E dx \\ &\quad -\lambda\int_a^c (y^{(1)}(x),\varphi^{(1)}_{m,b}(x))_E dx-\lambda\delta\int_c^b (y^{(2)}(x),\varphi^{(2)}_{m,b}(x))_E dx \\ &= \left(\lambda_{m,b}-\lambda\right)\gamma_m\left(\lambda\right)\ \ (m\in\mathbb{Z}). \end{aligned}$$

Then, we obtain

$$\gamma_m\left(\lambda\right)=\frac{c_m}{\lambda_{m,b}-\lambda},\ \ (m\in\mathbb{Z}).$$

and

$$y(x,\lambda)=\int_a^b G_b\left(x,t,\lambda\right)f\left(t\right)dt=\sum_{m=-\infty}^{\infty}\frac{c_m}{\lambda_{m,b}-\lambda}\varphi_{m,b}(x).$$

Hence, the expansion of the resolvent is

$$\left(R_b f\right)(x,z)=\sum_{m=-\infty}^{\infty}\frac{\varphi_{m,b}(x)\left\langle f\left(.\right),\varphi_{m,b}\left(.\right)\right\rangle_H}{\alpha^2_{m,b}\left(\lambda_{m,b}-z\right)} \tag{4.10}$$

$$=\int_{-\infty}^{\infty}\frac{\varphi(x,\lambda)}{\lambda-z}\left\langle f\left(.\right),\varphi\left(.\right)\right\rangle_H d\varrho_b\left(\lambda\right). \tag{4.11}$$

Lemma 4.2. *Let z be a non real number and x be a fixed number. Then we have*

$$\int_{-\infty}^{\infty}\left\|\frac{\varphi(x,\lambda)}{\lambda-z}\right\|^2_E d\varrho_b\left(\lambda\right)<K. \tag{4.12}$$

Proof. Putting $f(x)=\varphi_{m,b}(x)$ in (4.10), we get

$$\frac{1}{\alpha_{m,b}}\int_a^b G_b\left(x,t,z\right)\varphi_{m,b}\left(t\right)dt=\frac{\varphi_{m,b}(x)}{\alpha_{m,b}\left(\lambda_{m,b}-z\right)}, \tag{4.13}$$

since the eigenfunctions $\varphi_{m,b}(x)$ are orthogonal. Using (4.13), if we apply the Parseval equality to $G_b\left(x,t,z\right)$, we have

$$\begin{aligned} \int_a^b \left\|G_b\left(x,t,z\right)\right\|^2_E dx &= \sum_{m=-\infty}^{\infty}\frac{\left\|\varphi_{m,b}(x)\right\|^2_E}{\alpha^2_{m,b}\left|\lambda_{m,b}-z\right|^2} \\ &= \int_{-\infty}^{\infty}\left\|\frac{\varphi(x,\lambda)}{\lambda-z}\right\|^2_E d\varrho_b\left(\lambda\right). \end{aligned}$$

Since the last integral convergent, the statement of lemma follows. ∎

5. Integral representations for the resolvent operator in singular case

In this section, we will obtain integral representations for the resolvent in singular case.

Now, we recall that the following well-known theorems of Helly's.

Theorem 5.1 ([18]). *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real nondecreasing function on a finite interval $a \leq \lambda \leq b$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda),\ a \leq \lambda \leq b.$$

Theorem 5.2 ([18]). *Assume $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded, sequence of nondecreasing function on a finite interval $a \leq \lambda \leq b$, and suppose*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda),\ a \leq \lambda \leq b.$$

If f is any continuous function on $a \leq \lambda \leq b$, then

$$\lim_{n\to\infty} \int_a^b f(\lambda)\, dw_n(\lambda) = \int_a^b f(\lambda)\, dw(\lambda).$$

By Lemma 3, the set$\{\varrho_b(\lambda)\}$ is bounded. Using Theorems 5 and 6, we can find a sequence $\{b_k\}$ such that the function $\varrho_{b_k}(\lambda)$ converge to a monotone function $\varrho(\lambda)$. Then we have a lemma.

Lemma 5.3. *Let z be a non real number, and let $\varphi(x,\lambda)$ be as in (2.6). Then we have*

$$\int_{-\infty}^{\infty} \left\| \frac{\varphi(x,\lambda)}{\lambda - z} \right\|_E^2 d\varrho(\lambda) \leq K, \tag{5.1}$$

where x be a fixed number.

Proof. By the inequality (4.12), for arbitrary $\eta > 0$, we have

$$\int_{-\eta}^{\eta} \left\| \frac{\varphi(x,\lambda)}{\lambda - z} \right\|_E^2 d\varrho_b(\lambda) < K.$$

Letting $\eta \to \infty$ and $b \to \infty$, we get the desired result. ∎

Lemma 5.4. *For arbitrary $\eta > 0$, we have the following inequalities.*

$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{\lambda^2} < \infty,\ \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{\lambda^2} < \infty. \tag{5.2}$$

Proof. Since $\|\varphi_{m,b}(a,\lambda)\|_E^2 \neq 0$, putting $x = a$ in (5.1), we get

$$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|\lambda - z|^2} < \infty,$$

and the statement of lemma follows. ∎

Lemma 5.5. *Let $f(.) \in \mathcal{H}$, and let*

$$(Rf)(x,z) = \int_a^\infty G(x,t,z)\, f(t)\, dt,$$

where

$$G(x,t,z) = \begin{cases} \chi(x,z)\,\varphi^T(t,z), & t \le x,\ x \ne c,\ t \ne c \\ \varphi(x,z)\,\chi^T(t,z), & t > x,\ x \ne c,\ t \ne c. \end{cases}$$

Then

$$\int_a^c \|(Rf)(x,z)\|_E^2\, dx + \delta \int_c^\infty \|(Rf)(x,z)\|_E^2\, dx$$
$$\le \frac{1}{v^2}\left(\int_a^c \|f(x)\|_E^2\, dx + \delta \int_c^\infty \|f(x)\|_E^2\, dx\right),\ z = u + iv.$$

Proof. For each $b > c$, it follows from (4.10) and the Parseval equality that

$$\int_a^c \|(R_b f)(x,z)\|_E^2\, dx + \delta \int_c^b \|(R_b f)(x,z)\|_E^2\, dx$$
$$= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,b}^2 |\lambda_{m,b} - z|^2} \left\{\langle f(.), \varphi_{m,b}(.)\rangle_H\right\}^2$$
$$\le \frac{1}{v^2} \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,b}^2} \left\{\langle f(.), \varphi_{m,b}(.)\rangle_H\right\}^2$$
$$= \frac{1}{v^2}\left(\int_a^c \|f(x)\|_E^2\, dx + \delta \int_c^b \|f(x)\|_E^2\, dx\right).$$

Letting $b \to \infty$, we get the desired result. ∎

The main result of the paper is the following theorem.

Theorem 5.6. *For every nonreal z and for each $f(.) \in \mathcal{H}$, one has the following equality*

$$(Rf)(x,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F(\lambda)\, d\varrho(\lambda), \tag{5.3}$$

where

$$F(\lambda) = \int_a^c (f(x), \varphi^{(1)}(x,\lambda))_E dx + \delta \lim_{\xi\to\infty} \int_c^\xi (f(x), \varphi^{(2)}(x,\lambda))_E dx,$$

and $\varphi(x,\lambda)$ is defined by (2.6).

Proof. Let ς be an arbitrary positive number and the real-valued function

$$f_\xi(x)=\begin{pmatrix} f_{\xi1}(x)\\ f_{\xi2}(x)\end{pmatrix},\ f_{\xi1}(x)=\begin{cases} f_{\xi1}^{(1)}(x), & x\in[a,c)\\ f_{\xi1}^{(2)}(x), & x\in(c,\xi]\end{cases},$$
$$f_{\xi2}(x)=\begin{cases} f_{\xi2}^{(1)}(x), & x\in[a,c)\\ f_{\xi2}^{(2)}(x), & x\in(c,\xi]\end{cases}$$

satisfies the following conditions:

1) $f_\xi(x)$ vanishes outside the set $[a,c)\cup(c,\xi]$, $\xi>b$.
2) The function $f_\xi(x)$ has a continuous derivative.
3) $f_\xi(x)$ satisfy the conditions (2.2)-(2.4).

Set

$$F_\xi(\lambda)=\int_a^c(f_\xi^{(1)}(x),\varphi^{(1)}(x,\lambda))_E dx+\delta\int_c^\xi(f_\xi^{(2)}(x),\varphi^{(2)}(x,\lambda))_E dx.$$

From (4.11), we get

$$\begin{aligned}
&(R_bf_\xi)(x,z)\\
&=\int_{-\infty}^{\infty}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)\,d\varrho_b(\lambda)=\int_{-\infty}^{-\varsigma}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)\,d\varrho_b(\lambda)\\
&\quad+\int_{-\varsigma}^{\varsigma}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)\,d\varrho_b(\lambda)+\int_{\varsigma}^{\infty}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)\,d\varrho_b(\lambda)\\
&=I_1+I_2+I_3.
\end{aligned}\tag{5.4}$$

Now, we will estimate I_1. By (4.10), we get

$$\begin{aligned}
|I_1|&=\left|\int_{-\infty}^{-\varsigma}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)\,d\varrho_b(\lambda)\right|\\
&=\left|\sum_{\lambda_{k,b}<-\varsigma}\frac{\varphi_{k,b}(x)}{\alpha_{k,b}^2(\lambda_{k,b}-z)}\left\{\begin{array}{c}\int_a^c\left(f_\xi^{(1)}(x),\varphi_{k,b}^{(1)}(x)\right)_E dx\\ +\delta\int_c^\xi\left(f_\xi^{(2)}(x),\varphi_{k,b}^{(2)}(x)\right)_E dx\end{array}\right\}\right|\\
&\le\left(\sum_{\lambda_{k,b}<-\varsigma}\frac{\|\varphi_{k,b}(x)\|_E^2}{\alpha_{k,b}^2|\lambda_{k,b}-z|^2}\right)^{1/2}\\
&\quad\times\left(\sum_{\lambda_{k,b}<-\varsigma}\frac{1}{\alpha_{k,b}^2}\left[\begin{array}{c}\int_a^c\left(f_\xi^{(1)}(x),\varphi_{k,b}^{(1)}(x)\right)_E dx\\ +\delta\int_c^\xi\left(f_\xi^{(2)}(x),\varphi_{k,b}^{(2)}(x)\right)_E dx\end{array}\right]^2\right)^{1/2}.
\end{aligned}\tag{5.5}$$

By integration by parts, we find

$$\begin{aligned}
&\int_a^c \left(f_\xi^{(1)}(x),\varphi_{k,b}^{(1)}(x,\lambda)\right)_E dx+\delta\int_c^\xi \left(f_\xi^{(2)}(x),\varphi_{k,b}^{(2)}(x,\lambda)\right)_E dx \\
&=\frac{1}{\lambda_{k,b}}\left\{\begin{array}{c}\int_a^c f_{\xi1}^{(1)}(x)\left\{-\varphi_{k,b2}^{(1)\prime}(x)+p(x)\varphi_{k,b1}^{(1)}(x)\right\}dx\\ +\delta\int_c^\xi f_{\xi1}^{(2)}(x)\left\{-\varphi_{k,b2}^{(2)\prime}(x)+p(x)\varphi_{k,b1}^{(2)}(x)\right\}dx\end{array}\right\} \\
&\quad+\frac{1}{\lambda_{k,b}}\left\{\begin{array}{c}\int_a^c f_{\xi2}^{(1)}(x)\left\{\varphi_{k,b1}^{(1)\prime}(x)+r(x)\varphi_{k,b2}^{(1)}(x)\right\}dx\\ +\delta\int_c^\xi f_{\xi2}^{(2)}(x)\left\{\varphi_{k,b1}^{(2)\prime}(x)+r(x)\varphi_{k,b2}^{(2)}(x)\right\}dx\end{array}\right\} \\
&=\frac{1}{\lambda_{k,b}}\left\{\begin{array}{c}\int_a^c \varphi_{k,b1}^{(1)}(x)\left\{-f_{\xi2}^{(1)\prime}(x)+p(x)f_{\xi1}^{(1)}(x)\right\}dx\\ +\delta\int_c^\xi \varphi_{k,b1}^{(2)}(x)\left\{-f_{\xi2}^{(2)\prime}(x)+p(x)f_{\xi1}^{(2)}(x)\right\}dx\end{array}\right\} \\
&\quad+\frac{1}{\lambda_{k,b}}\left\{\begin{array}{c}\int_a^c \varphi_{k,b2}^{(1)}(x)\left\{f_{\xi1}^{(1)\prime}(x)+r(x)f_{\xi2}^{(1)}(x)\right\}dx\\ +\delta\int_c^\xi \varphi_{k,b2}^{(2)}(x)\left\{f_{\xi1}^{(2)\prime}(x)+r(x)f_{\xi2}^{(2)}(x)\right\}dx\end{array}\right\}.
\end{aligned}\tag{5.6}$$

By Lemma 4, we have

$$|I_1|\le\frac{K^{1/2}}{\varsigma^2}\left(\sum_{\lambda_{k,b}<-\varsigma}\frac{1}{\alpha_{k,b}^2}\left[\begin{array}{c}\int_a^c\left(h_\xi(x),\varphi_{k,b}(x)\right)_E dx\\ +\delta\int_c^\xi\left(h_\xi(x),\varphi_{k,b}(x)\right)_E dx\end{array}\right]^2\right)^{1/2},$$

where

$$h_\xi(t)=\begin{pmatrix}-f_{\xi2}'(x)+p(x)f_{\xi1}(x)\\ f_{\xi1}'(x)+r(x)f_{\xi2}(x)\end{pmatrix}.$$

Using Bessel inequality, we get

$$|I_1|\le\frac{K^{1/2}}{\varsigma}\left[\int_a^c\|h_\xi(x)\|_E^2\,dx+\delta\int_c^\xi\|h_\xi(x)\|_E^2\,dx\right]^{1/2}=\frac{C_1}{\varsigma}.$$

By similar method, one can prove that $|I_3|\le\frac{C_2}{\varsigma}$. Then I_1 and I_3 tend to zero as $\varsigma\to\infty$, uniformly in b. Using Theorems 5 and 6 in (5.4), we obtain

$$(Rf_\xi)(x,z)=\int_{-\infty}^{\infty}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)\,d\varrho(\lambda).\tag{5.7}$$

As is known, if $f(.)\in\mathcal{H}$, then one can find a sequence $\{f_\xi(x)\}_{\xi=1}^{\infty}$ which satisfy the previous conditions and tend to $f(x)$ as $\xi\to\infty$. From the Parseval equality, the sequence of Fourier transform converges to the transform of $f(x)$. By Lemmas 7 and 9, we can pass to the limit $\xi\to\infty$ in (5.7). So, we obtain the assertion of the theorem. ∎

6. The formulas for the Titchmarsh-Weyl function and the spectral function

In this section, we will derive formulas for the Titchmarsh-Weyl function $m(z)$ and the spectral function $\varrho(\lambda)$ with the help of the integral representation of the resolvent.

Firstly, we recall the Stieltjes inversion formula. Let $\sigma(\lambda) = \sigma_1(\lambda) + i\sigma_2(\lambda)$ be a complex valued function of bounded variation on the entire line. We put

$$\varphi(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{z-\lambda},\ \psi(\nu,\tau) = \frac{sgn\tau}{\pi}\frac{\varphi(z)-\varphi(\overline{z})}{2i}$$
$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\tau|\, d\sigma(\lambda)}{(\lambda-\nu)^2+\tau^2},\ z=\nu+i\tau.$$

Theorem 6.1 ([20]). *If the points a,b are points of continuity of $\sigma(\lambda)$, then we have*

$$\sigma(b)-\sigma(a) = -\lim_{\tau\to 0}\int_a^b \psi(\nu,\tau)\, d\nu.$$

Theorem 6.2. *(i) For any nonreal z, one has*

$$m(z)-m(z_0) = \int_{-\infty}^{\infty}\left[\frac{1}{\lambda-z}-\frac{1}{\lambda-z_0}\right] d\varrho(\lambda),\ (\operatorname{Im} z_0 \neq 0). \tag{6.1}$$

(ii) If λ and μ are points of continuity of $\varrho(\lambda)$, then one has

$$\varrho(\lambda)-\varrho(\mu) = -\frac{1}{\pi}\lim_{\tau\to 0}\int_\mu^\lambda \operatorname{Im}\{m(\nu+i\tau)\}\, d\nu,\ z=\nu+i\tau,\ \tau>0. \tag{6.2}$$

Proof. (i) Since $f(x)$ is arbitrary, it follows from (5.3) that

$$G(t,u,z) = \int_{-\infty}^{\infty} \frac{\varphi(t,\lambda)\varphi^T(u,\lambda)}{\lambda-z} d\varrho(\lambda).$$

Hence,

$$G(t,u,z)-G(t,u,z_0) = \int_{-\infty}^{\infty} \varphi(t,\lambda)\varphi^T(u,\lambda)\left[\frac{1}{\lambda-z}-\frac{1}{\lambda-z_0}\right] d\varrho(\lambda). \tag{6.3}$$

Since there are matrices on both sides in (6.3), their corresponding elements are equal. So, using (4.1) and the definition of the product $\varphi(t,\lambda)\varphi^T(u,\lambda)$, and then putting $t=u=a$ and taking the initial conditions (2.7), we get

$$\{\cos\beta + m(z)\sin\beta\}\sin\beta - \{\cos\beta + m(z_0)\sin\beta\}\sin\beta$$
$$= \int_{-\infty}^{\infty} \sin^2\beta\left[\frac{1}{\lambda-z}-\frac{1}{\lambda-z_0}\right] d\varrho(\lambda),$$

i.e.,

$$m(z) - m(z_0) = \int_{-\infty}^{\infty} \left[\frac{1}{\lambda - z} - \frac{1}{\lambda - z_0} \right] d\varrho(\lambda).$$

(ii) From (6.1), we get

$$\begin{aligned} \psi(\nu, \tau) &= \frac{sgn\tau}{\pi} \frac{m(z) - m(\overline{z})}{2i} \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\tau| \, d\varrho(\lambda)}{(\lambda - \nu)^2 + \tau^2}. \end{aligned}$$

By Theorem 11, we have

$$\varrho(\lambda) - \varrho(\mu) = -\lim_{\tau \to 0} \int_{\mu}^{\lambda} \psi(\nu, \tau) \, d\nu. \tag{6.4}$$

Since $m(\overline{z}) = \overline{m(z)}$, it follows that

$$\psi(\nu, \tau) = \frac{sgn\tau}{\pi} \frac{m(z) - m(\overline{z})}{2i} = \frac{sgn\tau}{\pi} \operatorname{Im}\{m(z)\}. \tag{6.5}$$

For $\tau > 0$, we obtain the formula (6.2) using (6.4) and (6.5). Thus, the theorem is proved. ∎

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Rezolventni operator singularnog Diracovog sustava s uvjetima transmisije

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Sažetak. U ovom članku se proučava rezolventni operator jednodimenzionalnog singularnog Diracovog operatora s uvjetima transmisije. Proučava se Titchmarsh-Weylova funkcija ovog problema. Također se konstruiraju Greenova funkcija i spektralna funkcija za regularni i singularni problem. Pomoću tih funkcija dobiva se razvoj u Fourierov red rezolvente u regularnom slučaju. Nadalje, dajemo integralnu reprezentaciju u terminima spektralne funkcije za rezolventu ovog operatora s uvjetima transmisije u singularnom slučaju. Naposljetku, dobivamo formulu za Titchmarsh-Weylovu funkciju u terminima spektralne funkcije singularnog Diracovog sustava.

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