GEOFIZIKA	Vol. 3	1986

Original scientific paper
UDC 550.342

Some properties of partial derivatives of the Love wave dispersion function

Anđelka Milošević

Geophysical Institute, Faculty of Science, University of Zagreb
Yugoslavia

Received 21 July, 1986, in final form 10 November, 1986.

The paper is systematically presents how, by applying the implicit function theorem, the problem of determining partial derivatives of the phase and group velocities of the Love surface wave is reduced to the derivation of the dispersion function with respect to independent variables. In addition to that, two relations are derived which are valid among the partial derivatives of the dispersion function of the Love wave phase velocity in a $n + 1$ layered solid medium according to independent variables. It is shown that these relations serve to check the numerical values of the partial derivatives, and they can be used for checking computer programmes of the linear inversion structure calculation.

Neka svojstva parcijalnih derivacija funkcije disperzije Loveova vala

Sustavno je izneseno kako se, primjenom teorema implicitne funkcije, problem određivanja parcijalnih derivacija fazne i grupne brzine Loveova površinskog vala svodi na deriviranje funkcije disperzije po nezavisnim varijablama. Osim toga, u radu su izvedene dvije relacije koje vrijede između parcijalnih derivacija funkcije disperzije fazne brzine Loveova vala u $n + 1$ slojnom čvrstom sredstvu po nezavisnim varijablama. Pokazano je da te relacije služe za kontrolu numeričkih vrijednosti parcijalnih derivacija, te se mogu upotrijebiti za provjeru programa linearne inverzije strukture na elektroničkom računalu.

1. Introduction

In order to solve the inverse problem of the structure, i. e. to determine the improved parameters of the initial Earth model from the empiric dispersion of phase or group velocities of surface seismic wave propagation, one should first establish the rectangular of coefficients matrix in the correction equations system (Dorman and Ewing, 1962., Milošević, 1982). It means that the partial derivatives of the corresponding velocity in relation to the given model parameters for each observation period need to be determined. Besides, to make the model evaluation as reliable as possible, in sense of the linear theory of inversion, partial derivatives have to be determined as precisely as possible. In the present investigation the implicit function method is applied for the determination of partial derivatives.

2. Love wave dispersion function in a $n + 1$ layered medium

This paper deals with a case of Love waves propagating in a solid medium where, above a homogeneous and isotropic half space (marked with the index $n + 1$) there are n welded flat surface homogeneous and isotropic layers. Mark j is the layer index, where j can take values from $1, 2, \ldots, n + 1$, counting from the free surface towards the inside of the solid medium. The thickness of the j^{th} layer is H_j*, and the physical parameters of the layer are μ_j (rigidity), ρ_j (density) and v_{tj} (transverse wave velocity), c is the phase velocity, ω is the circular frequency, and k – the wave number.

Taking the above symbols, the wave number is

$$k = \frac{\omega}{c} \tag{1}$$

and the rigidity becomes

$$\mu_j = \rho_j v_{tj}^2 \quad , \qquad j = 1, 2, \ldots, n + 1 \tag{2}$$

Further, with the notations

$$Q_j = kH_j s_j \quad , \qquad j = 1, 2, \ldots, n \tag{3}$$

and

$$Q_j' = kH_j r_j \quad , \qquad j = 2, 3, \ldots, n \tag{4}$$

where

$$s_j = \begin{cases} s_j = +\sqrt{\dfrac{c^2}{v_{tj}^2} - 1} \text{ , when } c > v_{tj} \text{ , } j = 1, 2, \ldots, n & \text{(5a)} \\[2ex] ir_j = +\, i\sqrt{1 - \dfrac{c^2}{v^2_{tj}}} \text{ , when } c < v_{tj} \text{ , } j = 2, 3, \ldots, n + 1 \text{ ,} & \text{(5b)} \end{cases}$$

and "i" is the imaginary unit.

* Thickness of the last $n + 1$ layer – half space $H_{n+1} = \infty$.

In case of increase of the transverse waves velocity v_{tj}, $j = 1, 2, \ldots, n + 1$ with the depth, in order to determine the phase velocity c of Love wave as a function of the wave number k for a complete range of values of c (i.e. $v_{tn+1} > c > v_{t1}$, $c \neq v_{tj}$, $j = 1, 2, \ldots, n+1$, what is derived from the condition of Love wave existence (Milošević, 1982), n corresponding equations have to be used which can be presented in the matrix form (Schwab and Knopoff, 1972; Milošević, 1982)

$$\mathbf{m}_{n+1}\,\mathbf{Y}_n\,\mathbf{Y}_{n-1}\ldots\mathbf{Y}_2\begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}_1 = 0\,, \quad \text{for } v_{tn+1} > c > v_{tn}\,. \tag{6}$$

Here

$$\mathbf{m}_{n+1} = [\,\mu_j r_j \quad -1\,]\,, \qquad j = n+1 \tag{7}$$

is the row vector containing only the parameters of the lower rigid half space ($n+1$ layer); it is therefore marked with the subscript $n+1$;

$$\mathbf{Y}_j = \begin{bmatrix} \cos Q_j & -\dfrac{\sin Q_j}{\mu_j s_j} \\ \mu_j s_j \sin Q_j & \cos Q_j \end{bmatrix} \qquad \text{for } c > v_{tj}\,,\ j = 1, 2, \ldots, n \tag{8}$$

is the Thomson-Haskell matrix of the j^{th} layer, changing to

$$\mathbf{Y}_j' = \begin{bmatrix} \mathrm{ch}Q_j' & -\dfrac{\mathrm{sh}Q_j'}{\mu_j r_j} \\ -\mu_j r_j\,\mathrm{sh}Q_j' & \mathrm{ch}Q_j' \end{bmatrix} \qquad \text{for } c < v_{tj}\,,\ j = 2, 3, \ldots, n, \tag{9}$$

and

$$\begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}_1 = \begin{bmatrix} \cos Q_1 \\ \mu_1 s_1 \sin Q_1 \end{bmatrix} \tag{10}$$

is the vector column relating to the first column of the matrix $\mathbf{Y}_j$ for $j = 1$.

Equation (6) is the characteristic equation for the phase velocity c of the Love wave propagation in the $n+1$ layered medium. Taking the meaning of values shown by relation (1), (2), (3), (4), (5a) and (5b) in the expressions (7), (8), (9) and (10), it can be concluded from (6) that the velocity c is a function of the transverse wave velocity v_{tj} and density ρ_j in all $n+1$ media ($j = 1, 2, \ldots, n+1$), the thickness H_j in n layers ($j = 1, 2, \ldots, n$) and the circular frequency ω. Obviously, due to the nature of the functions appearing in that equation (products, sums, trigonometric functions of different arguments etc.), the phase velocity c cannot be explicitly expressed by an analitical expression.

Therefore, formally, the phase velocity function of the surface wave, generally for a solid medium with $n+1$ layers

$$c = c(\omega, p_j) \ , \quad j = 1, 2, \ldots, n+1 \tag{11}$$

is defined implicitly by the dispersion equation

$$L(\omega, p_j \ , \ c(\omega, p_j)) = 0 \ , \quad j = 1, 2, \ldots, n+1 \ , \tag{12}$$

where $L(\omega, p_j, c(\omega, p_j))$ is the dispersion function, ω is the circular frequency, and p_j are the medium parameters. For Love wave $p_j = p_j(v_{tj}, \rho_j, H_j)$, $j=1, 2, \ldots, n$ and $p_j = p_j(v_{tj}, \rho_j)$, $j = n+1$.

3. Partial derivatives of phase velocity

The differentiation of equation (12) with respect to the independent variables ω and p_j according to formulae for the implicit function differentation (Bronštejn and Semendjajev, 1964) yields:

$$\left. \frac{\partial L}{\partial \omega} + \frac{\partial L}{\partial c} \frac{\partial c}{\partial \omega} \right|_{p_j} = 0 \ , \tag{12a}$$

and

$$\left. \frac{\partial L}{\partial p_j} + \frac{\partial L}{\partial c} \frac{\partial c}{\partial p_j} \right|_{\omega} = 0 \ , \tag{12b}$$

where (12a) yields

$$\left. \frac{\partial c}{\partial \omega} \right|_{p_j} = - \frac{\dfrac{\partial L}{\partial \omega}}{\dfrac{\partial L}{\partial c}} = c_1(\omega, p_j) \ . \tag{13a}$$

Analogously, (12b) yields

$$\left. \frac{\partial c}{\partial p_j} \right|_{\omega} = - \frac{\dfrac{\partial L}{\partial p_j}}{\dfrac{\partial L}{\partial c}} = c_2(\omega, p_j) \ . \tag{13b}$$

Accordingly, for a definite wave type and the given Earth model (in this case Love wave in $n+1$ layered medium) the dispersion function is defined (given explicitly by the left side of the characteristic equation (6)) and its partial derivatives are determined. Then for given values of the model parameters and the circular frequency ω, the phase velocity is computed from the dispersion equation and its partial derivatives are computed on the basis of the formulae (13a) and (13b).

Thus, for the dispersion function on the left side of equation (6) according to (7), (8), (9) it can be written

$$\mathbf{m}_{n+1} = g_a(p_j, c(\omega,p_j)),\ p_j = p_j(v_{tj}, \rho_j),\ j = n+1\ ,$$

$$\mathbf{Y}_j = g_b(\omega, p_j, c(\omega, p_j)),\ p_j = p_j(v_{tj}, \rho_j, H_j),\ j = 1, 2, \dots, n,\ \text{for } c > v_{tj},$$

while

$$\mathbf{Y}_j{}' = g_c(\omega, p_j, c(\omega, p_j)), p_j = p_j(v_{tj}, \rho_j, H_j),\ j = 2, 3, \dots, n,\ \text{for } c < v_{tj}\ .$$

When the dispersion function is expressed in the matrix form (left side of equation (6)), the rule analogous to the one for derivation of a function product applies to the matrices product, while the matrix derivation is understood as matrix derived from all elements of the matrix (Anđelić, 1962).

Thus, for the discussed case it can be written:

$$\frac{\partial L}{\partial c} = \frac{\partial \mathbf{m}_{n+1}}{\partial c} \mathbf{Y}_n \mathbf{Y}_{n-1} \cdots \mathbf{Y}_2 \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}_1 + \mathbf{m}_{n+1} \frac{\partial \mathbf{Y}_n}{\partial c} \mathbf{Y}_{n-1} \cdots \mathbf{Y}_2 \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}_1 +$$

$$+ \mathbf{m}_{n+1} \mathbf{Y}_n \frac{\partial \mathbf{Y}_{n-1}}{\partial c} \cdots \mathbf{Y}_2 \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}_1 + \mathbf{m}_{n+1} \mathbf{Y}_n \mathbf{Y}_{n-1} \cdots \frac{\partial \mathbf{Y}_2}{\partial c} \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}_1 +$$

$$+ \mathbf{m}_{n+1} \mathbf{Y}_n \mathbf{Y}_{n-1} \cdots \mathbf{Y}_2 \begin{bmatrix} \frac{\partial y_{11}}{\partial c} \\ \frac{\partial y_{21}}{\partial c} \end{bmatrix}_1,$$

analogously for $\frac{\partial L}{\partial \omega}$, while

$$\frac{\partial L}{\partial p_j} = \frac{\partial \mathbf{m}_{n+1}}{\partial p_j} \mathbf{Y}_n \mathbf{Y}_{n-1} \cdots \mathbf{Y}_2 \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}_1,\ p_j = p_j(v_{tj}, \rho_j),\ j = n+1,$$

$$\frac{\partial L}{\partial p_j} = \mathbf{m}_{n+1} \cdots \frac{\partial \mathbf{Y}_j}{\partial p_j} \cdots \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}_1,\ p_j = p_j(v_{tj}, \rho_j, H_j),\ j = 2, 3, \dots, n,$$

$$\frac{\partial L}{\partial p_j} = \mathbf{m}_{n+1} \mathbf{Y}_n \mathbf{Y}_{n-1} \cdots \mathbf{Y}_2 \begin{bmatrix} \frac{\partial y_{11}}{\partial p_j} \\ \frac{\partial y_{21}}{\partial p_j} \end{bmatrix}_1,\ p_j = p_j(v_{tj}, \rho_j, H_j)\ ,\ j = 1\ .$$

Therefore, generally

$$\frac{\partial L}{\partial c} = f_a(\omega, p_j, c(\omega, p_j)) \;, \qquad (*)$$

$$\frac{\partial L}{\partial \omega} = f_b(\omega, p_j, c(\omega, p_j)) \qquad (**)$$

and

$$\frac{\partial L}{\partial p_j} = f_c(\omega, p_j, c(\omega, p_j)) \;, \qquad (***)$$

for

$p_j = p_j(v_{tj}, \rho_j, H_j)$, $j=1, 2, \ldots, n$, and $p_j = p_j(v_{tj}, \rho_j)$,

$j = n+1$ respectively.

4. Calculation of theoretical group velocity and its partial derivatives

The expression for the group velocity U is:

$$U = \frac{c}{1 - \frac{\omega}{c} \frac{\partial c}{\partial \omega}\Big|_{p_j}} \;, \qquad (14)$$

i.e. generally

$$U = U(\omega, c(\omega, p_j), \frac{\partial c}{\partial \omega}\Big|_{p_j}(\omega, p_j)) = U(\omega, p_j) \;. \qquad (15)$$

If the phase velocity c is derived from the dispersion equation for a definite frequency ω (or period T), for a definite wave type and for a given Earth model, then, according to the above description, the partial derivative of the phase velocity with respect to frequency (relation (13a)) can be computed by the implicit function method. Subsequently, the corresponding group velocity (relation (14)) can be computed.

Starting from function (15), by applying the implicit function theorem and the chain rule, it is easily proved that the group velocity derivatives with respect to the independent variables ω and p_j are:

$$\frac{\partial U}{\partial \omega}\Big|_{p_j} = \frac{U}{c\omega}\left(\frac{U}{c} - 1\right) + \frac{U}{c}\left(2 - \frac{U}{c}\right)\frac{\partial c}{\partial \omega}\Big|_{p_j} + \frac{U^2}{c^2}\,\omega\,\frac{\partial^2 c}{\partial \omega^2}\Big|_{p_j} \qquad (16)$$

and

$$\frac{\partial U}{\partial p_j}\Big|_{\omega} = \frac{U}{c}\left(2 - \frac{U}{c}\right)\frac{\partial c}{\partial p_j}\Big|_{\omega} + \frac{U^2}{c^2}\,\omega\,\frac{\partial}{\partial p_j}\left(\frac{\partial c}{\partial \omega}\Big|_{p_j}\right)\Big|_{\omega} \;. \qquad (17)$$

As-can be seen in relation (16) and (17), second partial derivatives of the phase velocity appear and can be determined by solving the relations (12a) and (12b) with regard to the independent variables ω and p_j, keeping in mind the functions (*), (**), (***), (13a) and (13b). The result is:

$$\frac{\partial^2 c}{\partial \omega^2} = - \frac{\frac{\partial^2 L}{\partial \omega^2} + 2 \frac{\partial^2 L}{\partial \omega \partial c} \frac{\partial c}{\partial \omega} + \frac{\partial^2 L}{\partial c^2} (\frac{\partial c}{\partial \omega})^2}{\frac{\partial L}{\partial c}} , \tag{18}$$

and

$$\frac{\partial^2 c}{\partial \omega \partial p_j} = \frac{\frac{\partial^2 L}{\partial \omega \partial p_j} + (\frac{\partial^2 L}{\partial \omega \partial c} + \frac{\partial^2 L}{\partial c^2} \frac{\partial c}{\partial \omega}) \frac{\partial c}{\partial p_j} + \frac{\partial^2 L}{\partial c \partial p_j} \frac{\partial c}{\partial \omega}}{\frac{\partial L}{\partial c}} , \tag{19}$$

where

$p_j = p_j(v_{tj}, \rho_j, H_j), j = 1, 2, \ldots, n$, and $p_j = p_j(v_{tj}, \rho_j)$, $j = n+1$ respectively.

In other words, to compute partial derivatives of the group velocity, the second derivatives of the phase velocity (formulae (18) and (19)) must be known. In order to achieve this, when applying the implicit function theorem, the second derivatives of the dispersion function $L = L(\omega, p_j, c(\omega, p_j))$ with respect to the independent variables ω and p_j must be determined first.

This results in the conclusion that in case of applying the implicit function theorem, the problem of determining partial derivatives of the phase velocity c and group velocity U is reduced to the derivation of the dispersion function. Some properties of partial derivatives of Love wave dispersion function (in $n+1$ solid medium according to independent variables) will be presented in the following.

5. The property of mixed partial second derivatives of the dispersion function

It is well known (Novotný, 1970), and can easily be proved that for instance

$$\frac{\partial^2 L}{\partial \omega \partial H_j} = \frac{\partial^2 L}{\partial H_j \partial \omega} , \quad j = 1, 2, \ldots, n \quad ,$$

which means that the second partial derivatives of the dispersion function are independent of the order of differentiation. This is analogous for other combinations of mixed second partial derivatives of the dispersion function.

This property was therefore used to control the analytical expressions obtained.

Due to these properties of the Love wave dispersion function it is found:

$$\frac{\partial^2 c}{\partial p_j \partial \omega} = \frac{\partial^2 c}{\partial \omega \partial p_j} \quad ,$$

for $p_j = p_j(v_{tj}, \rho_j, H_j)$, $j=1,2,\ldots,n$, and $p_j = p_j(v_{tj}, \rho_j)$, $j=n+1$ respectively.

(The same is valid for the group velocity function.)

6. Another property of partial derivatives of the Love wave dispersion function according to independent variables

The derivation of expressions for partial derivatives of Love wave phase and group velocity leads to some other conclusions: it will be shown that among partial derivatives of the Love wave dispersion function $L = L(\omega, p_j, c(\omega, p_j))$ according to independent variables for a n-layered solid medium above a solid half space $(n+1)$, the following relations are true:

$$c\frac{\partial L}{\partial c} = \sum_{j=1}^{n+1}\left(2\rho_j\frac{\partial L}{\partial \rho_j} - v_{tj}\frac{\partial L}{\partial v_{tj}}\right) - \sum_{j=1}^{n} H_j\frac{\partial L}{\partial H_j} \tag{20}$$

and

$$\omega\frac{\partial L}{\partial \omega} = \sum_{j=1}^{n} H_j\frac{\partial L}{\partial H_j} \quad . \tag{21}$$

The above relationships are arrived at by applying the implicit function theorem and the chain rule:

Starting from the Love wave dispersion function $L = L(\mu_j, s_j, r_j, Q_j)$ in a $n+1$ layered medium (left side of equation (6)), we have according to (2), (3), (5a) and (5b)

$$\begin{aligned}
\mu_j &= \mu_j(\rho_j, v_{tj}), & & j=1,2,\ldots,n+1 \quad , \\
s_j &= s_j(v_{tj}, c), & c > v_{tj}, \quad & j=1,2,\ldots,n \quad , \\
r_j &= r_j(v_{tj}, c), & c < v_{tj}, \quad & j=n+1\ ^{*} \quad ,
\end{aligned}$$

and

$$Q_j = Q_j(k(\omega, c), H_j, s_j(v_{tj}, c)) , \quad j=1,2,\ldots,n\ ^{**} \quad ,$$

where again according to (1)

$$k = k(\omega, c)$$

---

* Generally for the variable r_j (relation (5b)), $j = 2, 3, \ldots, n+1$.

** Analogously, from (4), the same is valid for the variable Q_j', i.e. $Q_j' = Q_j'(k(\omega, c), H_j, r_j(v_{tj}, c))$, $j = 2, 3, \ldots, n$.

First we define partial derivatives of the above variables according to their independent variables

$$\frac{\partial \mu_j}{\partial \rho_j} = \frac{v_{tj}^2 \rho_j}{\rho_j} \qquad \text{i.e.} \quad \rho_j \frac{\partial \mu_j}{\partial \rho_j} = \mu_j \,, \quad j=1,2,\ldots,n+1 \tag{22}$$

and

$$\frac{\partial \mu_j}{\partial v_{tj}} = \frac{2\rho_j v_{tj}^2}{v_{tj}} \qquad \text{i.e.} \quad v_{tj} \frac{\partial \mu_j}{\partial v_{tj}} = 2\mu_j, \quad j=1,2,\ldots,n+1. \tag{23}$$

Also from (22) and (23)

$$v_{tj} \frac{\partial \mu_j}{\partial v_{tj}} = 2\rho_j \frac{\partial \mu_j}{\partial \rho_j} \,, \qquad j=1,2,\ldots,n+1 \quad .$$

Then,

$$\frac{\partial s_j}{\partial v_{tj}} = -\frac{c^2}{s_j v_{tj}^2 v_{tj}} \qquad \text{i.e.} \quad v_{tj} \frac{\partial s_j}{\partial v_{tj}} = -\frac{c^2}{s_j v_{tj}^2} \,, j=1,2,\ldots,n$$

and

$$\frac{\partial s_j}{\partial c} = \frac{c^2}{c s_j v_{tj}^2} \qquad \text{i.e.} \quad c \frac{\partial s_j}{\partial c} = \frac{c^2}{s_j v_{tj}^2} \,, \; j=1,2,\ldots,n \,,$$

so

$$c \frac{\partial s_j}{\partial c} = -v_{tj} \frac{\partial s_j}{\partial v_{tj}} \,, \qquad j=1,2,\ldots,n \quad . \tag{24}$$

Analogously

$$c \frac{\partial r_j}{\partial c} = -v_{tj} \frac{\partial r_j}{\partial v_{tj}} \,, \qquad j=n+1 \quad . \tag{25}$$

Further,

$$\frac{\partial Q_j}{\partial k} = H_j s_j \frac{k}{k} \qquad \text{i.e.} \quad k \frac{\partial Q_j}{\partial k} = Q_j \,, \; j=1,2,\ldots,n \,, \tag{26}$$

$$\frac{\partial Q_j}{\partial H_j} = k s_j \frac{H_j}{H_j} \qquad \text{i.e.} \quad H_j \frac{\partial Q_j}{\partial H_j} = Q_j, \; j=1,2,\ldots,n \,, \tag{27}$$

and

$$\frac{\partial Q_j}{\partial s_j} = k H_j \frac{s_j}{s_j} \qquad \text{i.e.} \quad s_j \frac{\partial Q_j}{\partial s_j} = Q_j, \; j=1,2,\ldots,n \,, \tag{28}$$

yielding

$$k\frac{\partial Q_j}{\partial k} = H_j\frac{\partial Q_j}{\partial H_j} = s_j\frac{\partial Q_j}{\partial s_j}\ , \qquad j=1,2,\ldots,n \quad .$$

There is still

$$\frac{\partial k}{\partial \omega} = \frac{\omega}{c\omega}\ , \qquad \text{i.e.} \quad \omega\frac{\partial k}{\partial \omega} = k\ , \tag{29}$$

while

$$\frac{\partial k}{\partial c} = -\frac{\omega}{c^2}\ , \qquad \text{i.e.} \quad c\frac{\partial k}{\partial c} = -k\ , \tag{30}$$

Also

$$c\frac{\partial k}{\partial c} = -\omega\frac{\partial k}{\partial \omega}\ .$$

After this, we determine partial derivatives of the dispersion function $L = L(\mu_j, s_j, r_j, Q_j)$ according to the independent variables:

$$\frac{\partial L}{\partial \rho_j} = \frac{\partial L}{\partial \mu_j}\frac{\partial \mu_j}{\partial \rho_j}\ , \qquad j=1,2,\ldots,n+1 \quad ,$$

which due to (22) is

$$\rho_j\frac{\partial L}{\partial \rho_j} = \mu_j\frac{\partial L}{\partial \mu_j}\ , \qquad j=1,2,\ldots,n+1 \quad , \tag{31}$$

Similarly

$$\frac{\partial L}{\partial H_j} = \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial H_j}\ , \qquad j=1,2,\ldots,n \quad ,$$

which due to (27) is

$$H_j\frac{\partial L}{\partial H_j} = Q_j\frac{\partial L}{\partial Q_j}\ , \qquad j=1,2,\ldots,n \tag{32}$$

Then,

$$\frac{\partial L}{\partial \omega} = \sum_{j=1}^{n}\frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial k}\frac{\partial k}{\partial \omega}\ ,$$

and due to (26) and (30)

$$\omega\frac{\partial L}{\partial \omega} = \sum_{j=1}^{n} Q_j\frac{\partial L}{\partial Q_j}\ ,$$

and due to (32)

$$\omega \frac{\partial L}{\partial \omega} = \sum_{j=1}^{n} H_j \frac{\partial L}{\partial H_j} \quad .$$

The formula (21) has thus been derived.

Further

$$\frac{\partial L}{\partial v_{tj}} = \frac{\partial L}{\partial \mu_j} \frac{\partial \mu_j}{\partial v_{tj}} + \frac{\partial L}{\partial s_j} \frac{\partial s_j}{\partial v_{tj}} + \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial s_j} \frac{\partial s_j}{\partial v_{tj}} \ , \quad j=1, 2, \ldots, n$$

and

$$\frac{\partial L}{\partial v_{tj}} = \frac{\partial L}{\partial \mu_j} \frac{\partial \mu_j}{\partial v_{tj}} + \frac{\partial L}{\partial r_j} \frac{\partial r_j}{\partial v_{tj}} \ , \qquad j=n+1 \quad ,$$

or due to (23) and (28)

$$\frac{\partial L}{\partial v_{tj}} = \frac{\partial L}{\partial \mu_j} \frac{2\mu_j}{v_{tj}} + \frac{\partial L}{\partial s_j} \frac{\partial s_j}{\partial v_{tj}} + \frac{\partial L}{\partial Q_j} \frac{Q_j}{s_j} \frac{\partial s_j}{\partial v_{tj}} \ , \quad j=1, 2, \ldots, n$$

and

$$\frac{\partial L}{\partial v_{tj}} = \frac{\partial L}{\partial \mu_j} - \frac{2\mu_j}{v_{tj}} + \frac{\partial L}{\partial r_j} \frac{\partial r_j}{\partial v_{tj}} \ , \qquad j=n+1 \quad ,$$

whence due to (31)

$$v_{tj} \frac{\partial L}{\partial v_{tj}} - 2\rho_j \frac{\partial L}{\partial \rho_j} = \frac{\partial L}{\partial s_j} v_{tj} \frac{\partial s_j}{\partial v_{tj}} + \frac{\partial L}{\partial Q_j} \frac{Q_j}{s_j} v_{tj} \frac{\partial s_j}{\partial v_{tj}} \ , \ j=1,2,\ldots,n, \tag{33a}$$

and

$$v_{tj} \frac{\partial L}{\partial v_{tj}} - 2\rho_j \frac{\partial L}{\partial \rho_j} = \frac{\partial L}{\partial r_j} v_{tj} \frac{\partial r_j}{\partial v_{tj}} \ , \qquad j=n+1 \quad . \tag{33b}$$

Analogously as earlier

$$\frac{\partial L}{\partial c} = \sum_{j=1}^{n} \left(\frac{\partial L}{\partial s_j} \frac{\partial s_j}{\partial c} + \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial s_j} \frac{\partial s_j}{\partial c} + \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial k} \frac{\partial k}{\partial c}\right) + \left.\frac{\partial L}{\partial r_j} \frac{\partial r_j}{\partial c}\right|_{j=n+1} .$$

When applying (24), (25), (26), (28) and (30)

$$\frac{\partial L}{\partial c} = - \sum_{j=1}^{n} \left(\frac{\partial L}{\partial s_j} \frac{v_{tj}}{c} \frac{\partial s_j}{\partial v_{tj}} + \frac{\partial L}{\partial Q_j} \frac{Q_j}{s_j} \frac{v_{tj}}{c} \frac{\partial s_j}{\partial v_{tj}} + \frac{\partial L}{\partial Q_j} \frac{Q_j}{k} \frac{k}{c}\right) -$$

$$- \left.\frac{\partial L}{\partial r_j} \frac{v_{tj}}{c} \frac{\partial r_j}{\partial v_{tj}}\right|_{j=n+1} ,$$

and finally respecting (33a), (33b) and (32), we obtain

$$c\frac{\partial L}{\partial c} = -\sum_{j=1}^{n+1}\left(2\rho_j\frac{\partial L}{\partial \rho_j} - v_{tj}\frac{\partial L}{\partial v_{tj}}\right) - \sum_{j=1}^{n} H_j\frac{\partial L}{\partial H_j} .$$

Formula (20) has been proved valid.

Further derivation of the expressions yields relations valid for second and higher partial derivatives.

Formulae (20) and (21) serve to check the expressions for partial derivatives of Love wave phase velocity dispersion function, as well as for checking the derived numerical values of these derivatives, and they can be used for checking computer programmes of the inversion calculation.

These relations were used to test a part of linear inversion structure calculation programme written in FORTRAN IV language, adapted to the computer KOPA 1500. The accuracy of the numerical values of the first and second derivatives of the dispersion function was tested according to formulae (20) and (21), as well as of their second derivatives. Differences between the numerical values on the left and the right side of the equations ranged between 10^{-8} and 10^{-6}. The programme was tested for a two--layered and three-layered solid medium.

Table 1. Structure parameters of Earth crust and the upper mantle approximated with a three-layered solid medium

Layer j No	Thickness H_j (km)	Transverse wave velocity v_{tj} (km/s)	Density ρ_j (10^{-9} kg/m^3)
1	17	3.3	2.65
2	17	3.7191	2.9018
3		4.3692	3.1906

Table 1 approximates the Earth as a three-layered solid medium.

For the period T = 20s, with the Earth model parameters as shown in Table 1, for the Love wave the phase velocity was determined with the accuracy of 10^{-6} km/s and is:

$$c = 3.736788 \text{ km/s}$$

while

$$\frac{\partial L}{\partial c} = 90.73858830 \quad \text{kg/m}^2\text{s}$$

$$\frac{\partial L}{\partial \rho_1} = 8.405785109 \quad \text{m}^2/\text{s}^2$$

$$\frac{\partial L}{\partial \rho_2} = -\ 3.938327652 \quad m^2/s^2$$

$$\frac{\partial L}{\partial \rho_3} = -\ 3.399702760 \quad m^2/s^2$$

$$\frac{\partial L}{\partial v_{t1}} = -63.79911529 \quad kg/m^2 s$$

$$\frac{\partial L}{\partial v_{t2}} = -33.21920944 \quad kg/m^2 s$$

$$\frac{\partial L}{\partial v_{t3}} = -11.7276459 \quad kg/m^2 s$$

$$\frac{\partial L}{\partial H_1} = 1.992536915 \quad kg/m^2 s^2$$

$$\frac{\partial L}{\partial H_2} = 0.728188515 \quad kg/m^2 s^2 ,$$

and, according to the relation (20) the difference between the left and the right side of the equation is 1.10^{-7} Pa.

The accuracy of the partial derivatives of the phase and the group velocity is thus 10^{-6} .

7. Conclusion

For the first time this paper offers a way of checking the accuracy of partial derivatives of the Love wave dispersion function from relationships among partial derivatives of that function.

This provides a very useful tool for testing the part of the programme calculating partial derivatives of Love wave phase or group velocity in the inversion of the structure.

Besides it presumably provides a possibility for some further physical research of Love wave dispersion peculiarities.

References

Anđelić, T.P. (1962), Matrice, Univerzitet u Beogradu, Naučna knjiga, Beograd, 266 pp.

Bronštejn, I.N. i Semendjajev, K.A. (1964), Matematički priručnik za inženjere i studente, prijevod s ruskog u redakciji D. Blanuše, Tehnička knjiga, Zagreb, 695 pp.

Dorman, J. and Ewing, M. (1962), Numerical inversion of surface wave dispersion data and crust-mantle structure in the New York Pennsylvania area, Journal of Geophysical Research, **67**, 5227–5241.

Milošević, A. (1982), Loveovi valovi u troslojnom sredstvu, Sveučilište u Zagrebu, Post-diplomski studij prirodnih znanosti, Zagreb, 109 pp. + 8 pp. (magistarski rad).

Novotný, O. (1970), Partial derivatives of dispersion curves of Love waves in a layered medium, Studia Geophysica et Geodaetica, **14**, 36–50.

Schwab, F.A. and Knopoff, L. (1972), Fast surface wave and free mode computations in "Methods in computational Physics, Advances in research and applications". **11**, Academic Press, New York and London, 87 + 180.

Author's address: A. Milošević, Geophysical Institute, Faculty of Science, University of Zagreb, P. O. Box 224, 41001 Zagreb, Yugoslavia.