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On the approximation by Bézier-Păltănea operators based on Gould-Hopper polynomials

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Abstract. In the present article, we give a Bézier variant of Păltănea operators, which involves Gould-Hopper polynomials. First, we investigate the rate of convergence by using Ditzian-Totik modulus of smoothness, weighted modulus of continuity and also for a class of Lipschitz function. Furthermore, we obtain the quantitative Voronovskaja type theorem in terms of Ditzian-Totik modulus of smoothness. In the last section, we study the rate of point-wise convergence for the functions having a derivative of bounded variation.

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1. Introduction

In approximation by positive linear operators, Szász operators play a significant role. Several authors have investigated many amusing properties of these operators. In order to approximate integrable functions, Durrmeyer and Kantorovich type amendments have been explored as well. Jakimovski and Leviatan generalized Szász operators in [10] by using Appell polynomials. Afterwards, Ismail [9] presented another generalization of Szász operators by means of Sheffer polynomials. A link between positive linear operator and orthogonal polynomials was introduced by Varma *et al.* in 2012. In [24], they formulated Szász operators concerning Brenke polynomials and demonstrated that these polynomials incorporate Appell polynomials and Gould-Hopper polynomials. Later on, many adaptations of Szász operators have been recognized by using different orthogonal polynomials. Sucu [21] generalized the Szász operators by utilizing Boas-Buck type polynomials. Varma and Taşdelen [25] determined a link between Szász operators and Charlier polynomials and established approximation and convergence results. Also, Varma gave another generalization of Szász operators by means of multiple Appell polynomials in [26]. Taşdelen *et al.* [23] introduced a Kantorovich type modification of Szász operators based on Brenke type polynomials and examined the rate of convergence. Mursaleen *et al.* considered Cholodowsky type modifications of Szász

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operators involving Brenke polynomials [15] and Durremeyer-Jakimovski-Leviatan operators [17] and studied their approximation properties. They also gave another generalization of Szász operators involving (p, q)-integers, q-Appell polynomials and multiple Appell polynomials in [16], [18] and [1], respectively. Mishra and Gandhi [13, 14] presented a summation integral type modification of Szász operators and studied simultaneous approximation and convergence properties of these operators.

It is well known that Bézier curves are the mathematically defined curves successively used in computer-aided geometric design (CAGD), image processing and curve fitting. The miscellaneous Bézier variant of operators is crucial subject matter in approximation theory. In 1983, Chang [3] pioneered the Bernstein-Bézier operators. Afterwards, several researchers established the Bézier variant of various operators [7, 8, 12].

For $\alpha > 0$, $\rho > 0$ and $x \in \mathbb{R}_0^+ = [0, \infty)$, Păltănea [20] recognized the two parameter summation-integral type modification of Szász operators as

$$L_\alpha^\rho(f;x) = \sum_{m=1}^{\infty} s_{\alpha,m}(x) \int_0^\infty \Theta_{\alpha,m}^\rho(u) f(u) du + e^{-\alpha x} f(0), \tag{1}$$

where

$$s_{\alpha,m}(x) = e^{-\alpha x} \frac{(\alpha x)^m}{m!},$$
$$\Theta_{\alpha,m}^\rho(u) = \frac{\alpha\rho}{\Gamma(m\rho)} \cdot e^{-\alpha\rho u} (\alpha\rho u)^{m\rho-1}$$

and f is an integrable function for which formula (1) is well defined for every $x \geq 0$.

Recently, in [6], the authors acknowledged the Jakimovski-Leviatan-Păltănea-Bézier operators and presented some direct approximation theorems and the rate of convergence for the functions having a derivative of bounded variation. Motivated by this, we introduce the Bézier-Păltănea operators based on Gould-Hopper polynomials and study further in this direction.

A generating function of the Gould-Hopper polynomials is given by (see [5])

$$e^{ht^{d+1}} e^{xt} = \sum_{k=0}^{\infty} g_k^{d+1}(x,h) \frac{t^k}{k!}, \tag{2}$$

where

$$g_k^{d+1}(x,h) = \sum_{s=0}^{[\frac{k}{d+1}]} \frac{k!}{s!(k-(d+1)s)!} h^s x^{k-(d+1)s},$$

$h \geq 0$ and $[.]$ denotes the integer part.

For $n \in \mathbb{N}$, $\rho > 0$, $\theta \geq 1$ and all real-valued continuous and bounded functions f on $\mathbb{R}_0^+$, we propose Bézier-Păltănea operators based on Gould-Hopper polynomials as

$$G_{n,h,\rho}^{d,\theta}(f;x) = \sum_{k=1}^{\infty} X_{n,h,k}^{d,\theta}(x) \int_0^\infty \Theta_{n,k}^\rho(u) f(u) du + X_{n,h,0}^{d,\theta}(x) f(0), \tag{3}$$

where

$$X^{d,\theta}_{n,h,k}(x) = [\zeta^d_{n,h,k}(x)]^\theta - [\zeta^d_{n,h,k+1}(x)]^\theta,$$

$$\zeta^d_{n,h,k}(x) = \sum_{j=k}^{\infty} l^d_{n,h,j}(x),\ \ k = 0, 1, 2, 3, \ldots,$$

$$l^d_{n,h,k}(x) = e^{-nx-h} \cdot \frac{g^{d+1}_k(nx, h)}{k!} \tag{4}$$

and

$$\Theta^\rho_{n,k}(u) = \frac{n\rho}{\Gamma(k\rho)} \cdot e^{-n\rho u}(n\rho u)^{k\rho-1}. \tag{5}$$

$\zeta^d_{n,h,k}(x)$ satisfies the following important properties:

1. $\zeta^d_{n,h,k}(x) - \zeta^d_{n,h,k+1}(x) = l^d_{n,h,k}(x),\ \ k = 0, 1, 2, 3, \ldots,$
2. $\zeta^d_{n,h,0}(x) > \zeta^d_{n,h,1}(x) > \cdots > \zeta^d_{n,h,k}(x) > \zeta^d_{n,h,k+1}(x) > \cdots,\ \forall x \in \mathbb{R}^+_0.$

Also, the operators $G^{d,\theta}_{n,h,\rho}$ have the integral representation

$$G^{d,\theta}_{n,h,\rho}(f; x) = \int_0^\infty K^{d,\theta}_{n,h,\rho}(x, u) f(u) du, \tag{6}$$

where $K^{d,\theta}_{n,h,\rho}(x, u)$ is the kernal defined by

$$K^{d,\theta}_{n,h,\rho}(x, u) = \sum_{k=1}^{\infty} X^{d,\theta}_{n,h,k}(x)\Theta^\rho_{n,k}(u) + X^{d,\theta}_{n,h,0}(x)\delta(u), \tag{7}$$

$\delta(u)$ is the Dirac-delta function.

If we take $\theta = 1$, then operator (3) reduces to operators $G^{d,1}_{n,h,\rho}(f; x) = G^d_{n,h,\rho}(f; x)$ given by

$$G^d_{n,h,\rho}(f; x) = \sum_{k=1}^{\infty} l^d_{n,h,k}(x) \int_0^\infty \Theta^\rho_{n,k}(u) f(u) du + l^d_{n,h,0}(x) f(0), \tag{8}$$

where $l^d_{n,h,k}(x)$ and $\Theta^\rho_{n,k}(u)$ are defined as in (4) and (5), respectively.

The main objective of this article is to study the rate of convergence of our constructed operators (3) by using Ditzian-Totik modulus of smoothness, weighted modulus of continuity and for functions having a derivative of bounded variation. Furthermore, we have also established the quantitative Voronovskaja type theorem.

2. Auxiliary results

Lemma 1. *Let $G^d_{n,h,\rho}$ be the operators defined by* (8)*. Then, we have*

1)
$$G^d_{n,h,\rho}((u-x);x)=\frac{h(d+1)}{n},$$

2)
$$G^d_{n,h,\rho}((u-x)^2;x)=\frac{x}{n\rho}(1+\rho)+\frac{h(d+1)}{n^2\rho}(\rho(d+1)(h+1)+1),$$

3)
$$\begin{aligned}G^d_{n,h,\rho}((u-x)^4;x)=&\frac{3x^2}{n^2}\left(1+\frac{2}{\rho}+\frac{1}{\rho^2}\right)+\frac{x}{n^3}\Bigg(6h^2(d+1)^2-4h(d+1)^3\\&+14h(d+1)^2+4hd^2(d+1)+1+\frac{1}{\rho}(6h(h+1)(d+1)^2\\&+18h(d+1)+6)+\frac{1}{\rho^2}(14h(d+1)+11)+\frac{1}{\rho^3}\Bigg)\\&+\frac{1}{n^4}\Bigg(h(d+1)^4(h^3+6h^2+7h+1)+\frac{6h}{\rho}(d+1)^3\\&\times(h^2+3h+1)+\frac{11h}{\rho^2}(h+1)(d+1)^2+\frac{6}{\rho^3}h(d+1)\Bigg).\end{aligned}$$

Proof. From the generating function for Gould-Hopper polynomials and by using the properties of the gamma function and linearity of operators, we get the above result. □

Throughout this article, let $C_B(\mathbb{R}_0^+)$ denote the space of all functions f on $\mathbb{R}_0^+$ which are bounded and continuous according to the norm
$$\|f\|=\sup_{x\in\mathbb{R}_0^+}|f(x)|.$$

Lemma 2. *For $f\in C_B(\mathbb{R}_0^+)$, we have*
$$\|G^d_{n,h,\rho}f\|\leq\|f\|.$$

Remark 1. *We have*
$$\begin{aligned}G^{d,\theta}_{n,h,\rho}(1;x)&=\sum_{k=1}^{\infty}X^{d,\theta}_{n,h,k}(x)\int_0^{\infty}\Theta^{\rho}_{n,k}(u)du+X^{d,\theta}_{n,h,0}(x)\\&=\sum_{k=0}^{\infty}X^{d,\theta}_{n,h,k}(x)=\left(\sum_{k=0}^{\infty}l^d_{n,h,k}(x)\right)^{\theta}\\&=(1)^{\theta}=1.\end{aligned}$$

Lemma 3. *Let $G^{d,\theta}_{n,h,\rho}$ be the operator defined by* (3)*; then*

1)
$$G^{d,\theta}_{n,h,\rho}((u-x);x)\leq\theta\frac{h(d+1)}{n},$$

2)

$$G_{n,h,\rho}^{d,\theta}((u-x)^2;x) \leq \theta\bigg\{\frac{x}{n\rho}(1+\rho)+\frac{h(d+1)}{n^2\rho}(\rho(d+1)(h+1)+1)\bigg\},$$

3)

$$\begin{aligned}G_{n,h,\rho}^{d,\theta}((u-x)^4;x) \leq &\theta\bigg\{\frac{3x^2}{n^2}\bigg(1+\frac{2}{\rho}+\frac{1}{\rho^2}\bigg)+\frac{x}{n^3}\bigg(6h^2(d+1)^2-4h(d+1)^3\\ &+14h(d+1)^2+4hd^2(d+1)+1+\frac{1}{\rho}(6h(h+1)(d+1)^2\\ &+18h(d+1)+6)+\frac{1}{\rho^2}(14h(d+1)+11)+\frac{1}{\rho^3}\bigg)\\ &+\frac{1}{n^4}\bigg(h(d+1)^4(h^3+6h^2+7h+1)+\frac{6h}{\rho}(d+1)^3\\ &\times(h^2+3h+1)+\frac{11h}{\rho^2}(h+1)(d+1)^2+\frac{6}{\rho^3}h(d+1)\bigg)\bigg\}.\end{aligned}$$

Proof. From equation (3), we have

$$\begin{aligned}G_{n,h,\rho}^{d,\theta}(f;x) &= \sum_{k=1}^{\infty}X_{n,h,k}^{d,\theta}(x)\int_0^{\infty}\Theta_{n,k}^{\rho}(u)f(u)du+X_{n,h,0}^{d,\theta}(x)f(0),\\ &= \sum_{k=1}^{\infty}([\zeta_{n,h,k}^{d}(x)]^{\theta}-[\zeta_{n,h,k+1}^{d}(x)]^{\theta})\int_0^{\infty}\Theta_{n,k}^{\rho}(u)f(u)du\\ &\quad+([\zeta_{n,h,0}^{d}(x)]^{\theta}-[\zeta_{n,h,1}^{d}(x)]^{\theta})f(0).\end{aligned}$$

Using the well-known inequality $|c^{\beta}-d^{\beta}| \leq \beta|c-d|$ with $0 \leq c, d \leq 1$, $\beta \geq 1$ and property (1) of $\zeta_{n,h,k}^{d}(x)$, we have

$$\begin{aligned}G_{n,h,\rho}^{d,\theta}(f;x) &\leq \theta\bigg\{\sum_{k=1}^{\infty}l_{n,h,k}^{d}(x)\int_0^{\infty}\Theta_{n,k}^{\rho}(u)f(u)du+l_{n,h,0}^{d}(x)f(0)\bigg\},\\ &\leq \theta G_{n,h,\rho}^{d}(f;x).\end{aligned}$$

In view of Lemma 1, we get all inequalities. □

Remark 2. *We have the following inequalities:*

1) *For* $C_1>1,\ \rho>0,\ \theta\geq 1,\ x\in(0,+\infty)$ *and sufficiently large* n

$$G_{n,h,\rho}^{d,\theta}((u-x)^2;x) \leq C_1\theta\frac{x(1+\rho)}{n\rho}.$$

2) *For* $C_2>1,\ \rho>0,\ \theta\geq 1,\ x\in(0,+\infty)$ *and sufficiently large* n

$$G_{n,h,\rho}^{d,\theta}((u-x)^4;x) \leq C_2\theta\bigg(\frac{x(1+\rho)}{n\rho}\bigg)^2.$$

Lemma 4. *Let $f \in C_B(\mathbb{R}_0^+)$; then*

$$\|G_{n,h,\rho}^{d,\theta} f\| \leq \theta \|f\|.$$

Proof. In view of equation (3), we have

$$\begin{aligned} |G_{n,h,\rho}^{d,\theta}(f;x)| &= \left| \sum_{k=1}^{\infty} X_{n,h,k}^{d,\theta}(x) \int_0^{\infty} \Theta_{n,k}^{\rho}(u) f(u) du + X_{n,h,0}^{d,\theta}(x) f(0) \right| \\ &\leq \left(\sum_{k=1}^{\infty} X_{n,h,k}^{d,\theta}(x) \int_0^{\infty} \Theta_{n,k}^{\rho}(u) du + X_{n,h,0}^{d,\theta}(x) \right) \|f\| \\ &\leq \theta G_{n,h,\rho}^{d}(1;x) \|f\| = \theta \|f\|, \end{aligned}$$

which completes the proof. □

3. Direct approximation

In this part, initially we recall the definition of the well-known Ditizian-Totik modulus of smoothness $\omega_{\phi^\tau}(.;.)$ and Peetre's $K-$functional [4].
Let $\phi(x) = \sqrt{x}$ and $f \in C_B(\mathbb{R}_0^+)$. For $0 \leq \tau \leq 1$, define

$$\omega_{\phi^\tau}(f;t) = \sup_{0 \leq h \leq t} \sup_{x \pm \frac{h\phi^\tau(x)}{2} \in \mathbb{R}_0^+} \left| f\left(x + \frac{h\phi^\tau(x)}{2}\right) - f\left(x - \frac{h\phi^\tau(x)}{2}\right) \right| \tag{9}$$

and the $K-$functional

$$K_{\phi^\tau}(f;t) = \inf_{g \in W_\tau} \left\{ \|f - g\| + t\|\phi^\tau g'\| \right\},$$

where

$$W_\tau = \{g : \quad g \in AC_{loc}; \quad \|\phi^\tau g'\| < \infty\}$$

and $g \in AC_{loc}$ means that g is an absolutely continuous function on every finite subinterval of $\mathbb{R}_0^+$. Also,

$$\omega_{\phi^\tau}(f;t) \sim K_{\phi^\tau}(f;t),$$

which means that there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\phi^\tau}(f;t) \leq K_{\phi^\tau}(f;t) \leq C\omega_{\phi^\tau}(f;t). \tag{10}$$

Lemma 5. *Let $\phi(x) = \sqrt{x}$ and $0 \leq \tau \leq 1$. Then for $f \in W_\tau$ and $y, x > 0$, we have*

$$\left| \int_x^y f'(u) du \right| \leq 2^\tau x^{-\frac{\tau}{2}} |y - x| \|\phi^\tau f'\|.$$

Proof. For a proof, see [6, Lemma 5]. □

Theorem 1. *For $f \in C_B(\mathbb{R}_0^+)$, we have*

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq C\omega_{\phi^\tau}\left(f; \frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right),$$

where ω_{ϕ^τ} is given by (9) and C is a constant free from the choice of n and x.

Proof. Let $g \in W_\tau$. Using Lemma 4, we have

$$\begin{aligned}|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| &\leq |G_{n,h,\rho}^{d,\theta}(f-g;x)| + |f(x) - g(x)| + |G_{n,h,\rho}^{d,\theta}(g;x) - g(x)| \\ &\leq (\theta+1)\|f-g\| + |G_{n,h,\rho}^{d,\theta}(g;x) - g(x)|. \qquad (11)\end{aligned}$$

Since $g(y) = g(x) + \int_x^y g'(u)du$ and $G_{n,h,\rho}^{d,\theta}(1;x) = 1$, we obtain

$$|G_{n,h,\rho}^{d,\theta}(g;x) - g(x)| = \left|G_{n,h,\rho}^{d,\theta}\left(\int_x^y g'(u)du; x\right)\right|.$$

By Lemma 5, we get

$$|G_{n,h,\rho}^{d,\theta}(g;x) - g(x)| \leq 2^\tau x^{-\frac{\tau}{2}} \|\phi^\tau g'\| G_{n,h,\rho}^{d,\theta}(|y-x|;x). \qquad (12)$$

Applying the Cauchy-Schwarz inequality and using Remark 2, we have

$$\begin{aligned}G_{n,h,\rho}^{d,\theta}(|y-x|;x) &\leq \sqrt{G_{n,h,\rho}^{d,\theta}(1;x)}\sqrt{G_{n,h,\rho}^{d,\theta}((y-x)^2;x)} \\ &\leq \sqrt{\frac{C_1\theta x(1+\rho)}{n\rho}}. \qquad (13)\end{aligned}$$

Combining (11)-(13), we get

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq (1+\theta)\|f-g\| + C_3\|\phi^\tau g'\| \frac{\phi^{1-\tau}(x)}{\sqrt{n}}.$$

Taking infimum over all $g \in W_\tau$, we have

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq C_4 K_{\phi^\tau}\left(f; \frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right).$$

Using (10), we get

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq C\omega_{\phi^\tau}\left(f; \frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right),$$

which is the required result. □

Remark 3. *Taking $\tau = 0$, we get error estimation in terms of the classical modulus of continuity, i.e.,*

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq C\omega\left(f; \sqrt{\frac{x}{n}}\right).$$

Now, we give the following local approximation result for the function belonging to Lipschitz-type space:

For $a \geq 0$, $b > 0$ to be fixed, the class of two parameteric Lipschitz type functions ([19]) is defined as

$$Lip_M^{a,b}(\beta)=\left\{f\in C_B(\mathbb{R}_0^+):|f(u)-f(x)|\leq M\frac{|u-x|^\beta}{(u+ax^2+bx)^{\frac{\beta}{2}}} \text{ and } x,u\in(0,\infty)\right\}, \tag{14}$$

where M is any positive constant and $0 < \beta \leq 1$. For $a = 0$ and $b = 1$, space $Lip_M^{0,1}(\beta)$ is the space $Lip_M^*(\beta)$ given by O. Szász [22].

Theorem 2. *Let $f \in Lip_M^{a,b}(\beta)$. Then for every $n \in \mathbb{N}$, $\rho > 0$, $\theta \geq 1$ and $x \in (0,+\infty)$, we have*

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq M\left(\frac{\theta G_{n,h,\rho}^{d}((u-x)^2;x)}{ax^2+bx}\right)^{\frac{\beta}{2}},$$

where $G_{n,h,\rho}^{d}((u-x)^2;x)$ is given in Lemma 1.

Proof. Consider $f \in Lip_M^{a,b}(\beta)$ and $x \in (0,+\infty)$; then

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| &= |G_{n,h,\rho}^{d,\theta}(f(u) - f(x);x)| \\
&\leq G_{n,h,\rho}^{d,\theta}(|f(u) - f(x)|;x) \\
&\leq G_{n,h,\rho}^{d,\theta}\left(M\frac{|u-x|^\beta}{\sqrt{(u+ax^2+bx)^{\frac{\beta}{2}}}};x\right) \\
&\leq \frac{M}{\sqrt{ax^2+bx}}G_{n,h,\rho}^{d,\theta}(|u-x|^\beta;x).
\end{aligned}$$

First of all, consider the case $\beta = 1$. Applying the Cauchy-Schwarz inequality and using the fact $G_{n,h,\rho}^{d,\theta}(1;x) = 1$, we have

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| &\leq \frac{M}{\sqrt{ax^2+bx}}(G_{n,h,\rho}^{d,\theta}((u-x)^2;x))^{\frac{1}{2}} \\
&\leq \frac{M}{\sqrt{ax^2+bx}}(\theta G_{n,h,\rho}^{d}((u-x)^2;x))^{\frac{1}{2}} \\
&\leq M\left(\frac{\theta G_{n,h,\rho}^{d}((u-x)^2;x)}{ax^2+bx}\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence the result holds for $\beta = 1$. Now assume that $0 < \beta < 1$. Applying Hölder's inequality with $p = \frac{1}{\beta}$ and $q = \frac{1}{1-\beta}$, we have

$$|G_{n,h,\rho}^{d,\theta}(f;x) - f(x)| \leq \frac{M}{(ax^2+bx)^{\frac{\beta}{2}}}(G_{n,h,\rho}^{d,\theta}(|u-x|;x))^\beta.$$

Finally, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| &\leq \frac{M}{(ax^2+bx)^{\frac{\beta}{2}}}(G^{d,\theta}_{n,h,\rho}((u-x)^2;x))^{\frac{\beta}{2}}\\ &\leq \frac{M}{(ax^2+bx)^{\frac{\beta}{2}}}(\theta G^{d}_{n,h,\rho}((u-x)^2;x))^{\frac{\beta}{2}}\\ &\leq M\left(\frac{\theta G^{d}_{n,h,\rho}((u-x)^2;x)}{ax^2+bx}\right)^{\frac{\beta}{2}},\end{aligned}$$

which is the desired result. □

We now evaluate the rate of convergence of operators (3) in the context of suitable weighted function spaces. More precisely, let us consider the space

$$B_2(\mathbb{R}_0^+)=\left\{f:|f(x)|\leq M_f(1+x^2),\ M_f \text{ is a constant connected with } f\right\}.$$

Introduce

$$\begin{aligned}C_2(\mathbb{R}_0^+)&=\left\{f\in B_2(\mathbb{R}_0^+):\ f \text{ is continuous}\right\},\\ C_2^*(\mathbb{R}_0^+)&=\left\{f\in C_2(\mathbb{R}_0^+):\exists\lim_{x\to\infty}\frac{|f(x)|}{1+x^2}<\infty\right\}.\end{aligned}$$

These spaces are endowed with the norm

$$\|f\|_2=\sup_{x\in\mathbb{R}_0^+}\frac{|f(x)|}{1+x^2}.$$

For $f\in C(\mathbb{R}_0^+)$ and $\delta>0$ modulus of continuity $\omega(f;\delta)$ has the property that $\omega(f;\delta)\to 0$ with $\delta\to 0$ on $[a,b]\subset\mathbb{R}_0^+$, but in general this property does not hold true on $\mathbb{R}_0^+$. So we use the following weighted modulus of continuity [27]:

$$\Omega(f;\delta)=\sup_{x\geq 0}\sup_{|t|<\delta}\frac{|f(x+t)-f(x)|}{1+(x+t)^2}. \tag{15}$$

Theorem 3. *Let $f\in C_2^*(\mathbb{R}_0^+)$ and $\Omega(f;.)$ as in (15). Then for $x\in\mathbb{R}_0^+$, $\rho,\delta>0$, $\theta\geq 1$ and for large enough n, we have*

$$\begin{aligned}|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)| \leq& 2(1+x^2)\Omega\left(f;\frac{1}{\sqrt{n}}\right)\left(1+C_1\theta\frac{x(1+\rho)}{n\rho}+\sqrt{\theta C_1}\left(\frac{x(1+\rho)}{\rho}\right)^{\frac{1}{2}}\right.\\ &\left.\times\left(1+\sqrt{\theta C_2}\frac{x(1+\rho)}{n\rho}\right)\right),\end{aligned}$$

where $C_1,C_2>1$ are constants free from the choice of x and n.

Proof. For $u, x \in \mathbb{R}_0^+$, $\delta > 0$ and by the definition of weighted modulus of continuity, we have

$$|f(u) - f(x)| \leq 2(1 + x^2)(1 + (u - x)^2)\left(1 + \frac{|u - x|}{\delta}\right)\Omega(f; \delta). \tag{16}$$

Applying $G^{d,\theta}_{n,h,\rho}$ to inequality (16) and then the Cauchy-Schwarz inequality in the last term, we obtain

$$\begin{aligned}
G^{d,\theta}_{n,h,\rho}(|f(u) - f(x)|; x) &\leq 2(1 + x^2)\Omega(f; \delta)\bigg(G^{d,\theta}_{n,h,\rho}(1 + (u - x)^2; x) \\
&\quad + G^{d,\theta}_{n,h,\rho}\left(\frac{(1 + (u - x)^2)|u - x|}{\delta}; x\right)\bigg) \\
&\leq 2(1 + x^2)\Omega(f; \delta)\bigg(G^{d,\theta}_{n,h,\rho}(1; x) + G^{d,\theta}_{n,h,\rho}((u - x)^2; x) \\
&\quad + G^{d,\theta}_{n,h,\rho}\left(\frac{(1 + (u - x)^2)|u - x|}{\delta}; x\right)\bigg) \\
&\leq 2(1 + x^2)\Omega(f; \delta)\bigg(1 + G^{d,\theta}_{n,h,\rho}((u - x)^2; x) \\
&\quad + \frac{1}{\delta}(G^{d,\theta}_{n,h,\rho}((u - x)^2; x))^{\frac{1}{2}} + \frac{1}{\delta}(G^{d,\theta}_{n,h,\rho}((u - x)^4; x))^{\frac{1}{2}} \\
&\quad \times (G^{d,\theta}_{n,h,\rho}((u - x)^2; x))^{\frac{1}{2}}\bigg).
\end{aligned} \tag{17}$$

Taking Remark 2 into account and choosing $\delta = \frac{1}{\sqrt{n}}$, we get the required result. □

4. Quantitative Voronovskaja-type asymptotic formula

In this section, we give the Voronovskaja-type asymptotic theorem for $G^{d,\theta}_{n,h,\rho}$. By using Ditzian-Totik modulus of smoothness of first order, we will prove this theorem:

Theorem 4. *Let $f \in C_B(\mathbb{R}_0^+)$ such that $f', f'' \in C_B(\mathbb{R}_0^+)$. Then*

$$\begin{aligned}
&\left|n\left\{G^{d,\theta}_{n,h,\rho}(f; x) - f(x) - f'(x)G^{d,\theta}_{n,h,\rho}((u - x); x) - \frac{1}{2}f''(x)G^{d,\theta}_{n,h,\rho}((u - x)^2; x)\right\}\right| \\
&\qquad \leq C\theta\frac{x(1 + \rho)}{\rho}\omega_{\phi^\tau}\left(f''; \frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right),
\end{aligned}$$

where C is a constant free from the choice of n and x.

Proof. By Taylor's formula, we write

$$f(u) = f(x) + (u - x)f'(x) + \int_x^u (u - v)f''(v)dv.$$

Thus,

$$f(u) - f(x) - (u - x)f'(x) - \frac{1}{2}(u - x)^2 f''(x) = \int_x^u (u - v)(f''(v) - f''(x))dv, \tag{18}$$

operating $G^{d,\theta}_{n,h,\rho}(.;x)$ to both sides of the above relation, we get

$$\begin{aligned}&\left|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)-f'(x)G^{d,\theta}_{n,h,\rho}((u-x);x)-\frac{1}{2}f''(x)G^{d,\theta}_{n,h,\rho}((u-x)^2;x)\right|\\&\quad=\left|G^{d,\theta}_{n,h,\rho}\left(\int_x^u(u-v)(f''(v)-f''(x))dv;x\right)\right|\\&\quad\leq G^{d,\theta}_{n,h,\rho}\left(\left|\int_x^u(u-v)(f''(v)-f''(x))dv\right|;x\right).\end{aligned}\tag{19}$$

Also, for $g\in W_\tau$, we have

$$\left|\int_x^u(u-v)(f''(v)-f''(x))dv\right|\leq\|f''-g\|(u-x)^2+2^\tau\phi^{-\tau}(x)\|\phi^\tau g'\||u-x|^3.\tag{20}$$

From (19), we have

$$\begin{aligned}&\left|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)-f'(x)G^{d,\theta}_{n,h,\rho}((u-x);x)-\frac{1}{2}f''(x)G^{d,\theta}_{n,h,\rho}((u-x)^2;x)\right|\\&\quad\leq\|f''-g\|G^{d,\theta}_{n,h,\rho}((u-x)^2;x)+2^\tau\phi^{-\tau}(x)\|\phi^\tau g'\|G^{d,\theta}_{n,h,\rho}(|u-x|^3;x).\end{aligned}$$

In view of Remark 2 and using the Cauchy-Schwarz inequality in the last term, we obtain

$$\begin{aligned}&\left|G^{d,\theta}_{n,h,\rho}(f;x)-f(x)-f'(x)G^{d,\theta}_{n,h,\rho}((u-x);x)-\frac{1}{2}f''(x)G^{d,\theta}_{n,h,\rho}((u-x)^2;x)\right|\\&\quad\leq\|f''-g\|G^{d,\theta}_{n,h,\rho}((u-x)^2;x)+2^\tau\phi^{-\tau}(x)\|\phi^\tau g'\|(G^{d,\theta}_{n,h,\rho}((u-x)^2;x))^{\frac{1}{2}}\\&\qquad\times(G^{d,\theta}_{n,h,\rho}((u-x)^4;x))^{\frac{1}{2}}\\&\quad\leq\|f''-g\|C_1\theta\frac{x(1+\rho)}{n\rho}+2^\tau\phi^{-\tau}(x)\|\phi^\tau g'\|\left(C_1\theta\frac{x(1+\rho)}{n\rho}\right)^{\frac{1}{2}}\\&\qquad\times\left(C_2\theta\left(\frac{x(1+\rho)}{n\rho}\right)^2\right)^{\frac{1}{2}}\\&\quad\leq C_1\theta\frac{x(1+\rho)}{n\rho}\left\{\|f''-g\|+M^*\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\|\phi^\tau g'\|\right\}.\end{aligned}$$

Taking the infimum on the right-hand side of the above relations over $g\in W_\tau$, we get

$$\begin{aligned}&\left|n\left\{G^{d,\theta}_{n,h,\rho}(f;x)-f(x)-f'(x)G^{d,\theta}_{n,h,\rho}((u-x);x)-\frac{1}{2}f''(x)G^{d,\theta}_{n,h,\rho}((u-x)^2;x)\right\}\right|\\&\quad\leq C_1\theta\frac{x(1+\rho)}{\rho}K_{\phi^\tau}\left(f'';M^*\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right).\end{aligned}$$

Now using inequality (10), the theorem is proved. □

5. Rate of convergence for functions of bounded variation

The rate of convergence for functions having a derivative of bounded variation is a fascinating topic of research. Zeng and Piriou [28] established the rate of convergence of Bernstein-Bèzier operators for a function of bounded variation. Thereafter, many researchers have studied the Bèzier variant of different operators [2, 11, 29, 30]. In this section, we would like to obtain the rate of convergence of $G^{d,\theta}_{n,h,\rho}(.;.)$ for functions having a derivative of bounded variation.

Let $DBV(\mathbb{R}^+_0)$ be the space of functions on $\mathbb{R}^+_0$ having a derivative of bounded variation on every finite subinterval of $\mathbb{R}^+_0$. Since the domain of the function f is unbounded, i.e. $\mathbb{R}^+_0$, we consider the space

$$D_{x^2}BV(\mathbb{R}^+_0) = \{f : \mathbb{R}^+_0 \to \mathbb{R} : f \in DBV(\mathbb{R}^+_0) \text{ and } |f(x)| \le M_f(1+x^2) \text{ for some } M_f > 0\}.$$

Taking into consideration the fundamental theorem of calculus, we observe that the function $f \in D_{x^2}BV(\mathbb{R}^+_0)$ possesses a representation

$$f(x) = \int_0^x g(u)du + f(0),$$

where g is a function of bounded variation on each finite subinterval of $\mathbb{R}^+_0$.

Lemma 6. *Let $x \in \mathbb{R}^+_0$ and let $K^{d,\theta}_{n,h,\rho}(x,u)$ be the kernel defined by (7). Then for $C_1 > 1$ and for n large enough, we have*

1) $\xi^\theta_{n,\rho}(x,y) = \int_0^y K^{d,\theta}_{n,h,\rho}(x,u)du \le \frac{\theta C_1 x(1+\rho)}{n\rho}\frac{1}{(x-y)^2}, \quad 0 \le y < x.$

2) $1 - \xi^\theta_{n,\rho}(x,z) = \int_z^\infty K^{d,\theta}_{n,h,\rho}(x,u)du \le \frac{\theta C_1 x(1+\rho)}{n\rho}\frac{1}{(z-x)^2}, \quad x < z < \infty.$

Proof. Using Lemma 3, we get

$$\begin{aligned}
\xi^\theta_{n,\rho}(x,y) &= \int_0^y K^{d,\theta}_{n,h,\rho}(x,u)du \\
&\le \int_0^y \left(\frac{x-u}{x-y}\right)^2 K^{d,\theta}_{n,h,\rho}(x,u)du \\
&\le \frac{1}{(x-y)^2} G^{d,\theta}_{n,h,\rho}((u-x)^2;x) \\
&\le \frac{\theta C_1 x(1+\rho)}{n\rho}\frac{1}{(x-y)^2}.
\end{aligned}$$

Similarly, we can show the second part; hence the proof is omitted. □

Theorem 5. *Let $f \in D_{x^2}BV(\mathbb{R}^+_0)$ and for every $x \in (0,+\infty)$ consider the function f'_x defined by*

$$f'_x(u) = \begin{cases} f'(u) - f'(x^-), & \text{if } 0 \le u < x, \\ 0, & \text{if } u = x, \\ f'(u) - f'(x^+), & \text{if } x < u < \infty. \end{cases} \tag{21}$$

Let us denote by $\bigvee_c^d f'_x$ *the total variation of* f'_x *on* $[c,d] \subset \mathbb{R}_0^+$. *Then, for every* $x \in (0,+\infty)$ *and large* n,

$$\begin{aligned}|G_{n,h,\rho}^{d,\theta}(f;x)-f(x)| \leq\ & \frac{\sqrt{\theta}}{\theta+1}|f'(x^+)+\theta f'(x^-)|\bigg(\frac{C_1x(1+\rho)}{n\rho}\bigg)^{\frac{1}{2}} \\ &+\frac{\theta^{\frac{3}{2}}}{\theta+1}|f'(x^+)-f'(x^-)|\bigg(\frac{C_1x(1+\rho)}{n\rho}\bigg)^{\frac{1}{2}} \\ &+\theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right) \\ &+\frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x\right)+\theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{k}} f'_x\right).\end{aligned}$$

Proof. For any $f \in D_{x^2}BV(\mathbb{R}_0^+)$, from the definition of $f'_x(u)$, we can write

$$\begin{aligned}f'(u) =\ & \frac{1}{\theta+1}(f'(x^+)+\theta f'(x^-))+f'_x(u)+\frac{1}{2}(f'(x^+)-f'(x^-)) \\ &\times\bigg(\operatorname{sgn}(u-x)+\frac{\theta-1}{\theta+1}\bigg)+\delta_x(u)\bigg(f'(x)-\frac{1}{2}(f'(x^+)+f'(x^-))\bigg), \quad (22)\end{aligned}$$

where

$$\delta_x(u)=\begin{cases}1, & \text{if } x=u,\\ 0, & \text{if } x\neq u.\end{cases}$$

Now since $G_{n,h,\rho}^{d,\theta}(1;x)=1$, we have

$$\begin{aligned}G_{n,h,\rho}^{d,\theta}(f;x)-f(x) &= G_{n,h,\rho}^{d,\theta}(f(u)-f(x);x) \\ &= \int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)(f(u)-f(x))du \\ &= \int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)\bigg(\int_x^u f'(v)dv\bigg)du.\end{aligned}$$

From (22), we obtain

$$\begin{aligned}G_{n,h,\rho}^{d,\theta}(f;x)-f(x) = \int_0^\infty & K_{n,h,\rho}^{d,\theta}(x,u)\bigg(\int_x^u \bigg\{\frac{1}{\theta+1}(f'(x^+)+\theta f'(x^-))+f'_x(v) \\ &+\frac{1}{2}(f'(x^+)-f'(x^-))\bigg(\operatorname{sgn}(v-x)+\frac{\theta-1}{\theta+1}\bigg) \\ &+\delta_x(v)\bigg(f'(x)-\frac{1}{2}(f'(x^+)+f'(x^-))\bigg)\bigg\}dv\bigg)du. \quad (23)\end{aligned}$$

From the definition of $\delta_x(u)$, it is clear that

$$\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)\bigg(\int_x^u \bigg(f'(x)-\frac{1}{2}(f'(x^+)+f'(x^-))\bigg)\delta_x(v)dv\bigg)du=0. \quad (24)$$

Now, consider

$$\begin{aligned}
&\Big|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)\Big(\int_x^u \frac{1}{\theta+1}(f'(x^+)+\theta f'(x^-))dv\Big)du\Big|\\
&\quad=\Big|\frac{1}{\theta+1}(f'(x^+)+\theta f'(x^-))\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)(u-x)du\Big|\\
&\quad\leq \frac{1}{\theta+1}|f'(x^+)+\theta f'(x^-)|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)|u-x|du.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality and Remark 2 to adequately large n, we have

$$\begin{aligned}
&\Big|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)\Big(\int_x^u \frac{1}{\theta+1}(f'(x^+)+\theta f'(x^-))dv\Big)du\Big|\\
&\quad\leq \frac{1}{\theta+1}|f'(x^+)+\theta f'(x^-)|\sqrt{G_{n,h,\rho}^{d,\theta}((u-x)^2;x)}\\
&\quad\leq \frac{1}{\theta+1}|f'(x^+)+\theta f'(x^-)|\Big(\frac{\theta C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}\\
&\quad\leq \frac{\sqrt{\theta}}{\theta+1}|f'(x^+)+\theta f'(x^-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}. \qquad (25)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\Big|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)\Big(\int_x^u \frac{1}{2}(f'(x^+)-f'(x^-))\Big(\operatorname{sgn}(u-x)+\frac{\theta-1}{\theta+1}\Big)dv\Big)du\Big|\\
&\quad\leq \frac{\theta}{\theta+1}|f'(x^+)-f'(x^-)|\Big(\frac{\theta C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}\\
&\quad\leq \frac{\theta^{\frac{3}{2}}}{\theta+1}|f'(x^+)-f'(x^-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}. \qquad (26)
\end{aligned}$$

Considering (23)-(26), we obtain the following estimates

$$\begin{aligned}
|G_{n,h,\rho}^{d,\theta}(f;x)-f(x)| &\leq \Big|\int_0^\infty K_{n,h,\rho}^{d,\theta}(x,u)\Big(\int_x^u f_x'(v)dv\Big)du\Big|\\
&\quad+\frac{\sqrt{\theta}}{\theta+1}|f'(x^+)+\theta f'(x^-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}\\
&\quad+\frac{\theta^{\frac{3}{2}}}{\theta+1}|f'(x^+)-f'(x^-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}\\
&\leq \Big|A_{n,h,\rho}^{d,\theta}(f_x';x)+B_{n,h,\rho}^{d,\theta}(f_x';x)\Big|\\
&\quad+\frac{\sqrt{\theta}}{\theta+1}|f'(x^+)+\theta f'(x^-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}\\
&\quad+\frac{\theta^{\frac{3}{2}}}{\theta+1}|f'(x^+)-f'(x^-)|\Big(\frac{C_1 x(1+\rho)}{n\rho}\Big)^{\frac{1}{2}}, \qquad (27)
\end{aligned}$$

where

$$A_{n,h,\rho}^{d,\theta}(f'_x;x)=\int_0^x\left(\int_x^u f'_x(v)dv\right)K_{n,h,\rho}^{d,\theta}(x,u)du$$

and

$$B_{n,h,\rho}^{d,\theta}(f'_x;x)=\int_x^\infty\left(\int_x^u f'_x(v)dv\right)K_{n,h,\rho}^{d,\theta}(x,u)du.$$

Now, we estimate the terms $A_{n,h,\rho}^{d,\theta}(f'_x;x)$ and $B_{n,h,\rho}^{d,\theta}(f'_x;x)$. Using the definition of $\xi_{n,\rho}^{\theta}(.,.)$ given in Lemma 6 and integrating by parts, we can write

$$\begin{aligned}A_{n,h,\rho}^{d,\theta}(f'_x;x)&=\int_0^x\left(\int_x^u f'_x(v)dv\right)\frac{\partial\xi_{n,\rho}^{\theta}(x,u)}{\partial u}du\\&=\int_0^x f'_x(u)\xi_{n,\rho}^{\theta}(x,u)du.\end{aligned}$$

Thus,

$$\begin{aligned}|A_{n,h,\rho}^{d,\theta}(f'_x;x)|&=\left|\int_0^x f'_x(u)\xi_{n,\rho}^{\theta}(x,u)du\right|\\&\leq\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(u)|\xi_{n,\rho}^{\theta}(x,u)du+\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(u)|\xi_{n,\rho}^{\theta}(x,u)du.\end{aligned}$$

Since $f'_x(x)=0$ and $\xi_{n,\rho}^{\theta}(x,u)\leq 1$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(u)|\xi_{n,\rho}^{\theta}(x,u)dt&=\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(u)-f'_x(x)|\xi_{n,\rho}^{\theta}(x,u)du\\&\leq\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(u)-f'_x(x)|du\\&\leq\int_{x-\frac{x}{\sqrt{n}}}^x\left(\bigvee_u^x f'_x\right)du\\&\leq\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).\end{aligned}$$

Now, considering $\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(u)|\xi_{n,\rho}^{\theta}(x,u)du$ and using Lemma 6, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(u)|\xi_{n,\rho}^{\theta}(x,u)du&\leq\theta\frac{C_1x(1+\rho)}{n\rho}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{|f'_x(u)|}{(x-u)^2}du\\&=\theta\frac{C_1x(1+\rho)}{n\rho}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{|f'_x(u)-f'_x(x)|}{(x-u)^2}du\\&\leq\theta\frac{C_1x(1+\rho)}{n\rho}\int_0^{x-\frac{x}{\sqrt{n}}}\left(\bigvee_u^x f'_x\right)\frac{du}{(x-u)^2}.\end{aligned}$$

Assuming $u = x - \frac{x}{v}$, we have

$$\int_0^{x-\frac{x}{\sqrt{n}}} |f_x'(u)|\xi_{n,\rho}^{\theta}(x,u)du \leq \theta\frac{C_1(1+\rho)}{n\rho}\int_1^{\sqrt{n}}\left(\bigvee_{x-\frac{x}{v}}^{x} f_x'\right)dv$$
$$\leq \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f_x'\right).$$

Therefore,

$$|A_{n,h,\rho}^{d,\theta}(f_x';x)| \leq \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f_x'\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\right). \tag{28}$$

Considering

$$|B_{n,h,\rho}^{d,\theta}(f_x';x)| = \left|\int_x^{\infty}\left(\int_x^u f_x'(v)dv\right)K_{n,h,\rho}^{d,\theta}(x,u)du\right|,$$

using integration by parts and applying Lemma 6 with $z = x + \frac{x}{\sqrt{n}}$, we have

$$|B_{n,h,\rho}^{d,\theta}(f_x';x)| \leq \left|\int_x^z f_x'(u)(1-\xi_{n,\rho}^{\theta}(x,u))du\right| + \left|\int_z^{\infty} f_x'(u)(1-\xi_{n,\rho}^{\theta}(x,u))du\right|$$
$$\leq \int_x^z \bigvee_x^u f_x' du + \theta\frac{C_1x(1+\rho)}{n\rho}\int_z^{\infty}\bigvee_x^u f_x'\frac{1}{(u-x)^2}du$$
$$\leq \frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right) + \theta\frac{C_1x(1+\rho)}{n\rho}\int_{x+\frac{x}{\sqrt{n}}}^{\infty}\bigvee_x^u f_x'(u-x)^{-2}du. \tag{29}$$

Putting $u = x + \frac{x}{v}$, we get

$$\theta\frac{C_1x(1+\rho)}{n\rho}\int_{x+\frac{x}{\sqrt{n}}}^{\infty}\bigvee_x^u f_x'(u-x)^{-2}du = \theta\frac{C_1(1+\rho)}{n\rho}\int_0^{\sqrt{n}}\bigvee_x^{x+\frac{x}{v}} f_x'dv$$
$$\leq \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{k}} f_x'. \tag{30}$$

Combining (29) and (30), we have

$$|B_{n,h,\rho}^{d,\theta}(f_x';x)| \leq \frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right) + \theta\frac{C_1(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}} f_x'\right). \tag{31}$$

Now, assembling estimates (27), (28) and (31), we get the required result. □

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