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Characteristic of unique positive solution for a fractional q-difference equation with multistrip boundary conditions

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Abstract. We study a new fractional q-difference equation m-point boundary value problem supplemented with the Riemann-Stieltjes integral, while the nonlinear terms are increasing. Using a fixed point theorem in cones, good characteristics of a unique positive solution dependent on the parameter $\lambda > 0$ are established. An example demonstrating the main results is presented.

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1. Introduction

In this paper, we consider the following m-point boundary value problem of fractional q-difference equation

$$\begin{cases} D_q^\alpha x(t) + \lambda p(t) f(t, x(t)) = 0, \ t \in (0,1), \\ D_q^i x(0) = 0, \ 0 \le i \le n-2, \\ \mu D_q^{\alpha-1} x(1) - \nu\alpha[x] = \sum\limits_{j=1}^{m} \beta_j D_q^\gamma x(\xi_j), \end{cases} \tag{1}$$

where D_q^α is the standard Riemann-Liouville fractional q-derivative, $0 < q < 1$, $\alpha \in (n-1, n]$ is a real number, $n \ge 3$, $\alpha - \gamma - 1 > 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $\beta_j \ge 0$, $j = 1, 2, \dots, m$, $\lambda > 0$ is a parameter, $\mu, \nu > 0$, $p : [0,1] \to \mathbb{R}^+$, f is a continuous function. $\alpha[x]$ denotes the Riemann-Stieltjes integral $\alpha[x] = \int_0^1 x(t) d\Lambda(t)$ with respect to the function $\Lambda : [0,1] \to \mathbb{R}$ of bounded variation.

The original works on q-difference equations considered by Adams [1], Agarwal [2], Al-Salam [6] and Jackson [10] have drawn widespread attention to various aspects of biomathematics, physics and technical engineering due to intensive applications of mathematical modeling. The nanofluids anomalous mass transfer problem or the viscoelastic fluid boundary layer problem for example, the governing equations bear on the anomalous motion of nanoparticles or the spatial fractional derivative plays a key role in these problems; however, fractional differential equations can describe

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them well by similarity transformation, which can be used for further research in the interdisciplinary areas. The details can be seen in [11, 14] and related themes.

Recently, the existence and multiplicity of positive, sign-changing or nontrivial solutions for different kinds of boundary value problems such as integral, nonlocal, multiple-point boundary problems and some others can be found in [3, 4, 7, 8, 9, 12, 13, 15, 16, 17]. In reality, an integral boundary condition is more convenient to employ as it can be able to cover nonlocal and multiple-point conditions as exceptional cases. Many scholars deal with fractional differential equations involving Stieltjes integral boundary conditions including many kinds of boundary conditions:

$$\alpha[x] = \int_0^1 x(t)p(t)dt,\ p \in C([0,1],\mathbb{R}),$$
$$\alpha[x] = \sum_{i=1}^{n} \kappa_i x(\delta_i),\ 0 < \delta_i < 1,\ i = 1,2,\ldots,n.$$

Obviously, $\alpha[x]$ is not always positive for all positive x because κ_i, $p(t)$ can be negative.

For instance, Mao et al.[13] discussed a nonlocal boundary value problem as follows:

$$\begin{cases} D^q x(t) + f(t,x(t)) = 0,\ t \in (0,1), \\ x(1) = x'(1) = 0, \\ x(0) = \int_0^1 x(t)d\Lambda(t). \end{cases}$$

Here $2 < q \le 3$, $\lambda[x] = \int_0^1 x(t)d\Lambda(t)$ denotes a Stieltjes integral, $D^q x$ is a left-handed Riemann-Liouville derivative. The existence of a unique positive solution is derived in view of iterative methods. Further, an estimation of the approximation error and a convergence rate are also enunciated.

Liu et al.[12] investigated a singular nonlinear fractional differential equation

$$D_{0+}^{\alpha} u(t) + a(t) f(t, u(t), D_{0+}^{\beta_1} u(t), \ldots, D_{0+}^{\beta_{n-1}} u(t)) = 0,$$

with the conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)} u(0) = 0, \\ D_{0+}^{\beta} u(1) = \int_0^1 l(t) D_{0+}^{\beta_{n-1}} u(t) dA(t), \end{cases}$$

where $n-1 < \alpha \le n,\ i-1 < \beta_i \le i (i = 1,2,\ldots,n-1), \alpha - \beta_{n-1} > \alpha - \beta > 1, a \in C((0,1),\mathbb{R}_+),\ f : [0,1] \times (0,+\infty)^n \to \mathbb{R}_+$ and f, g may be singular. $\int_0^1 l(t)u(t)dA(t)$ denotes the Riemann-Stieltjes integral with a signed measure. By applying spectral analysis of relevant linear operators and Gelfand's formula, the existence of positive solutions for this problem is established.

This paper considers the existence of a unique positive solution for problem (1) by using a recent fixed point theorem in cones. It should be noted that m-point boundary value problems of fractional q-difference equations supplemented with the Stieltjes integral have not been encountered in previous works. In comparison with the existing results, the new features are shown in this paper as follows: Firstly, the form of problem (1) is new and the boundary condition is more general as it includes a multistrip fractional q-derivative of unknown function. Secondly, instead of applying the method of upper-lower solutions, we use a recent fixed point theorem in normal cones to get the main existence results. Thirdly, successive iterative schemes starting at any initial point will be constructed to converge uniformly to the unique solution. Most importantly, we can give some good characteristic of a unique positive solution dependent on the parameter $\lambda > 0$.

2. Preliminaries

In this section, we recall some useful definitions and lemmas, which we can refer to [1, 5, 10].

Definition 1 (see [2])**.** *Let $\alpha \geq 0$ and f be a function defined on* $[0,1]$. *The fractional q-integral of Riemann-Liouville type is $I_q^0 f(t) = f(t)$ and*

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s, \ \alpha > 0.$$

Note that $I_q^\alpha f(t) = I_q f(t)$ when $\alpha = 1$.

Definition 2 (see [2])**.** *The fractional q-derivative of Riemann-Liouville type of order $\alpha \geq 0$ is defined by*

$$D_q^\alpha f(t) = D_q^k I_q^{k-\alpha} f(t), \quad t \in [0,1],$$

where k is the smallest integer greater than or equal to α.

Moreover, let $\alpha > 0$, k be the smallest integer greater than or equal to α. Then

$$I_q^\alpha D_q^\alpha f(t) = f(t) - \sum_{n=0}^{k-1} \frac{t^{\alpha-k+n}}{\Gamma_q(\alpha-k+n+1)} D_q^\alpha f(0), \quad k \in \mathbb{N}. \tag{2}$$

First, we consider the following boundary value problem:

$$\begin{cases} D_q^\alpha x(t) + y(t) = 0, \ \alpha \in (n-1,n], \ t \in (0,1), \\ D_q^i x(0) = 0, \ 0 \leq i \leq n-2, \\ \mu D_q^{\alpha-1} x(1) - \nu \int_0^1 x(t) d\Lambda(t) = \sum\limits_{j=1}^m \beta_j D_q^\gamma x(\xi_j). \end{cases} \tag{3}$$

We require the following assumption:

(F_0) $\Lambda : [0,1] \to \mathbb{R}$ is a function of bounded variation and

$$\varpi := \mu\Gamma_q(\alpha) - \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)} \sum_{j=1}^m \beta_j \xi_j^{\alpha-\gamma-1} - \nu A > 0,$$

$$A := \int_0^1 t^{\alpha-1} d\Lambda(t) \geq 0,$$

$$\zeta(s) := \int_0^1 H(t,qs) d\Lambda(t) \geq 0$$

for

$$H(t,qs)=\frac{1}{\Gamma_q(\alpha)}\begin{cases} t^{\alpha-1}-(t-qs)^{(\alpha-1)}, 0\le qs\le t\le 1,\\ t^{\alpha-1}, \qquad\qquad\qquad 0\le t\le qs\le 1.\end{cases} \tag{4}$$

Let

$$K(\xi_j,qs)=\frac{1}{\Gamma_q(\alpha-\gamma)}\begin{cases} \xi_j^{\alpha-\gamma-1}-(\xi_j-qs)^{(\alpha-\gamma-1)}, 0\le qs\le \xi_j\le 1,\\ \xi_j^{\alpha-\gamma-1}, \qquad\qquad\qquad 0\le \xi_j\le qs\le 1.\end{cases} \tag{5}$$

Lemma 1. *Assume* (F_0) *holds and* $y\in C[0,1]$. *Then problem* (3) *has a unique solution*

$$x(t)=\int_0^1 G(t,qs)y(s)d_qs, \tag{6}$$

where

$$G(t,qs)=H(t,qs)+\frac{\sum\limits_{j=1}^{m}\beta_j t^{\alpha-1}}{\varpi}K(\xi_j,qs)+\frac{\nu t^{\alpha-1}}{\varpi}\zeta(s). \tag{7}$$

Proof. For $t\in[0,1]$, q-integrating from 0 to t for the first equation in (3), and by means of definitions 1, 2 and (2), one can see that

$$x(t)=c_1t^{\alpha-1}-I_q^{\alpha}y(t),\ c_1\in\mathbb{R}.$$

Since

$$D_q^{\alpha-1}x(1)=c_1\Gamma_q(\alpha)-\int_0^1 y(s)d_qs, \tag{8}$$

$$D_q^{\gamma}x(\xi_j)=c_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)}\xi_j{}^{\alpha-\gamma-1}-\frac{1}{\Gamma_q(\alpha-\gamma)}\int_0^{\xi_j}(\xi_j-qs)^{(\alpha-\gamma-1)}y(s)d_qs. \tag{9}$$

Combining (8) and (9), we obtain

$$\begin{aligned} c_1=&\frac{1}{\varpi}\left[\mu\int_0^1 y(s)d_qs-\frac{\nu}{\Gamma_q(\alpha)}\int_0^1\int_0^t(t-qs)^{(\alpha-1)}y(s)d_qsd\Lambda(t)\right.\\ &\left.-\frac{1}{\Gamma_q(\alpha-\gamma)}\sum_{j=1}^{m}\beta_j\int_0^{\xi_j}(\xi_j-qs)^{(\alpha-\gamma-1)}y(s)d_qs\right],\end{aligned}$$

and then we deduce that

$$
\begin{aligned}
x(t)&\\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}y(s)d_qs - \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}y(s)d_qs\\
&\quad + \frac{\nu t^{\alpha-1}}{\varpi\Gamma_q(\alpha)}\left[\int_0^1\int_0^t\left(t^{\alpha-1}-(t-qs)^{(\alpha-1)}\right)y(s)d_qsd\Lambda(t)+\int_0^1\int_t^1 t^{\alpha-1}y(s)d_qsd\Lambda(t)\right]\\
&\quad + \frac{t^{\alpha-1}}{\varpi\Gamma_q(\alpha-\gamma)}\left[\sum_{j=1}^m\beta_j\int_0^1 {\xi_j}^{\alpha-\gamma-1}y(s)d_qs - \sum_{j=1}^m\beta_j\int_0^{\xi_j}(\xi_j-qs)^{(\alpha-\gamma-1)}y(s)d_qs\right]\\
&= \int_0^1 H(t,qs)y(s)d_qs + \int_0^1 \frac{\nu t^{\alpha-1}}{\varpi}\zeta(s)y(s)d_qs + \int_0^1 \frac{\sum\limits_{j=1}^m \beta_j t^{\alpha-1}}{\varpi}K(\xi_j,qs)y(s)d_qs.
\end{aligned}
$$

The proof is completed. □

Lemma 2. *Green's functions $H(t,qs)$, $K(t,qs)$ and $G(t,qs)$ have the following properties:*

(i)

$$H(t,qs) > 0, t, s \in (0,1),$$

and

$$H(t,qs) \le \frac{1}{\Gamma_q(\alpha)}t^{\alpha-1} \le \frac{1}{\Gamma_q(\alpha)},\ \ \forall t,s\in[0,1];$$

(ii)

$$
\begin{aligned}
t^{\alpha-1}H(1,qs) \le H(t,qs) \le H(1,qs),\ \ \forall t,s\in[0,1];\\
t^{\alpha-\gamma-1}K(1,qs) \le K(t,qs) \le K(1,qs),\ \ \forall t,s\in[0,1];
\end{aligned}
$$

(iii)

$$\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}P(s) \le G(t,qs) \le \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}(P+P_m),$$

where

$$
\begin{aligned}
P(s) &= [1-(1-qs)^{(\alpha-1)}]P + [1-(1-qs)^{(\alpha-\gamma-1)}]P_1,\\
P_i &= \frac{\Gamma_q(\alpha)\sum\limits_{j=1}^m \beta_j {\xi_i}^{\alpha-\gamma-1}}{\varpi\Gamma_q(\alpha-\gamma)},\ \ i=1,m,\ \ P = 1+\frac{\nu A}{\varpi},\ \ \forall t,s\in[0,1].
\end{aligned}
$$

Proof. The conclusions (i) and (ii) are clearly established, so we only prove (iii).

By means of (i) and (ii), we obtain

$$\begin{aligned}G(t,qs) =&H(t,qs)+\frac{1}{\varpi}\sum_{j=1}^{m}\beta_j t^{\alpha-1}K(\xi_j,qs)+\frac{\nu t^{\alpha-1}}{\varpi}\zeta(s)\\ \le&\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}+\frac{\nu t^{\alpha-1}}{\varpi\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}d\Lambda(t)+\frac{\sum_{j=1}^{m}\beta_j t^{\alpha-1}{\xi_m}^{\alpha-\gamma-1}}{\varpi\Gamma_q(\alpha-\gamma)}\\ =&\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\left[1+\frac{\nu A}{\varpi}+\frac{\Gamma_q(\alpha)\sum_{j=1}^{m}\beta_j{\xi_m}^{\alpha-\gamma-1}}{\varpi\Gamma_q(\alpha-\gamma)}\right]=\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}(P+P_m).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&G(t,qs)\\ &\ge t^{\alpha-1}H(1,qs)+\frac{1}{\varpi}\sum_{j=1}^{m}\beta_j t^{\alpha-1}\xi_1^{\alpha-\gamma-1}K(1,qs)+\frac{\nu t^{\alpha-1}}{\varpi}\int_0^1 t^{\alpha-1}H(1,qs)d\Lambda(t)\\ &\ge\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\left[\left(1+\frac{\nu A}{\varpi}\right)\left(1-(1-qs)^{(\alpha-1)}\right)+\frac{\Gamma_q(\alpha)\sum_{j=1}^{m}\beta_j\xi_1^{\alpha-\gamma-1}}{\varpi\Gamma_q(\alpha-\gamma)}\left(1-(1-qs)^{(\alpha-\gamma-1)}\right)\right]\\ &=\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}P(s).\end{aligned}$$

□

Let $(E,\|\cdot\|)$ be a real Banach space, which is partially ordered by a cone $Q\subset E$. For fixed $h>\theta$, *i.e.*, $h\ge\theta$, $h\ne\theta$, a set Q_h is given by $Q_h=\{x\in E\mid x\sim h\}$. Here the notation $x\sim y$ means that for any $x,y\in E$, there exist $\mu>0$ and $\nu>0$ such that $\mu x\le y\le\nu x$. Obviously, $Q_h\subset Q$.

Next, we give a recent fixed point theorem presented in [18], that will play a key role in the following analysis.

Lemma 3. *Let Q be a normal cone in a real Banach space E with $h>\theta$, $T:Q\to Q$ an increasing operator satisfying the following:*

(i) there is $h_0\in Q_h$ such that $Th_0\in Q_h$;

(ii) for any $x\in Q$, $r\in(0,1)$, there exists $\varphi(r)\in(r,1)$ such that $T(rx)(t)\ge\varphi(r)Tx(t)$.

Then:

1. *the operator equation $Tx=x$ has a unique solution x^* in Q_h;*
2. *for any $x_0\in Q_h$, by constructing the sequence $x_n=Tx_{n-1}$, $n=1,2,\dots$, we get $x_n\to x^*$ as $n\to\infty$.*

Remark 1. *It is noteworthy that if $\varphi(r) = r,\ r \in (0,1)$ in Lemma 3. Then the operator $T : Q \to Q$ is said to be sub-homogeneous; if $\varphi(r) = r^\gamma,\ 0 \le \gamma < 1$ in Lemma 3, $T : Q \to Q$ is said to be γ-concave.*

Lemma 4. *On the basis of Lemma 3. If x_λ is the unique solution of operator equation $Tx = \lambda x$ for $\lambda > 0$, then we obtain:*

(i) *x_λ is strictly decreasing in λ; namely, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$;*

(ii) *if there exists $\gamma \in (0,1)$ such that $\varphi(r) \ge r^\gamma$ for $r \in (0,1)$, then x_λ is continuous in λ; namely, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|x_\lambda - x_{\lambda_0}\| \to 0$;*

(iii) $\lim_{\lambda\to+\infty} \|x_\lambda\| = 0,\ \lim_{\lambda\to0^+} \|x_\lambda\| = +\infty$.

3. Unique positive solution

In this paper, we work in Banach space $E = C[0,1]$, equipped with the norm $\|x\| = \max\{|x(t)| : t \in [0,1]\}$. Define the standard cone $Q = \{x \in C[0,1] | x(t) \ge 0,\ t \in [0,1]\}$. Then Q is normal. The operator $T : E \to E$ is given by

$$Tx(t) = \int_0^1 G(t,qs)p(s)f(s,x(s))d_qs,\ t \in [0,1]. \tag{10}$$

Theorem 1. *Suppose that (F_0) holds. In addition,*

(F_1) $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+),\ p \in C([0,1], \mathbb{R}^+),\ f(t,0) \not\equiv 0,\ p(t) \not\equiv 0$ *for* $t \in [0,1]$;

(F_2) *$f(t,x)$ is increasing in x for each $t \in [0,1]$;*

(F_3) *for any $r \in (0,1)$, there exists $\varphi(r) \in (r,1)$ such that $f(t,rx) \ge \varphi(r)f(t,x)$, $\forall t \in [0,1],\ x \in \mathbb{R}^+$.*

Then:

1. *For any fixed $\lambda > 0$, problem (1) has a unique positive solution $x_\lambda^* \in Q_h$, where $h(t) = t^{\alpha-1},\ t \in [0,1]$. In addition, for any initial value $x_0 \in Q_h$, by constructing the sequence*

$$x_n(t) = \lambda \int_0^1 G(t,qs)p(s)f(s,x_{n-1}(s))d_qs,\ n = 1,2,\ldots,$$

   one has $\lim_{n\to+\infty} x_n(t) = x_\lambda^*(t)$, $t \in [0,1]$;

2. *x_λ^* is strictly increasing in λ, that is, $0 < \lambda_1 < \lambda_2$ means $x_{\lambda_1}^* < x_{\lambda_2}^*$;*
3. *If there exists $\gamma \in (0,1)$ and $\varphi(t) \ge t^\gamma,\ t \in (0,1)$, then x_λ^* is continuous in λ;*
4. $\lim_{\lambda\to+\infty} \|x_\lambda\| = +\infty,\ \lim_{\lambda\to0^+} \|x_\lambda\| = 0$.

Proof. It is easy to see that $x(t)$ is the solution of problem (1) if and only if $x(t) = \lambda Tx(t)$. From Lemma 2 and (F_1), $T : Q \to Q$ is obvious. We can also easily get that $T : Q \to Q$ is increasing by condition (F_2).

Now we check that T satisfies all assumptions of Lemma 3. First, for the condition (i) of Lemma 3, we take $h(t) = t^{\alpha-1} \geq 0$, that is, $h \in Q$. Now we show that $Th \in Q_h$. Set

$$\tau_1 = \int_0^1 \frac{P(s)}{\Gamma_q(\alpha)} p(s) f(s,0) d_q s,\ \ \tau_2 = \int_0^1 \frac{P + P_m}{\Gamma_q(\alpha)} p(s) f(s,1) d_q s,$$

since $\mu \geq 0$, f is increasing with $f(t,0) \not\equiv 0$, $p(t) \not\equiv 0$, in line with Lemma 2, we can easily get $0 < \tau_1 \leq \tau_2$. From (F_2), we have

$$\begin{aligned} Th(t) &= \int_0^1 G(t,qs) p(s) f(s, s^{\alpha-1}) d_q s \\ &\leq \int_0^1 \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} (P + P_m) p(s) f(s,1) d_q s \\ &= \int_0^1 \frac{P + P_m}{\Gamma_q(\alpha)} p(s) f(s,1) d_q s \cdot t^{\alpha-1} = \tau_2 h(t). \end{aligned}$$

Moreover, we get

$$\begin{aligned} Th(t) &\geq \int_0^1 \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} P(s) \cdot p(s) f(s,0) d_q s \\ &= \int_0^1 \frac{P(s)}{\Gamma_q(\alpha)} p(s) f(s,0) d_q s \cdot t^{\alpha-1} = \tau_1 h(t). \end{aligned}$$

Hence, $\tau_1 h(t) \leq Th(t) \leq \tau_2 h(t)$, $t \in [0,1]$. That is, $\tau_1 h \leq Th \leq \tau_2 h$, then $Th \in Q_h$.

Next, for $r \in (0,1),\ x \in Q$, it follows from (F_3) that

$$\begin{aligned} T(rx)(t) = &\int_0^1 G(t,qs) p(s) f(s, rx(s)) d_q s \geq \varphi(r) \int_0^1 G(t,qs) p(s) f(s, x(s)) d_q s \\ =&\varphi(r) Tx(t),\ t \in [0,1]. \end{aligned}$$

Then we have $T(rx) \geq \varphi(r) Tx$, $\forall x \in Q$, $r \in (0,1)$. Therefore, all conditions of Lemma 3 are satisfied. Finally, by Lemma 4, there exists a unique $x_\lambda^* \in Q_h$ such that $Tx_\lambda^* = \frac{1}{\lambda} x_\lambda^*$, i.e., $\lambda T x_\lambda^* = x_\lambda^*$. Then

$$x_\lambda^*(t) = \lambda \int_0^1 G(t,qs) p(s) f(s, x_\lambda^*(s)) d_q s,\ t \in [0,1].$$

Considering Lemma 1, x_λ^* is a unique positive solution of problem (1) for given $\lambda > 0$. From Lemma 4 (i), x_λ^* is strictly increasing in λ, i.e., $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1}^* \leq x_{\lambda_2}^*$, $x_{\lambda_1}^* \neq x_{\lambda_2}^*$. And one has $\lim_{n\to+\infty} \|x_\lambda^*\| = +\infty$, $\lim_{\lambda\to0^+} \|x_\lambda^*\| = 0$. Further, if there exists $\gamma \in (0,1)$ such that $\varphi(r) \geq r^\gamma$ for $r \in (0,1)$, Lemma 4 shows that x_λ^* is continuous in λ, i.e. $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$ as $\lambda \to \lambda_0$ $(\lambda_0 > 0)$.

Let $T_\lambda = \lambda T$. For T_λ, all conditions of Lemma 3 are satisfied. Thus, for any initial value $x_0 \in Q_h$, constructing the sequence $x_n = T_\lambda x_{n-1},\ n = 1, 2, \ldots$, one has $x_n \to x_\lambda^*$ as $n \to +\infty$. That is,

$$x_n(t) = \lambda \int_0^1 G(t, qs) q(s) f(s, x_{n-1}(s)) d_q s,\ n = 1, 2, \ldots,$$

and $\lim_{n\to+\infty} x_n(t) = x_\lambda^*(t), t \in [0, 1]$. □

Remark 2. *For condition (F_3), fix $\varphi(r) = r^\gamma,\ \gamma \in (0, 1)$, then for any initial value $x_0 \in Q_h$, $x_n(t) \to x_\lambda^*(t)$ as $n \to \infty$; further from the proof of Theorem 2.1 [18], we have the error estimation*

$$\|x_n - x_\lambda^*\| = \circ(1 - \rho^{\gamma^n}),\ n \to \infty,$$

here $\rho \in (0, 1)$ is a constant hinge on x_0.

Corollary 1. *Assume $(F_0) - (F_3)$ hold. Then the following fractional differential equation with m-point boundary conditions*

$$\begin{cases} D_q^\alpha x(t) + p(t) f(t, x(t)) = 0,\ t \in (0, 1), \\ D_q^i x(0) = 0,\ 0 \le i \le n - 2, \\ \mu D_q^{\alpha-1} x(1) - \nu\alpha[x] = \sum\limits_{j=1}^m \beta_j D_q^\gamma x(\xi_j), \end{cases} \tag{11}$$

has a unique positive solution x^* in Q_h, where $h(t) = t^{\alpha-1},\ t \in [0, 1]$. Moreover, for $x_0 \in Q_h$, constructing the sequence

$$x_n(t) = \int_0^1 G(t, qs) p(s) f(s, x_{n-1}(s)) d_q s,\ n = 1, 2, \ldots,$$

then $\lim_{n\to+\infty} x_n(t) = x^*(t),\ t \in [0, 1]$.

Corollary 2. *Assume $(F_1) - (F_3)$ hold. Then the following fractional differential equation with m-point boundary conditions*

$$\begin{cases} D_q^\alpha x(t) + \lambda f(t, x(t)) = 0,\ t \in (0, 1), \\ D_q^i x(0) = 0,\ 0 \le i \le n - 2,\ \mu D_q^{\alpha-1} x(1) = 0, \end{cases} \tag{12}$$

has a unique positive solution x_λ^* in Q_h, where $h(t) = t^{\alpha-1},\ t \in [0, 1]$. Moreover, for $x_0 \in Q_h$, constructing the sequence

$$x_n(t) = \lambda \int_0^1 H(t, qs) f(s, x_{n-1}(s)) d_q s,\ n = 1, 2, \ldots,$$

and $\lim_{n\to+\infty} x_n(t) = x_\lambda^*(t),\ t \in [0, 1]$.

Remark 3. *The results like Theorem 1 and Corollaries 1, 2 of fractional q-difference equations supplemented with the Stieltjes integral condition have not been found in previous references. Not only the new existence criteria of a unique solution with respect to λ of problem (1) are gained, but also an iterative scheme can be constructed to approach the unique solution.*

Remark 4. *If $q \to 1^-$, then problem* (1) *becomes the usual fractional differential equation and we can obtain the corresponding results, which are also new.*

In the end, an interesting example is given to show the practicability of the main result.

Example 1. *Consider the boundary value problem:*

$$\begin{cases} D_q^{\frac{9}{2}} x(t) + \lambda \ln \frac{1}{t} \left(\sqrt{2}t^2 + [x(t)]^{\frac{3}{2}} t + 1\right)^{\frac{1}{2}} = 0, \ t \in (0,1), \\ x(0) = D_q x(0) = D_q^2 x(0) = D_q^3 x(0) = 0, \\ \mu D_q^{\frac{7}{2}} x(1) - \nu \alpha[x] = \sum_{j=1}^{2} \beta_j D_q^{\frac{5}{2}} x(\xi_j); \end{cases} \tag{13}$$

here $n = 5,\ m = 2,\ q = \frac{1}{2},\ \alpha = \frac{9}{2},\ \gamma = \frac{5}{2},\ p(t) = \ln\frac{1}{t} > 0,\ \mu = 16,\ \nu = 1,\ \beta_1 = \frac{1}{2},\ \beta_2 = \frac{1}{3},\ \xi_1 = \frac{1}{4},\ \xi_2 = \frac{1}{7}$, *and*

$$f(t,x) = \left(\sqrt{2}t^2 + x^{\frac{3}{2}} t + 1\right)^{\frac{1}{2}}, \ t \in (0,1).$$

We find that $f(t,0) = \left(\sqrt{2}t^2 + 1\right)^{\frac{1}{2}} > 0$, $f(t,x)$ *is increasing in* x *for* $x \in [0,+\infty),\ t \in [0,1]$. *Then the conditions* (F_1) *and* (F_2) *hold. Given* $\varphi(r) = r^{\frac{3}{4}}$, *we have* $\varphi(r) \in (r,1),\ r \in (0,1)$. *Then for* $x \in \mathbb{R}^+$, *we get*

$$f(t,rx) = \left(\sqrt{2}t^2 + [rx(t)]^{\frac{3}{2}} t + 1\right)^{\frac{1}{2}} \geq r^{\frac{3}{4}} \left(\sqrt{2}t^2 + [x(t)]^{\frac{3}{2}} t + 1\right)^{\frac{1}{2}} = \varphi(r) f(t,x).$$

We can discuss several cases about $\alpha[x]$:

1. *Let* $\alpha[x] = 0$; *then we have* $A = 0,\ \mu,\nu > 0$, *and*

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)} \sum_{j=1}^{m} \beta_j \xi_j^{\alpha-\gamma-1} = \Gamma_q(9/2)(\beta_1\xi_1 + \beta_2\xi_2) \approx 0.0145 < \mu\Gamma_q(9/2), i.e., \varpi > 0.$$

2. *Let* $\alpha[x] = \frac{1}{2}x(\frac{1}{2}) - \frac{1}{5}x(\frac{1}{3})$; *then* $A = \alpha[t^{\frac{7}{2}}] = \frac{1}{16\sqrt{2}} - \frac{1}{135\sqrt{3}} \approx 0.0399 > 0$ *and*

$$\Gamma_q(\alpha) \sum_{j=1}^{2} \beta_j \xi_j + \nu A = \frac{29}{168} \Gamma_q(9/2) + A \approx 0.0544 < \mu\Gamma_q(9/2) \approx 1.3391, i.e., \varpi > 0.$$

3. *Let* $\alpha[x] = \int_0^1 (3t-1)x(t)dt$ *and the function* $3t-1$ *changes the sign for* $t \in [0,1]$. *Then we have*

$$A = \int_0^1 (3t-1) t^{\frac{7}{2}} dt = \frac{32}{99} > 0,\ \Gamma_q(\alpha) \sum_{j=1}^{2} \beta_j \xi_j + \nu A \approx 0.3377 < \mu\Gamma_q(9/2).$$

All the conditions of Theorem 1 are satisfied, so we can claim that problem (13) *has a unique positive solution* $x_\lambda^* \in Q_h$, *here* $h(t) = t^{\frac{7}{2}}$. *Further, for* $x_0 \in Q_h$, *construct the sequence*

$$x_n(t) = \lambda \int_0^1 G(t, qs) \ln \frac{1}{s} \left(\sqrt{2}s^2 + [x_{n-1}(s)]^{\frac{3}{2}} s + 1\right)^{\frac{1}{2}} d_q s, \ n = 1, 2, \ldots$$

Then $x_n(t) \to x_\lambda^*(t)$ *as* $n \to \infty$.

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References

[1] C. Adams, *On the linear ordinary q-difference equation*, American Math. Ser. Proc. Camb. Philo. Soc. **30**(1929), 195–205.

[2] R. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Camb. Philo. Soc. **66**(1969), 365–370.

[3] B. Ahmad, S. Etemad, M. Ettefagh, S. Rezapour, *On the existence of solutions for fractional q-difference inclusions with q-antiperiodic boundary conditions*, Bull. Math. Soc. Sci. Math. Roumanie. **59**(2016), 119–134.

[4] B. Ahmad, M. Alghanmia, S. Ntouyasb, A. Alsaedia, *Fractional differential equations involving generalized derivative with Stieltjes and fractional integral boundary conditions*, Appl. Math. Lett. **84**(2018), 111–117.

[5] B. Ahmad, S. Ntouyas, A. Alsaedi, H. Al-Hutami, *Nonlinear q-fractional differential equations with nonlocal and sub-strip type boundary conditions*, Electron. J. Qual. Theo. Differ. Equ. **26**(2014), 1–12.

[6] W. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proc. Edinb. Math. Soc. **15**(1966), 135–140.

[7] W. Cheng, J. Xu, Y. Cui, *Positive solutions for a system of nonlinear semipositone fractional q-difference equations with q-integral boundary conditions*, J. Nonlinear. Sci. Appl. **10**(2017), 4430–4440.

[8] Y. Cui, W. Ma, Q. Sun, X. Su, *New uniqueness results for boundary value problem of fractional differential equation*, Nonl. Anal. Model. Control. **23**(2018), 31–39.

[9] J. Graef, L. Kong, Q. Kong, M. Wang, *Uniqueness of positive solutions of fractional boudary value problems with non-homogeneous integral boundary condition*, Fract. Calc. Appl. Anal. **15**(2012), 509–528.

[10] F. Jackson, *On q-functions and a certain difference operator*, Trans. Roy. Soc. Edinb. **46**(1908), 253–281.

[11] A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Science, B.V. Amsterdam, 2006.

[12] X. Liu, L. Liu, Y. Wu, *Existence of positive solutions for a singular nonlinear fractional differential equation with integral boundary conditions involving fractional derivatives*, Bound. Value Probl. 2018:24.

[13] J. Mao, Z. Zhao, C. Wang, *The exact iterative solution of fractional differential equation with nonlocal boundary value conditions*, J. Funct. Spaces. Volume 2018, Article ID 8346398, 6 pages.

[14] F. MAINARDI, *Fractional calculus and waves in linear viscoelasticity*, World Scientific, Singapore, 2010.

[15] J. REN, C. ZHAI, *A fractional q-difference equation with integral boundary conditions and comparison theorem*, Int. J. Nonlin. Sci. Num. **18**(2017), 575–583.

[16] C. ZHAI, J. REN, *Positive and negative solutions of a boundary value problem for a fractional q-difference equation*, Adv. Differ. Equ. 2017:82.

[17] J. REN, C. ZHAI, *The unique solution for a fractional q-difference equation with three-point boundary conditions*, Indagat. Math. New. Ser. **29**(2018), 948–961.

[18] C. ZHAI, F. WANG, *Properties of positive solutions for the operator equation $Ax = \lambda x$ and applications to fractional differential equations with integral boundary conditions*, Adv. Differ. Equ. 2015:366.