

Convolution and radius problems of analytic functions associated with the tilted Carathéodory functions*

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Abstract. The concept of convolution is applied to investigate some subordination results for the normalized analytic functions whose first derivative belongs to the class of the tilted Carathéodory functions. The sharp radius of starlikeness of order α of the product of two normalized analytic functions satisfying certain specified conditions is computed. In addition, various sharp radius constants such as the radius of lemniscate of Bernoulli starlikeness, the radius of parabolic starlikeness and several other radius constants of product of two normalized analytic functions are also determined. Relevant connections of our results with the existing results are also pointed out.

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1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and \mathcal{A} denote the class of analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let \mathcal{A}_0 denote the class of analytic functions f with the normalization $f(0) = 1$. For analytic functions f and g , we say that f is subordinate to g , denoted by $f \prec g$, if there is a function $w : \mathbb{D} \rightarrow \mathbb{D}$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. The subclass of \mathcal{A} containing univalent functions is denoted by \mathcal{S} . The famous Bieberbach conjecture led researchers to explore various subclasses of the class \mathcal{S} . The class \mathcal{S}^* of starlike functions is the collection of functions $f \in \mathcal{S}$ for which $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all

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$z \in \mathbb{D}$. The class \mathcal{K} of convex functions consists of all functions $f \in \mathcal{S}$ for which $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ for all $z \in \mathbb{D}$. These subclasses are closely related to the class of Carathéodory functions and play a very important role in the study of many classes. The class \mathcal{P} of Carathéodory functions is a collection of analytic functions $p \in \mathcal{A}_0$ satisfying $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$. The function $p_0(z) = (1+z)/(1-z)$ univalently maps the unit disk onto the right-half plane, and is a leading example of the class \mathcal{P} which acts as an extremal function for many problems. Thus, it is clear that the classes of starlike and convex functions can be alternatively defined as $zf'(z)/f(z) \in \mathcal{P}$ and $1 + zf''(z)/f'(z) \in \mathcal{P}$, respectively. This association of the normalized analytic functions with the Carathéodory functions is helpful in the study of coefficient problems, radius problems, growth and distortion theorems and several other problems.

In 2012, Wang [32] studied a class of titled Carathéodory functions, which is a generalization of Carathéodory functions. For an analytic function $p \in \mathcal{A}_0$ and $\lambda \in (-\pi/2, \pi/2)$, let

$$\mathcal{P}_\lambda := \{p \in \mathcal{A}_0 : \operatorname{Re}(e^{i\lambda}p(z)) > 0\}$$

denote the class of titled Carathéodory functions of angle λ . The class \mathcal{P}_λ is convex and compact. Note that $\mathcal{P}_0 =: \mathcal{P}$. The function $p_\lambda(z) = (1 + e^{-2i\lambda}z)/(1 - z)$ maps univalently \mathbb{D} onto tilted right-half plane $\operatorname{Re}(e^{i\lambda}w) > 0$ and acts as an extremal function for many problems. Wang [32] established several types of relations between the functions in the class \mathcal{P}_λ and the class \mathcal{P} . In addition, it was proved that modulus of the n th coefficient of functions in this class is bounded above by $2 \cos \lambda$. Further, for the functions $p \in \mathcal{P}_\lambda$, the sharp estimates on $|zp'(z)/p(z)|$ and $|p'(z)|$ were also derived. These results are very helpful in determining radius properties. Some more results related to the class can be found in [13, 14, 18, 23].

Using the concept of convolution and subordination, Shanmugam [27] studied the unified class $\mathcal{S}_g^*(\varphi)$, defined as $\mathcal{S}_g^*(\varphi) := \{f \in \mathcal{A} : z(f * g)'(z)/(f * g)(z) \prec \varphi(z)\}$, where φ is a convex function and g is a fixed function in class \mathcal{A} . For a general analytic function φ with $\operatorname{Re} \varphi(z) > 0$ ($z \in \mathbb{D}$) and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$, the classes $\mathcal{S}_{z/(1-z)}^*(\varphi) =: \mathcal{S}^*(\varphi)$ and $\mathcal{S}_{z/(1-z)^2}^*(\varphi) =: \mathcal{K}(\varphi)$ were studied by Ma and Minda [19]. For special choices of φ , the class $\mathcal{S}^*(\varphi)$ gives well-known subclasses of starlike functions. For some recent development related to the convolution, giving another unified treatment of starlike and convex functions the reader may refer to the papers [5, 35]. By restricting the values of $\zeta := zf'/f$ to lie in the precise domain of the right-half plane, several authors have defined many new subclasses of starlike functions in recent years. For example, $\mathcal{S}^*[A, B] := \mathcal{S}^*((1 + Az)/(1 + Bz))$ and $\mathcal{K}[A, B] := \mathcal{K}((1 + Az)/(1 + Bz))$ ($-1 \leq B < A \leq 1$) are the familiar classes of Janowski starlike and convex functions, see [11]. Moreover, for $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha) := \mathcal{S}^*[1 - 2\alpha, -1]$ and $\mathcal{K}(\alpha) := \mathcal{K}[1 - 2\alpha, -1]$ are the classes of starlike and convex functions of order α , respectively. For more details of these classes, see [10, 24, 8]. If $\varphi(z) = 1 + (2/\pi^2)(\log(1 - \sqrt{z})/(1 + \sqrt{z}))^2$, then the classes $\mathcal{S}^*(\varphi)$ reduce to the class \mathcal{S}_P^* of parabolic starlike functions, introduced by Rønning [25]. Analytically, $f \in \mathcal{S}_P^*$ if and only if ζ lies in the disk $|w - 1| < \operatorname{Re} w$. In 1994, Uralegaddi [31] studied the class defined by $\mathcal{M}(\beta) := \{f \in \mathcal{A} : \operatorname{Re}(zf'(z)/f(z)) < \beta, \beta > 1\}$. The following special cases of the class $\mathcal{S}^*(\varphi)$ were considered and studied by several

researchers:

- (1) $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$, see [30];
- (2) $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$, see [20];
- (3) $\mathcal{S}_c^* := \mathcal{S}^*(1 + 4z/3 + 2z^2/3)$, see [29];
- (4) $\mathcal{S}_R^* := \mathcal{S}^*(\varphi_0)$, where $\varphi_0(z) := 1 + (z/k)((k+z)/(k-z))$, $k = \sqrt{2}+1$, see [15];
- (5) $\mathcal{S}_q^* := \mathcal{S}^*(z + \sqrt{1+z^2})$, see [22];
- (6) $\mathcal{S}_s^* := \mathcal{S}^*(1 + \sin z)$, see [6];
- (7) $\mathcal{BS}^*(\alpha) := \mathcal{S}^*(1 + z/(1 - \alpha z^2))$ ($0 \leq \alpha < 1$), see [12].

Motivated by the works [20, 6, 7, 9, 33, 34, 32, 16, 4], convolution, subordination and radius properties of normalized analytic functions associated with the tilted Carathéodory functions are investigated in the present paper. The paper is arranged as follows. Section 2 provides several properties of the normalized analytic functions whose first derivative belongs to the class of the tilted Carathéodory functions. For fixed normalized analytic functions f and g , in Section 3, we determine the value of r^* so that $F(z) = g(z)h(z)/z$ is starlike of order α in the disk $|z| < r^*$. In addition, we also compute the $\mathcal{M}(\beta)$, $\mathcal{S}^*(\alpha)$, \mathcal{S}_P^* , \mathcal{S}_L^* , \mathcal{S}_R^* , \mathcal{S}_q^* , \mathcal{S}_c^* , \mathcal{S}_e^* , \mathcal{S}_s^* and $\mathcal{BS}^*(\alpha)$ -radii for the classes associated with the tilted Carathéodory functions. We also point out relevant connections of our results with the existing ones.

2. Convolution properties

The following results were proved by Wang and using some of them we derive our main results.

Lemma 1 ([32, Lemma 1, p. 673]). *For any real constant $\lambda \in (-\pi/2, \pi/2)$, the following conditions are equivalent for a function $p \in \mathcal{A}_0$:*

- (a) $p \in \mathcal{P}_\lambda$,
- (b) $(e^{i\lambda}p - i \sin \lambda)/\cos \lambda \in \mathcal{P}$,
- (c) *there exists a Borel probability measure μ on $\partial\mathbb{D}$ such that p can be represented as*

$$p(z) = \int_{\partial\mathbb{D}} \frac{1 + e^{-2i\lambda}xz}{1 - xz} d\mu(x),$$

- (d) $p \prec p_\lambda$ in \mathbb{D} .

Lemma 2 ([32, Theorem 2, p. 675]). *Let $p \in \mathcal{P}_{\lambda_1}$ and $q \in \mathcal{P}_{\lambda_2}$ with $\lambda_i \in (-\pi/2, \pi/2)$ ($i = 1, 2$). Then*

$$\operatorname{Re} \left(e^{i(\lambda_1 + \lambda_2)}(p * q) \right) > -\cos(\lambda_1 - \lambda_2).$$

*In particular, if $\cos(\lambda_1 - \lambda_2)$ is negative, then $p * q \in \mathcal{P}_{\lambda_1 + \lambda_2}$.*

The following lemma of Ruscheweyh and Stankiewicz is needed for the proof of our results.

Lemma 3 (see [26]). *Let F and G be any convex functions in \mathbb{D} and if $f \prec F$ and $g \prec G$, then $f * g \prec F * G$ in \mathbb{D} .*

The following theorem gives the properties of normalized analytic functions whose derivative belongs to the class \mathcal{P}_λ .

Theorem 1. *Let $f, g \in \mathcal{A}$ with $f' \in \mathcal{P}_{\lambda_1}$ and $g' \in \mathcal{P}_{\lambda_2}$, $\lambda_i \in (-\pi/2, \pi/2)$ ($i = 1, 2$). Then the following holds:*

- (1) $\operatorname{Re} (e^{i(\lambda_1+\lambda_2)}(f' * g')(z)) > -\cos(\lambda_1 - \lambda_2)$
- (2) $\frac{f(z)}{z} \prec 1 + (1 + e^{2i\lambda_1}) \sum_{n=1}^\infty \frac{z^n}{n+1}$
- (3) $(f * g)'(z) \prec 1 + (1 + e^{2i\lambda_1}) (1 + e^{2i\lambda_2}) \sum_{n=1}^\infty \frac{z^n}{n+1}$
- (4) $\frac{(f * g)(z)}{z} \prec 1 + (1 + e^{2i\lambda_1}) (1 + e^{2i\lambda_2}) \sum_{n=1}^\infty \frac{z^n}{(n+1)^2}$
- (5) $\left| \frac{f(z)}{z} \right| \leq 1 + \sqrt{2} (1 + \cos(2\lambda_1))^{1/2} \sum_{n=1}^\infty \frac{|z|^n}{n+1}$
- (6) *If $f(z) = z + a_2 z^2 + \dots$, then $|a_n| \leq \frac{2 \cos \lambda_1}{n}$.*

Proof. Since $f' \in \mathcal{P}_{\lambda_1}$ and $g' \in \mathcal{P}_{\lambda_2}$, it follows that $f' \prec p_{\lambda_1}$ and $g' \prec p_{\lambda_2}$, where the functions p_{λ_1} and p_{λ_2} are convex functions defined by $p_{\lambda_1}(z) = (1 + e^{-2i\lambda_1} z)/(1 - z)$ and $p_{\lambda_2}(z) = (1 + e^{-2i\lambda_2} z)/(1 - z)$.

(1) The first result follows immediately from Lemma 2 by setting $f' = p$ and $g' = q$.

(2) The function $k : \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$k(z) = -\frac{2[z + \log(1 - z)]}{z} = \sum_{n=1}^\infty \frac{z^n}{n + 1}$$

is a convex function, and by the definition of convolution, we have

$$(f * k)(z) = 2 \sum_{n=1}^\infty \frac{a_n}{n + 1} z^n = \frac{2}{z} \int_0^z f(t) dt.$$

It should be noted that the function $l(z) = 1 + k(z)$ is also convex, and the functions p_{λ_1} and p_{λ_2} can be written as

$$p_{\lambda_1}(z) = 1 + (1 + e^{2i\lambda_1}) \sum_{n=1}^\infty z^n \quad \text{and} \quad p_{\lambda_2}(z) = 1 + (1 + e^{2i\lambda_2}) \sum_{n=1}^\infty z^n.$$

A computation yields

$$(p_{\lambda_1} * l)(z) = -1 + \frac{2}{z} \int_0^z p_{\lambda_1}(t) dt. \tag{1}$$

Since $f' \prec p_{\lambda_1}$ and l is a convex function, it follows from Lemma 3 that

$$\begin{aligned} (f' * l)(z) &\prec (p_{\lambda_1} * l)(z) \\ &= -1 + \frac{2}{z} \int_0^z \left[1 + (1 + e^{2i\lambda_1}) \sum_{n=1}^\infty t^n \right] dt. \end{aligned} \tag{2}$$

In view of (1), we have

$$(f' * l)(z) = -1 + \frac{2}{z} \int_0^z f'(t) dt. \tag{3}$$

From (2) and (3), we get

$$\begin{aligned} \frac{f(z)}{z} &< \frac{2}{z} \int_0^z \left[1 + (1 + e^{2i\lambda_1}) \sum_{n=1}^{\infty} t^n \right] dt \\ &= 1 + (1 + e^{2i\lambda_1}) \sum_{n=1}^{\infty} \frac{z^n}{n+1}. \end{aligned} \tag{4}$$

(3) It is well-known that for any two functions $f, g \in \mathcal{A}$, we have $(f * g)'(z) = g'(z) * (f(z)/z)$. Since $g' \in \mathcal{P}_{\lambda_2}$, so $g' \prec p_{\lambda_2}$, and the function p_{λ_2} is convex. So in the of Lemma 3, subsequent upon convoluting (4) and $g' \prec p_{\lambda_2}$ side by side, we have

$$g'(z) * \frac{f(z)}{z} \prec \left(1 + (1 + e^{2i\lambda_2}) \sum_{n=1}^{\infty} z^n \right) * \left(1 + (1 + e^{2i\lambda_1}) \sum_{n=1}^{\infty} \frac{z^n}{n+1} \right)$$

or equivalently,

$$(f * g)'(z) \prec 1 + (1 + e^{2i\lambda_1})(1 + e^{2i\lambda_2}) \sum_{n=1}^{\infty} \frac{z^n}{n+1}. \tag{5}$$

(4) Let $\varphi(z) := (f * g)(z)$. Then (5) can be written as

$$\varphi'(z) \prec 1 + (1 + e^{2i\lambda_1})(1 + e^{2i\lambda_2}) \sum_{n=1}^{\infty} \frac{z^n}{n+1}. \tag{6}$$

Convoluting both sides of (6) with the convex function l , we have

$$\varphi'(z) * l(z) \prec \left(1 + (1 + e^{2i\lambda_1})(1 + e^{2i\lambda_2}) \sum_{n=1}^{\infty} \frac{z^n}{n+1} \right) * \left(1 + \sum_{n=1}^{\infty} 2 \frac{z^n}{n+1} \right)$$

or equivalently,

$$\frac{1}{z} \int_0^z \varphi'(t) dt \prec 1 + (1 + e^{2i\lambda_1})(1 + e^{2i\lambda_2}) \sum_{n=1}^{\infty} \frac{z^n}{(n+1)^2}.$$

This gives the desired result.

(5) The fifth implication is obtained from part (2), using the triangle inequality and the fact $|1 + e^{2i\lambda_1}| = \sqrt{2} (1 + \cos(2\lambda_1))^{1/2}$. The upper bound is sharp.

(6) Let $f(z) = z + a_2 z^2 + \dots$. Since $f' \in \mathcal{P}_{\lambda_1}$, it follows that there exists $p(z) = 1 + p_1 z + p_2 z^2 + \dots \in \mathcal{P}_{\lambda_1}$ such that $f'(z) = p(z)$. Thus, we have $na_n = p_n$. The desired result follows by using the fact that $|p_n| \leq 2 \cos \lambda_1$.

This completes the proof. □

3. Radius properties

For two sub-classes Υ_1 and Υ_2 of \mathcal{A} , the Υ_1 radius of Υ_2 , is the largest number $\rho \in (0, 1)$ such that $r^{-1}f(rz) \in \Upsilon_1$, $0 < r \leq \rho$ for all $f \in \Upsilon_2$. In recent years, authors [15, 6, 29, 7, 17, 20, 1] have studied the radius properties for many well-known classes of analytic functions. For $\lambda \in (-\pi/2, \pi/2)$, we define the class \mathcal{SP}_λ as a collection of the normalized analytic functions f such that $zf'(z)/f(z) \in \mathcal{P}_\lambda$. Such functions are called λ -spirallike functions. It should be noted that $\mathcal{SP}_0 =: \mathcal{S}^*$. For a survey on spirallike and related classes of functions the reader may refer to the paper [2]. A function $f \in \mathcal{A}$ is called the λ -Robertson function if $zf' \in \mathcal{SP}_\lambda$. Some subordination problems related to the λ -Robertson functions were discussed by Wang [33]. Coefficient estimates for close-to-convex functions with argument β were discussed in [34].

The next theorem provides the value of constant r^* so that the function $F(z) = g(z)h(z)/z$ is starlike of order α in the disk $|z| < r^*$.

Theorem 2. *Let the functions $g \in \mathcal{SP}_{\lambda_1}$ and $h \in \mathcal{SP}_{\lambda_2}$, $\lambda_i \in (-\pi/2, \pi/2)$ ($i = 1, 2$). Then the function $F(z) = g(z)h(z)/z$ is starlike of order $\alpha \in [0, 1)$ in the disk $|z| < r^*$, where*

$$r^* = \frac{\cos \lambda_1 + \cos \lambda_2 + \sqrt{(\cos \lambda_1 + \cos \lambda_2)^2 + (\alpha - 1)(2 \cos(\lambda_1 + \lambda_2) \cos(\lambda_1 - \lambda_2) + \alpha + 1)}}{2 \cos(\lambda_1 + \lambda_2) \cos(\lambda_1 - \lambda_2) + \alpha + 1}.$$

The result is sharp.

We need the following result to derive Theorem 2.

Lemma 4 ([32, Theorem 5, p. 677], see also [23]). *Let $p \in \mathcal{P}_\lambda$. Then*

$$\operatorname{Re} p(z) \geq \frac{1 + r^2 \cos 2\lambda - 2r \cos \lambda}{1 - r^2}.$$

The result is sharp.

Proof of Theorem 2. Define the function $F : \mathbb{D} \rightarrow \mathbb{C}$ by $F(z) = g(z)h(z)/z$, where $g \in \mathcal{SP}_{\lambda_1}$ and $h \in \mathcal{SP}_{\lambda_2}$. Then the function F is analytic in \mathbb{D} . A logarithmic differentiation gives

$$\frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} - 1.$$

Thus, by making use of Lemma 4, we have

$$\begin{aligned} \operatorname{Re} \frac{zF'(z)}{F(z)} &\geq \frac{1 + r^2 \cos 2\lambda_1 - 2r \cos \lambda_1}{1 - r^2} + \frac{1 + r^2 \cos 2\lambda_2 - 2r \cos \lambda_2}{1 - r^2} - 1 \\ &= \frac{2 + 2r^2 \cos(\lambda_1 + \lambda_2) \cos(\lambda_1 - \lambda_2) - 2r(\cos \lambda_1 + \cos \lambda_2)}{1 - r^2} - 1 \\ &=: T. \end{aligned}$$

It follows that $F \in \mathcal{S}^*$ if $T \geq \alpha$, or equivalently, if

$$(2 \cos(\lambda_1 + \lambda_2) \cos(\lambda_1 - \lambda_2) + \alpha + 1)r^2 - 2(\cos \lambda_1 + \cos \lambda_2)r - \alpha + 1 \geq 0. \quad (7)$$

Now consider the functions g_0 and h_0 defined by

$$g_0(z) = \frac{z}{(1-xz)^{1+e^{-2i\lambda_1}}} \in \mathcal{SP}_{\lambda_1} \quad \text{and} \quad h_0(z) = \frac{z}{(1-xz)^{1+e^{-2i\lambda_2}}} \in \mathcal{SP}_{\lambda_2}$$

with $|x| = 1$. For the functions g_0 and h_0 defined above, we have

$$\frac{zg'_0(z)}{g_0(z)} = \frac{1+e^{-2i\lambda_1}xz}{1-xz} = p_{\lambda_1}(xz) \in \mathcal{P}_{\lambda_1} \quad \text{and} \quad \frac{zh'_0(z)}{h_0(z)} = \frac{1+e^{-2i\lambda_2}xz}{1-xz} = p_{\lambda_2}(xz) \in \mathcal{P}_{\lambda_2}.$$

Since the functions $p_{\lambda_i}(xz)$ ($i = 1, 2$) maximize the first inequality of Lemma 4 over the class \mathcal{P}_{λ_i} ($i = 1, 2$), it follows that the result asserted in Theorem 2 is sharp for the function

$$F_0(z) = \frac{g_0(z)h_0(z)}{z}.$$

This ends the proof. □

Remark 1. *By setting $\lambda_1 = 0 = \lambda_2$ and $\alpha = 0$, Theorem 2 reduces to the result [21, Corollary 2, p. 795].*

In order to prove further results related to radius properties, the following lemma is needed:

Lemma 5 ([32, Theorem 6, p. 678]). *Let $p \in \mathcal{P}_\lambda$, $\lambda \in (-\pi/2, \pi/2)$. Then*

$$\left| \frac{zp'(z)}{p(z)} \right| \leq M(\lambda, |z|),$$

where

$$M(\lambda, r) := \begin{cases} \frac{2r \cos \lambda}{1+r^2-2r|\sin \lambda|}, & r < |\tan \lambda/2|; \\ \frac{2r}{1-r^2}, & r \geq |\tan \lambda/2|. \end{cases} \tag{8}$$

The equality holds for some point $z_0 = re^{i\theta}$, $0 < r < 1$, if and only if $p(z) = p_\lambda(xz)$, where $x = e^{i(\alpha-\theta)}$ with α satisfying

$$\alpha = \begin{cases} \frac{\pi}{2} + \lambda, & r < -\tan(\lambda/2); \\ -\frac{\pi}{2} + \lambda, & r < \tan(\lambda/2); \\ \arcsin\left(\frac{1+r^2}{r^2-1}\right) + \lambda, & r \geq |\tan(\lambda/2)|. \end{cases} \tag{9}$$

The following theorems give the sharp radius constants such as the radius of lemniscate of Bernoulli starlikeness, the radius of parabolic starlikeness and other radii for the product of two normalized analytic functions satisfying certain specified conditions. To derive these results, we extensively use Lemma 5. Before proceeding further, for $\lambda \in (-\pi/2, \pi/2)$, we define the class S_λ by

$$S_\lambda := \left\{ f \in \mathcal{A} : \frac{f(z)}{z} \in \mathcal{P}_\lambda \right\}.$$

Theorem 3. Let $g, h \in S_\lambda$, $\lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class \mathcal{S}_L^* in the disk $|z| = r < R_l$, where $R_l = r_l$ if $r < |\tan \lambda/2|$ and $R_l = r_l^*$ if $r \geq |\tan \lambda/2|$. Here r_l and r_l^* are given as

$$r_l = |\sin \lambda| + 2(\sqrt{2} + 1) \cos \lambda + \sqrt{((11 + 8\sqrt{2}) \cos \lambda + 4(\sqrt{2} + 1)|\sin \lambda|) \cos \lambda}$$

and

$$r_l^* = -(\sqrt{2} + 1) + \sqrt{4 + 2\sqrt{2}} \approx 0.20.$$

The result is sharp.

Proof. Since $g, h \in S_\lambda$, it follows that the functions $p(z) = g(z)/z$ and $q(z) = h(z)/z \in \mathcal{P}_\lambda$. In terms of p and q , the function $F(z) = g(z)h(z)/z$ can be written as: $F(z) = zp(z)q(z)$. A logarithmic differentiation gives

$$\frac{zF'(z)}{F(z)} - 1 = \frac{zp'(z)}{p(z)} + \frac{zq'(z)}{q(z)}. \tag{10}$$

In the view of equation (10) and Lemma 5, we get

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{zq'(z)}{q(z)} \right| \leq 2M(\lambda, |z|), \quad |z| = r, \tag{11}$$

where $M(\lambda, r)$ is given by (8). Thus, in view of the result [3, Lemma 2.2, p. 6559], $f \in \mathcal{S}_L^*$ if and only if $w := zF'(z)/F(z)$ lies in the disk $|w - a| < r_a$, contained in $|w^2 - 1| < 1$, for

$$r_a = \begin{cases} (\sqrt{1 - a^2} - (1 - a^2))^{1/2}, & 0 < a \leq 2\sqrt{2}/3; \\ \sqrt{2} - a, & 2\sqrt{2}/3 \leq a < \sqrt{2}. \end{cases}$$

Using the above result it is evident that $f \in \mathcal{S}_L^*$, if

$$2M(\lambda, r) \leq \sqrt{2} - 1. \tag{12}$$

Now if $r < |\tan \lambda/2|$, then (12) takes the form

$$\frac{2r \cos \lambda}{1 + r^2 - 2r|\sin \lambda|} \leq \frac{\sqrt{2} - 1}{2}$$

which simplifies to the following inequality:

$$r^2 - (2|\sin \lambda| + 4(\sqrt{2} + 1) \cos \lambda)r + 1 \geq 0. \tag{13}$$

In a similar way, if $r \geq |\tan \lambda/2|$, then (12) becomes $2r/(1 - r^2) \geq (\sqrt{2} - 1)$, which reduces to

$$r^2 + 2(\sqrt{2} + 1)r - 1 \leq 0. \tag{14}$$

In view of inequalities (12)–(14), we get the desired result.

Now consider the function $F_0(z) = g_0(z)h_0(z)/z$, where g_0 and h_0 are given by

$$h_0(z) = g_0(z) = \frac{z(1 + e^{-2i\lambda}xz)}{1 - xz}. \tag{15}$$

Then it is easy to see that $g_0(z)/z = p_\lambda(xz) \in \mathcal{P}_\lambda$ and $h_0(z)/z = q_\lambda(xz) \in \mathcal{P}_\lambda$. Thus, in view of Lemma 5, the equality in the result holds in case of the functions $F_0(z) = g_0(z)h_0(z)/z$ at some point $z_0 = re^{i\theta}$ ($0 < r < 1$), where $x = e^{i(\alpha-\theta)}$ with α given by (9). \square

Theorem 4. *Let $\beta > 1$ and $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{M}(\beta)$ in the disk $|z| = r < R_\beta$, where $R_\beta = r_\beta$ if $r < |\tan \lambda/2|$ and $R_\beta = r_\beta^*$ if $r \geq |\tan \lambda/2|$. Here r_β and r_β^* are given by*

$$r_\beta = \frac{(\beta - 1)|\sin \lambda| + 2 \cos \lambda + \sqrt{(3 - \beta^2 + 2\beta) \cos^2 \lambda + 4(\beta - 1)|\sin \lambda| \cos \lambda}}{\beta - 1}$$

and

$$r_\beta^* = \frac{2 + \sqrt{\beta^2 - 2\beta + 5}}{\beta - 1}.$$

The radius estimate is sharp.

Proof. Proceeding as in the proof of Theorem 2, we get inequality (11). From this, we see that

$$\operatorname{Re} \left(\frac{zF'(z)}{F(z)} \right) \leq 1 + 2M(\lambda, r) < \beta.$$

From the above, we see that $F \in \mathcal{M}(\beta)$ if the inequality

$$M(\lambda, r) < \beta - 1 \tag{16}$$

holds. Now if $r < |\tan \lambda/2|$, then (16) becomes

$$\frac{2r \cos \lambda}{1 + r^2 - 2r|\sin \lambda|} \leq \frac{1}{2}(\beta - 1),$$

which is equivalent to

$$(\beta - 1)r^2 - 2((\beta - 1)|\sin \lambda| + 2 \cos \lambda)r + (\beta - 1) \geq 0. \tag{17}$$

In a similar way, if $r \geq |\tan \lambda/2|$, then (16) can be written as $2r/(1-r^2) \geq (\beta-1)/2$, which reduces to

$$(\beta - 1)r^2 + 4r - (\beta - 1) \leq 0. \tag{18}$$

Using inequalities (16)–(18), we get the required sharp radius. The equality holds in the case of the function $F_0(z) = g_0(z)h_0(z)/z$, as described in the proof of Theorem 3, where g_0 and h_0 are given by (15). \square

Theorem 5. *Let $0 \leq \alpha < 1$ and $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{S}^*(\alpha)$ in the disk $|z| = r < R_\alpha$, where $R_\alpha = r_\alpha$ if $r < |\tan \lambda/2|$ and $R_\alpha = r_\alpha^*$ if $r \geq |\tan \lambda/2|$. Here r_α and r_α^* are given by*

$$r_\alpha = \frac{(1 - \alpha)|\sin \lambda| + 2 \cos \lambda + \sqrt{(3 - \alpha^2 + 2\alpha) \cos^2 \lambda + 4(1 - \alpha)|\sin \lambda| \cos \lambda}}{1 - \alpha}$$

and

$$r_\alpha^* = \frac{2 + \sqrt{\alpha^2 - 2\alpha + 5}}{1 - \alpha}.$$

The radius estimate is sharp.

Proof. Proceeding as in the proof of Theorem 2, we have inequality (11). From this, we see that

$$\operatorname{Re} \left(\frac{zF'(z)}{F(z)} \right) > 1 - 2M(\lambda, r) > \alpha$$

holds if

$$M(\lambda, r) < \frac{1}{2}(1 - \alpha). \tag{19}$$

Therefore, $F \in \mathcal{S}^*(\alpha)$ if (19) holds. Now if $r < |\tan \lambda/2|$, then (19) becomes

$$\frac{2r \cos \lambda}{1 + r^2 - 2r|\sin \lambda|} \leq \frac{1}{2}(1 - \alpha),$$

which can be equivalently written as

$$(1 - \alpha)r^2 - 2((1 - \alpha)|\sin \lambda| + 2 \cos \lambda)r + (1 - \alpha) \geq 0. \tag{20}$$

Similarly, if $r \geq |\tan \lambda/2|$, then (19) $2r/(1 - r^2) \geq (1 - \alpha)/2$, which reduces to

$$(1 - \alpha)r^2 + 4r - (1 - \alpha) \leq 0. \tag{21}$$

The desired result follows from inequalities (19)–(21). The sharpness of the result is confirmed by the function $F_0(z) = g_0(z)h_0(z)/z$, where g_0 and h_0 are given by (15). □

Theorem 6. *Let $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class \mathcal{S}_p^* in the disk $|z| = r < R_p$, where $R_p = r_p$ if $r < |\tan \lambda/2|$ and $R_p = r_p^*$ if $r \geq |\tan \lambda/2|$. Here r_p and r_p^* are given as*

$$r_p = |\sin \lambda| + 4 \cos \lambda + \sqrt{15 \cos^2 \lambda + 8|\sin \lambda| \cos \lambda}$$

and $r_p^* = (\sqrt{17} - 4)/2 \approx 0.062$. The radius estimate is sharp.

Proof. As in the proof of the previous theorem, we have (11). Now in light of the result [28, Lemma 3.3, p. 5], it is clear that $F \in \mathcal{S}_p^*$ if and only if $\Omega := \{w \in \mathbb{C} : |w - a| < r_a\} \subset \{w \in \mathbb{C} : |w - 1| < \operatorname{Re} w\}$, where r_a is given by

$$r_a = \begin{cases} a - \frac{1}{2}, & 1/2 < a \leq 3/2; \\ \sqrt{2a - 2}, & a \geq 3/2. \end{cases}$$

Using this result, we see that disk (11) lies inside the parabolic region Ω provided

$$M(\lambda, r) < \frac{1}{4}. \tag{22}$$

If $r < |\tan \lambda/2|$, then inequality (22) is equivalent to

$$\frac{2r \cos \lambda}{1 + r^2 - 2r|\sin \lambda|} \leq \frac{1}{4},$$

which implies

$$r^2 - 2(|\sin \lambda| + 4 \cos \lambda)r + 1 \geq 0. \tag{23}$$

Further, if $r \geq |\tan \lambda/2|$, then inequality (22) becomes $2r/(1 - r^2) \geq 1/4$. This gives the condition

$$r^2 + 8r - 1 \leq 0. \tag{24}$$

Inequalities (22) – (24) give the desired radius estimate. Sharpness can be verified for the function $F_0(z) = g_0(z)h_0(z)/z$, where g_0 and h_0 are given by (15). \square

Theorem 7. *Let $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{S}_{\mathcal{R}}^*$ in the disk $|z| = r < R_\rho$, where $R_\rho = r_\rho$ if $r < |\tan \lambda/2|$ and $R_\rho = r_\rho^*$ if $r \geq |\tan \lambda/2|$. Here r_ρ and r_ρ^* are given by*

$$r_\rho = |\sin \lambda| + 2(3 + 2\sqrt{2}) \cos \lambda + \sqrt{(67 + 48\sqrt{2}) \cos^2 \lambda + 4(3 + 2\sqrt{2})|\sin \lambda| \cos \lambda}$$

and

$$r_\rho^* = (\sqrt{21 - 12\sqrt{2}} - 2)/(3 - 2\sqrt{2}) \approx 0.043.$$

The radius estimate is sharp.

Proof. Proceeding as in the proof of Theorem 2, by using the result [15, Lemma 2.2, p. 4], which states that $\{w : |w - a| < r_a\} \subset \varphi_0(\mathbb{D})$, where $\varphi_0(z) = 1 + (z/k)((k+z)/(k-z))$, $k = \sqrt{2} + 1$ and r_a is given by

$$r_a = \begin{cases} a - 2(\sqrt{2} - 1) & \text{if } 2(\sqrt{2} - 1) < a \leq \sqrt{2}; \\ 2 - a & \text{if } \sqrt{2} \leq a < 2, \end{cases}$$

we see that $F \in \mathcal{S}_{\mathcal{R}}^*$ if

$$M(\lambda, r) < \frac{1}{2}(3 - 2\sqrt{2}). \tag{25}$$

Moreover, if $r < |\tan \lambda/2|$, then from (25), we have the following inequality:

$$\frac{2r \cos \lambda}{1 + r^2 - 2r|\sin \lambda|} \leq \frac{1}{2}(3 - 2\sqrt{2})$$

and this is equivalent to

$$(3 - 2\sqrt{2})r^2 - 2((3 - 2\sqrt{2})|\sin \lambda| + 2 \cos \lambda)r + (3 - 2\sqrt{2}) \geq 0. \tag{26}$$

Further, in the case when $r \geq |\tan \lambda/2|$, from (25) we have the inequality $2r/(1 - r^2) \geq (3 - 2\sqrt{2})/2$ which reduces to

$$(3 - 2\sqrt{2})r^2 + 4r - (3 - 2\sqrt{2}) \leq 0. \tag{27}$$

Thus, using inequalities (25)–(27), we get the desired result as stated in the theorem. The estimate is best possible as the equality occurs in the case of the function $F_0(z) = g_0(z)h_0(z)/z$, where the functions g_0 and h_0 are given by (15), as described in the proof of Theorem 3. \square

Theorem 8. *Let $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class \mathcal{S}_q^* in the disk $|z| = r < R_q$, where $R_q = r_q$ if $r < |\tan \lambda/2|$ and $R_q = r_q^*$ if $r \geq |\tan \lambda/2|$ such that r_q and r_q^* are given by*

$$r_q = |\sin \lambda| + (2 + \sqrt{2}) \cos \lambda + \sqrt{(5 + 4\sqrt{2}) \cos^2 \lambda + 2(2 + \sqrt{2})|\sin \lambda| \cos \lambda}$$

and

$$r_q^* = (\sqrt{10 - 4\sqrt{2}} - 2)/(2 - \sqrt{2}) \approx 0.14.$$

The radius estimate is sharp.

Proof. Proceeding as in the proof of Theorem 2, we get disk (11). Now in view of the result [9, Lemma 2.1, p. 3], which states that if $\sqrt{2} - 1 < a \leq \sqrt{2} + 1$ and $r_a = 1 - |\sqrt{2} - a|$, then $\{w : |w - a| < r_a\} \subset \{w : |w^2 - 1| < 2|w|\}$, we get the desired result. Sharpness occurs for the function $F_0(z) = g_0(z)h_0(z)/z$ as described in the proof of Theorem 3. \square

The following Theorems 9–12 can be proved using the results [29, Lemma 2.5, p. 4], [20, Lemma 2.2, p. 3], [6, Lemma 3.3, p. 5] and [7, Lemma 3.4, p. 10], respectively. Since the proofs of these theorems are similar to that of Theorem 3, they are omitted.

Theorem 9. *Let $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class \mathcal{S}_c^* in the disk $|z| = r < R_c$, where $R_c = r_c$ if $r < |\tan \lambda/2|$ and $R_c = r_c^*$ if $r \geq |\tan \lambda/2|$. Here r_c and r_c^* are given by*

$$r_c = |\sin \lambda| + 3 \cos \lambda + \sqrt{8 \cos^2 \lambda + 6|\sin \lambda| \cos \lambda}$$

and

$$r_c^* = \sqrt{10} - 3 \approx 0.16.$$

The radius estimate is sharp for the function $F_0(z) = g_0(z)h_0(z)/z$.

Theorem 10. *Let $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class \mathcal{S}_e^* in the disk $|z| = r < R_e$, where $R_e = r_e$ if $r < |\tan \lambda/2|$ and $R_e = r_e^*$ if $r \geq |\tan \lambda/2|$. Here r_e and r_e^* are given by*

$$r_e = |\sin \lambda| + \frac{2e}{e - 1} \cos \lambda + \sqrt{\frac{3e^2 + 2e - 1}{(e - 1)^2} \cos^2 \lambda + \frac{4e}{e - 1} |\sin \lambda| \cos \lambda}$$

and

$$r_e^* = \frac{\sqrt{5e^2 - 2e + 1} - 2e}{(e - 1)^2}.$$

The radius estimate is sharp for the function $F_0(z) = g_0(z)h_0(z)/z$.

Theorem 11. Let $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class S_s^* in the disk $|z| = r < R_s$, where $R_s = r_s$ if $r < |\tan \lambda/2|$ and $R_s = r_s^*$ if $r \geq |\tan \lambda/2|$. Here r_s and r_s^* are given by

$$r_s = |\sin \lambda| + \frac{2}{\sin(1)} \cos \lambda + \sqrt{\frac{4 - \sin^2(1)}{\sin^2(1)} \cos^2 \lambda + \frac{4}{\sin(1)} |\sin \lambda| \cos \lambda}$$

and

$$r_s^* = (\sqrt{4 + \sin^2(1)} - 2)/\sin(1) \approx 0.20.$$

The radius estimate is sharp for the function $F_0(z) = g_0(z)h_0(z)/z$.

Theorem 12. Let $g, h \in S_\lambda, \lambda \in (-\pi/2, \pi/2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{BS}^*(\alpha)$ ($0 \leq \alpha < 1$) in the disk $|z| = r < R_b$, where $R_b = r_b$ if $r < |\tan \lambda/2|$ and $R_b = r_b^*$ if $r \geq |\tan \lambda/2|$. Here r_b and r_b^* are given by

$$r_b = |\sin \lambda| + 2(1 + \alpha) \cos \lambda + \sqrt{(4\alpha^2 + 8\alpha - 3) \cos^2 \lambda + 4(1 + \alpha) |\sin \lambda| \cos \lambda}$$

and

$$r_b^* = \sqrt{4\alpha^2 + 8\alpha + 5} - \alpha - 1.$$

The radius estimate is sharp for the function $F_0(z) = g_0(z)h_0(z)/z$.

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