MATHEMATICAL COMMUNICATIONS 193
Math. Commun. **24**(2019), 193–210

A parameter uniform fitted mesh method for a weakly coupled system of two singularly perturbed convection-diffusion equations*

Saravana Sankar Kalaiselvan[1], John J. H. Miller[2] and Valarmathi Sigamani[1,†]

[1] *Department of Mathematics, Bishop Heber College, Tiruchirappalli, Tamil Nadu, 620 017, India*
[2] *Trinity College Dublin, The University of Dublin, Dublin 2, Ireland*

Received September 1, 2018; accepted March 4, 2019

Abstract. In this paper, a boundary value problem for a singularly perturbed linear system of two second order ordinary differential equations of convection-diffusion type is considered on the interval $[0, 1]$. The components of the solution of this system exhibit boundary layers at 0. A numerical method composed of an upwind finite difference scheme applied on a piecewise uniform Shishkin mesh is suggested to solve the problem. The method is proved to be first order convergent in the maximum norm uniformly in the perturbation parameters. Numerical examples are provided in support of the theory.

AMS subject classifications: 65L11, 65L12, 65L20, 65L70

Key words: Singular perturbation problems, system of convection-diffusion equations, boundary layers, Shishkin mesh, parameter uniform convergence

1. Introduction

Singular perturbation problems of convection-diffusion type arise in many areas of applied mathematics such as fluid dynamics, chemical reactor theory, etc. Also, linearising Navier-Stokes equations, which plays a vital role in the field of science, leads to a system of convection-diffusion equations.

For a broad introduction to singularly perturbed boundary value problems of convection-diffusion type one can refer to [9, 4, 5]. There, the authors suggest robust computational techniques to solve them. A class of systems of singularly perturbed reaction-diffusion equations has been examined by several authors in [8, 7, 11, 3].

Here, in this paper, a weakly coupled system of two singularly perturbed convection-diffusion equations with distinct perturbation parameters is studied both analytically and numerically. If the perturbation parameters are equal, then the arguments in [5] are sufficient to show that the suggested method is parameter uniform. But in general, boundary layers of unequal width are expected for the components of the solution because of the coupling of the components.

*The first author has been supported by the Junior Research Fellowship by UGC, India.
†Corresponding author. *Email addresses:* saravanasankar55@gmail.com (S. Sankar Kalaiselvan), jmiller@tcd.ie (J. J. H. Miller), valarmathi07@gmail.com (V. Sigamani)

 ©2019 Department of Mathematics, University of Osijek

In papers [1, 10], a class of strongly coupled systems of singularly perturbed convection-diffusion problems is examined. A coupled system of two singularly perturbed convection-diffusion equations is considered in [2]. In [6], the author analysed a coupled system of singularly perturbed convection-diffusion equations.

In this paper, the major assumptions $\varepsilon_1 \leq CN^{-1}$, $\varepsilon_2 \leq CN^{-1}$ in [2], are removed. Moreover, the analytical and numerical arguments are completely different from [2] and [6] in the following sense. The decomposition of the solution is based on the effect of each perturbation parameter on the components of the solution. Thus, we get more information about the components of the solution and its layer pattern. Also, it is to be noted that the decomposition of the smooth component in [2] is given a correct definition here in this paper.

Notations. For any real valued function y on D, the norm of y is defined as $\|y\|_D = \sup_{x\in D}|y(x)|$. For any vector valued function $\vec{z}(x) = (z_1(x), z_2(x))^T$, $|\vec{z}(x)| = \big(|z_1(x)|, |z_2(x)|\big)^T$, $(\vec{z}(x))_i = z_i(x)$ and $\|\vec{z}\|_D = \max\big\{\|z_1\|_D, \|z_2\|_D\big\}$. Also $\vec{z}(x) \geq \vec{0}$, if $z_1(x) \geq 0$ and $z_2(x) \geq 0$.

For any mesh function Y on $D^N = \big\{x_j\big\}_{j=0}^{N}$, $\|Y\|_{D^N} = \max_{0\leq j\leq N}|Y(x_j)|$ and for any vector valued mesh function $\vec{Z} = (Z_1, Z_2)^T$, $|\vec{Z}(x_j)| = (|Z_1(x_j)|, |Z_2(x_j)|)^T$, $\|\vec{Z}\|_{D^N} = \max\big\{\|Z_1\|_{D^N}, \|Z_2\|_{D^N}\big\}$.

Throughout this paper, C denotes a generic positive constant which is independent of the singular perturbation and discretization parameters.

2. Formulation of the problem

Consider the following system of equations

$$L\vec{u}(x) \equiv E\vec{u}''(x) + A(x)\vec{u}'(x) - B(x)\vec{u}(x) = \vec{f}(x), x \in \Omega \tag{1}$$

$$\vec{u}(0) = \vec{l}, \;\; \vec{u}(1) = \vec{r}, \tag{2}$$

where $\Omega = (0,1)$, $\vec{u}(x) = (u_1(x), u_2(x))^T$, $\vec{f}(x) = (f_1(x), f_2(x))^T$,

$$E = \begin{bmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{bmatrix}, \;\; A(x) = \begin{bmatrix}a_1(x) & 0\\ 0 & a_2(x)\end{bmatrix}, \;\; B(x) = \begin{bmatrix}b_{11}(x) & -b_{12}(x)\\ -b_{21}(x) & b_{22}(x)\end{bmatrix}.$$

Here, ε_1 and ε_2 are two distinct small positive parameters and, without loss of generality, we assume that $\varepsilon_1 < \varepsilon_2$. The coefficient functions are taken to be sufficiently smooth on $\overline{\Omega}$ and $a_i(x) \geq \alpha > 0$, $b_{ii}(x) - b_{ij}(x) \geq \beta > 0$, $b_{ij} > 0$, for $i, j = 1, 2$ and $i \neq j$.

The case $a_i(x) \leq \alpha < 0$, for $i = 1, 2$, is put into the form (1) by changing independent variable from x to $1 - x$.

Since the matrix B(x) is not diagonal and the matrix A(x) is diagonal, the system is weakly coupled. If the matrix A(x) is not diagonal, then the system becomes strongly coupled. If $a_1(x)$ and $a_2(x)$ are zero functions, then the above problem comes under the class considered in [8].

The reduced problem corresponding to (1)–(2) is

$$L_0\vec{u_0}(x) \equiv A(x)\vec{u_0}'(x) - B(x)\vec{u_0}(x) = \vec{f}(x),\ x \in \Omega \tag{3}$$

$$\vec{u_0}(1) = \vec{r}, \tag{4}$$

where, $\vec{u_0}(x) = (u_{01}(x), u_{02}(x))^T$.

A boundary layer of width $O(\varepsilon_2)$ is expected near $x = 0$ in the solution components u_1 and u_2 if $u_2(0) \neq u_{02}(0)$ and a boundary layer of width $O(\varepsilon_1)$ is expected near $x = 0$ in the solution component u_1 if $u_1(0) \neq u_{01}(0)$. Numerical illustrations provided for each case exhibit such layer patterns.

3. Analytical results

In this section, a maximum principle, a stability result and estimates of the derivatives of the solution of the system of equations (1)–(2) are presented.

Lemma 1 (Maximum principle)**.** *Let* $\vec{\psi} \in (C^2(\overline{\Omega}))^2$ *such that* $\vec{\psi}(0) \geq \vec{0},\ \vec{\psi}(1) \geq \vec{0},\ L\vec{\psi} \leq \vec{0}$ *on* $(0,1)$*; then* $\vec{\psi} \geq \vec{0}$ *on* $[0,1]$.

Proof. Let x^* and y^* be such that $\psi_1(x^*) = \min_{x\in\overline{\Omega}} \psi_1(x)$ and $\psi_2(y^*) = \min_{x\in\overline{\Omega}} \psi_2(x)$. Without loss of generality, we assume that $\psi_1(x^*) \leq \psi_2(y^*)$ and suppose $\psi_1(x^*) < 0$; then $x^* \notin \{0,1\},\ \psi_1'(x^*) = 0$ and $\psi_1''(x^*) \geq 0$.
$(L\vec{\psi})_1(x^*) \geq \varepsilon_1\psi_1''(x^*) + a_1(x^*)\psi_1'(x^*) - (b_{11}(x^*) - b_{12}(x^*))\psi_1(x^*) > 0$, contradiction to the assumption that $L\vec{\psi} \leq \vec{0}$ on $(0,1)$. Hence, $\vec{\psi}(x) \geq 0$, on $[0,1]$. □

An immediate consequence of the maximum principle is the following stability result.

Lemma 2 (Stability result)**.** *Let* $\vec{\psi} \in (C^2(\overline{\Omega}))^2$*; then for* $x \in \overline{\Omega}$ *and i=1,2*

$$|\psi_i(x)| \leq \max\left\{\|\vec{\psi}(0)\|,\ \|\vec{\psi}(1)\|,\ \frac{1}{\beta}\|L\vec{\psi}\|\right\}.$$

Corollary 1. *Let* $\vec{u}$ *be the solution of* $(1)-(2)$*; then*

$$|u_i(x)| \leq \max\left\{\|\vec{l}\|,\ \|\vec{r}\|,\ \frac{1}{\beta}\|\vec{f}\|\right\}.$$

Theorem 1. *Let* $\vec{u}$ *be the solution of (1)–(2); then for* $x \in \overline{\Omega}$ *and i=1,2*

$$|u_i^{(k)}(x)| \leq C\varepsilon_i^{-k}\left(\|\vec{u}\| + \varepsilon_i\|\vec{f}\|\right)\ for\ \ k = 1,2 \tag{5}$$

$$|u_1^{(3)}(x)| \leq C\varepsilon_1^{-3}\left(\|\vec{u}\| + \varepsilon_1\|\vec{f}\|\right) + \varepsilon_1^{-1}\|f_1'\| \tag{6}$$

$$|u_2^{(3)}(x)| \leq C\varepsilon_2^{-2}\varepsilon_1^{-1}\left(\|\vec{u}\| + \varepsilon_2\|\vec{f}\|\right) + \varepsilon_2^{-1}\|f_2'\| \tag{7}$$

Proof. For any $x \in [0,1]$, there exists $a \in [0, 1-\varepsilon_i]$ such that $x \in N_a = [a, a+\varepsilon_i]$. By the mean value theorem, there exists $y_i \in (a, a+\varepsilon_i)$ such that

$$u_i'(y_i) = \frac{u_i(a+\varepsilon_i) - u_i(a)}{\varepsilon_i}$$

and hence

$$|u_i'(y_i)| \leq C\varepsilon_i^{-1}\|\vec{u}\|.$$

Also,

$$u_i'(x) = u_i'(y_i) + \int_{y_i}^{x} u_i''(s)ds.$$

Substituting for $u_i''(s)$ from (1) and integrating by parts, we get

$$|u_i'(x)| \leq C\varepsilon_i^{-1}\Big(\|\vec{u}\| + \varepsilon_i\|\vec{f}\|\Big).$$

Again from (1),

$$|u_i''(x)| \leq C\varepsilon_i^{-2}\Big(\|\vec{u}\| + \varepsilon_i\|\vec{f}\|\Big).$$

Differentiating (1) once and substituting the above bounds lead to

$$|u_1^{(3)}(x)| \leq C\varepsilon_1^{-3}\Big(\|\vec{u}\| + \varepsilon_1\|\vec{f}\|\Big) + \varepsilon_1^{-1}\|f_1'\|$$
$$|u_2^{(3)}(x)| \leq C\varepsilon_2^{-2}\varepsilon_1^{-1}\Big(\|\vec{u}\| + \varepsilon_2\|\vec{f}\|\Big) + \varepsilon_2^{-1}\|f_2'\|.$$

□

3.1. Shishkin decomposition of the solution

The solution $\vec{u}$ of problem (1)–(2) can be decomposed into smooth and singular components $\vec{v}$ and $\vec{w}$ given by

$$\vec{u} = \vec{v} + \vec{w},$$

where

$$L\vec{v} = \vec{f}, \vec{v}(1) = \vec{r},\ \vec{v}(0) \text{ suitably chosen,} \tag{8}$$
$$L\vec{w} = \vec{0}, \vec{w}(0) = \vec{l} - \vec{v}(0), \vec{w}(1) = \vec{0}, \tag{9}$$

with $\vec{v} = (v_1, v_2)^T$ and $\vec{w} = (w_1, w_2)^T$.

Now, $\vec{v}$ is decomposed into $\vec{v} = \vec{y}_0 + \varepsilon_2\vec{y}_1 + \varepsilon_2^2\vec{y}_2$, where $\vec{y}_0 = (y_{01}, y_{02})^T$ is the solution of (10)–(12),

$$a_1(x)y_{01}'(x) - b_{11}(x)y_{01}(x) + b_{12}(x)y_{02}(x) = f_1(x) \tag{10}$$
$$a_2(x)y_{02}'(x) + b_{21}(x)y_{01}(x) - b_{22}(x)y_{02}(x) = f_2(x) \tag{11}$$
$$y_{01}(1) = r_1(1),\ y_{02}(1) = r_2(1), \tag{12}$$

$\vec{y}_1 = (y_{11}, y_{12})^T$ is the solution of (13)–(15),

$$a_1(x)y_{11}'(x) - b_{11}(x)y_{11}(x) + b_{12}(x)y_{12}(x) = -\frac{\varepsilon_1}{\varepsilon_2}y_{01}''(x) \tag{13}$$

$$a_2(x)y_{12}'(x) + b_{21}(x)y_{11}(x) - b_{22}(x)y_{12}(x) = -y_{02}''(x) \tag{14}$$

$$y_{11}(1) = 0,\ y_{12}(1) = 0. \tag{15}$$

$\vec{y}_2 = (y_{21}, y_{22})^T$ is the solution of (16)–(18),

$$\varepsilon_1 y_{21}''(x) + a_1(x)y_{21}'(x) - b_{11}(x)y_{21}(x) + b_{12}(x)y_{22}(x) = -\frac{\varepsilon_1}{\varepsilon_2}y_{11}''(x) \tag{16}$$

$$\varepsilon_2 y_{22}''(x) + a_2(x)y_{22}'(x) + b_{21}(x)y_{21}(x) - b_{22}(x)y_{22}(x) = -y_{12}''(x) \tag{17}$$

$$y_{21}(0) = p,\ y_{22}(0) = 0,\ y_{21}(1) = 0,\ y_{22}(1) = 0. \tag{18}$$

In (18), p is a constant to be chosen such that $|p| \leq C$.

From (10)–(15), it is not hard to see that for $0 \leq k \leq 3$,

$$\|\vec{y_0}^{(k)}\| \leq C,\ \ \|\vec{y_1}^{(k)}\| \leq C. \tag{19}$$

Now, consider equations (16)–(18) and using Lemma 2

$$\|\vec{y_2}\| \leq C. \tag{20}$$

Using estimate (5) from Theorem 1, we get

$$|y_{22}^{(k)}(x)| \leq C\varepsilon_2^{-k} \quad \text{for} \quad k = 1, 2. \tag{21}$$

From (16),

$$\varepsilon_1 y_{21}'' + a_1(x)y_{21}'(x) - b_{11}(x)y_{21}(x) = -\frac{\varepsilon_1}{\varepsilon_2}y_{11}''(x) - b_{12}(x)y_{22}(x). \tag{22}$$

Decompose y_{21} as $y_{21}(x) = z_0(x) + \varepsilon_1 z_1(x) + \varepsilon_1^2 z_2(x)$ with

$$a_1(x)z_0'(x) - b_{11}(x)z_0(x) = -\frac{\varepsilon_1}{\varepsilon_2}y_{11}''(x) - b_{12}(x)y_{22}(x), z_0(1) = 0, \tag{23}$$

$$a_1(x)z_1'(x) - b_{11}(x)z_1(x) = -z_0''(x), z_1(1) = 0, \tag{24}$$

$$\varepsilon_1 z_2''(x) + a_1(x)z_2'(x) - b_{11}(x)z_2(x) = -z_1''(x), z_2(0) = 0, z_2(1) = 0. \tag{25}$$

Estimating z_0 and z_1 from (23) and (24) and using Chapter 8 of [9] for the problem (25), the following estimates hold for $0 \leq k \leq 3$;

$$|z_0^{(k)}| < C(1 + \varepsilon_2^{(1-k)}),\ |z_1^{(k)}| < C(1 + \varepsilon_2^{-2}\varepsilon_1^{2-k}),\ |z_2^{(k)}| < C(1 + \varepsilon_2^{-2}\varepsilon_1^{-k}).$$

Then $p = z_0(0) + \varepsilon_1 z_1(0)$ and for $k = 0, 1, 2$,

$$|y_{21}^{(k)}(x)| \leq C\varepsilon_2^{-2}\ ,\ |y_{21}^{(3)}(x)| \leq C\varepsilon_1^{-1}\varepsilon_2^{-2}. \tag{26}$$

Differentiating (17) once and using (21) and (26)

$$|y_{22}^{(3)}(x)| \leq C\varepsilon_2^{-3}. \tag{27}$$

Hence, from (19)–(21) and (26)–(27), the estimates of the components $v_1 = y_{01} + \varepsilon_2 y_{11} + \varepsilon_2^2 y_{21}$ and $v_2 = y_{02} + \varepsilon_2 y_{12} + \varepsilon_2^2 y_{22}$ of $\vec{v}$ are as follows:

$$|v_1^{(k)}(x)| \le C, \;\; |v_2^{(k)}(x)| \le C \text{ for } 0 \le k \le 2, \tag{28}$$
$$|v_1^{(3)}(x)| \le C\varepsilon_1^{-1}, \;\; |v_2^{(3)}(x)| \le C\varepsilon_2^{-1}. \tag{29}$$

Theorem 2. *Let $\vec{w}(x)$ be the solution of (9), then for $x \in \overline{\Omega}$, the following estimates hold:*

$$|w_1(x)| \le C \exp\frac{-\alpha x}{\varepsilon_2}, \; |w_2(x)| \le C \exp\frac{-\alpha x}{\varepsilon_2}, \tag{30}$$
$$|w_1^{(k)}(x)| \le C\Big(\varepsilon_1^{-k} \exp\frac{-\alpha x}{\varepsilon_1} + \varepsilon_2^{-k} \exp\frac{-\alpha x}{\varepsilon_2}\Big), \; \textit{for} \;\; k = 1, 2, 3, \tag{31}$$
$$|w_2^{(k)}(x)| \le C\varepsilon_2^{-k} \exp\frac{-\alpha x}{\varepsilon_2}, \;\; \textit{for} \;\; k = 1, 2, \tag{32}$$
$$|w_2^{(3)}(x)| \le C\varepsilon_2^{-1}\Big(\varepsilon_1^{-1} \exp\frac{-\alpha x}{\varepsilon_1} + \varepsilon_2^{-2} \exp\frac{-\alpha x}{\varepsilon_2}\Big). \tag{33}$$

Proof. Estimates (30)–(32) follow from Lemma 4 of [2].

From (9), we have

$$\varepsilon_2 w_2''(x) + a_2(x) w_2'(x) + b_{21}(x) w_1(x) - b_{22}(x) w_2(x) = 0.$$

Differentiating the above equation once yields

$$|\varepsilon_2 w_2^{(3)}(x)| \le C\big(|w_2''(x)| + |w_1'(x)|\big)$$

and hence,

$$|w_2^{(3)}(x)| \le C\varepsilon_2^{-1}\Big(\varepsilon_1^{-1} \exp\frac{-\alpha x}{\varepsilon_1} + \varepsilon_2^{-2} \exp\frac{-\alpha x}{\varepsilon_2}\Big).$$

□

3.2. Improved estimates for the bounds of the singular components

Let $B_1(x)$ and $B_2(x)$ be the layer functions defined on $[0, 1]$ as follows:

$$B_1(x) = \exp\frac{-\alpha x}{\varepsilon_1}, \;\; B_2(x) = \exp\frac{-\alpha x}{\varepsilon_2}.$$

Using the arguments similar to those used in Lemma 5 of [11], it is not hard to see that there exists point $x_s \in (0, \frac{1}{2})$ such that

$$\frac{B_1(x_s)}{\varepsilon_1^s} = \frac{B_2(x_s)}{\varepsilon_2^s}, \; s = 1, 2, 3 \tag{34}$$

and

$$\frac{B_1(x)}{\varepsilon_1^s} > \frac{B_2(x)}{\varepsilon_2^s}, \text{ for } x \in [0, x_s), \;\; \frac{B_1(x)}{\varepsilon_1^s} < \frac{B_2(x)}{\varepsilon_2^s}, \text{ for } x \in (x_s, 1]. \tag{35}$$

Now the singular components $w_1(x)$ and $w_2(x)$ are decomposed as follows:

$$w_1(x) = w_{11}(x) + w_{12}(x), \quad w_2(x) = w_{21}(x) + w_{22}(x), \tag{36}$$

where, w_{11}, w_{12}, w_{21} and w_{22} are defined by

$$w_{11}(x) = \begin{cases} \sum_{k=0}^{3} \big((x-x_3)^k/k!\big) w_1^{(k)}(x_3), & \text{for } x \in [0, x_3) \\ w_1(x), & \text{for } x \in [x_3, 1] \end{cases} \tag{37}$$

$$w_{12}(x) = w_1(x) - w_{11}(x) \tag{38}$$

$$w_{21}(x) = \begin{cases} \sum_{k=0}^{3} \big((x-x_1)^k/k!\big) w_2^{(k)}(x_1), & \text{for } x \in [0, x_1) \\ w_2(x), & \text{for } x \in [x_1, 1] \end{cases} \tag{39}$$

$$w_{22}(x) = w_2(x) - w_{21}(x). \tag{40}$$

Lemma 3. *Let w_{11}, w_{12}, w_{21} and w_{22} be as defined in (37)–(40); then for $x \in \overline{\Omega}$, the following estimates hold.*

$$|w_{11}^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x), \quad |w_{12}''(x)| \le C\varepsilon_1^{-2} B_1(x), \tag{41}$$

$$|w_{21}^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x), \quad |w_{22}''(x)| \le C\varepsilon_2^{-2} B_1(x). \tag{42}$$

Proof. For $x \in [0, x_3)$, by the definition of $w_{11}(x)$ and using (31) and (34),

$$|w_{11}^{(3)}(x)| = |w_1^{(3)}(x_3)| \le C\varepsilon_2^{-3} B_2(x_3) \le C\varepsilon_2^{-3} B_2(x).$$

For $x \in [x_3, 1]$, by the definition of $w_{11}(x)$ and using (31) and (35),

$$|w_{11}^{(3)}(x)| = |w_1^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x).$$

Hence,

$$|w_{11}^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x), \text{on } \overline{\Omega}. \tag{43}$$

Similar arguments lead to

$$|w_{21}^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x), \text{on } \overline{\Omega}. \tag{44}$$

Using (38), (31), (45) and (35), it is not hard to see that for $x \in [0, x_3)$,

$$|w_{12}^{(3)}(x)| \le |w_1^{(3)}(x)| + |w_{11}^{(3)}(x)| \le C\varepsilon_1^{-3} B_1(x).$$

Since $w_{12}''(1) = 0$, it follows that for any $x \in [0, 1]$,

$$|w_{12}''(x)| = \Big| \int_x^1 w_{12}^{(3)}(t)dt \Big| \le C \int_x^1 \varepsilon_1^{-3} B_1(t)dt \le C\varepsilon_1^{-2} B_1(x).$$

Hence,

$$|w_{12}''(x)| \le C\varepsilon_1^{-2} B_1(x), \text{on } \overline{\Omega}. \tag{45}$$

Similar arguments lead to

$$|w_{22}''(x)| \le C\varepsilon_2^{-2} B_1(x), \text{on } \overline{\Omega}. \tag{46}$$

□

Now consider the alternate decomposition of the singular component $w_1(x)$ as below.

$$w_1(x) = w_{11}(x) + w_{12}(x), \tag{47}$$

where w_{11} and w_{12} are defined by

$$w_{11}(x) = \begin{cases} \sum_{k=0}^{2} \left((x-x_2)^k/k!\right) w_1^{(k)}(x_2), & \text{for } x \in [0, x_2) \\ w_1(x), & \text{for } x \in [x_2, 1] \end{cases} \tag{48}$$

$$w_{12}(x) = w_1(x) - w_{11}(x). \tag{49}$$

Then, arguments similar to Lemma 3 lead to

$$|w_{11}''(x)| \le C\varepsilon_2^{-2} B_2(x), \quad |w_{12}'(x)| \le C\varepsilon_1^{-1} B_1(x). \tag{50}$$

4. Numerical method

A piecewise uniform Shishkin mesh $\overline{\Omega}^N$ is defined on $[0,1]$, so as to resolve the layers in the neighbourhood of $x = 0$. Let N denote the number of mesh elements which is taken to be a multiple of 4. The interval $[0,1]$ is divided into three subintervals $[0,\tau_1]$, $[\tau_1,\tau_2]$ and $[\tau_2,1]$, where τ_1 and τ_2 are the transition parameters given by

$$\tau_2 = \min\Big\{\frac{1}{2}, \frac{2\varepsilon_2}{\alpha}\ln N\Big\}, \qquad \tau_1 = \min\Big\{\frac{\tau_2}{2}, \frac{2\varepsilon_1}{\alpha}\ln N\Big\}.$$

In each of the intervals $[0,\tau_1], [\tau_1,\tau_2]$, $N/4$ mesh elements are placed and $N/2$ mesh elements are placed in the interval $[\tau_2, 1]$ so that the mesh is piecewise uniform. The mesh becomes uniform when $\tau_2 = 1/2$ and $\tau_1 = \tau_2/2$.

Let H_1, H_2 and H_3 denote the step sizes in the intervals $[0,\tau_1]$, $[\tau_1,\tau_2]$ and $[\tau_2,1]$, respectively. Thus,

$$H_1 = \frac{4\tau_1}{N}, \;\; H_2 = \frac{4(\tau_2-\tau_1)}{N} \;\text{ and }\; H_3 = \frac{2(1-\tau_2)}{N}.$$

Therefore, possible four Shishkin meshes are represented by $\overline{\Omega}^N = \{x_j\}_{j=0}^N$, where

$$x_j = \begin{cases} jH_1, & \text{if } 0 \le j \le \frac{N}{4} \\ \tau_1 + (j - \frac{N}{4})H_2, & \text{if } \frac{N}{4} \le j \le \frac{N}{2} \\ \tau_2 + (j - \frac{N}{2})H_3, & \text{if } \frac{N}{2} \le j \le N. \end{cases}$$

To resolve the layers, the mesh is constructed in such a way that it condenses at the inner regions where the layers are exhibited and is coarse in the outer region, away from the layers.

To solve the BVP (1)–(2) numerically the following upwind classical finite difference scheme is applied on the mesh $\overline{\Omega}^N$.

$$L^N \vec{U}(x_j) \equiv E\delta^2 \vec{U}(x_j) + A(x_j)D^+\vec{U}(x_j) - B(x_j)\vec{U}(x_j) = \vec{f}(x_j), \tag{51}$$

$$\vec{U}(x_0) = \vec{l},\; \vec{U}(x_N) = \vec{r}, \tag{52}$$

where $\vec{U}(x_j) = (U_1(x_j), U_2(x_j))^T$ and for $1 \le j \le N-1$,

$$D^+U(x_j) = \frac{U(x_{j+1}) - U(x_j)}{h_{j+1}},$$
$$D^-U(x_j) = \frac{U(x_j) - U(x_{j-1})}{h_j},$$
$$\delta^2 U(x_j) = \frac{1}{\overline{h_j}}\Big(D^+U(x_j) - D^-U(x_j)\Big),$$

with

$$h_j = x_j - x_{j-1}, \ \ \overline{h_j} = \frac{(h_j + h_{j+1})}{2}.$$

5. Error analysis

In this section, a discrete maximum principle, a discrete stability result and the first order convergence of the proposed numerical method are established.

Lemma 4 (Discrete maximum principle)**.** *Assume that the vector valued mesh function $\vec{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j))^T$ satisfies $\vec{\psi}(x_0) \ge \vec{0}$ and $\vec{\psi}(x_N) \ge \vec{0}$. Then $L^N\vec{\psi}(x_j) \le \vec{0}$ for $1 \le j \le N-1$ implies that $\vec{\psi}(x_j) \ge \vec{0}$ for $0 \le j \le N$.*

Proof. Let k_1 and k_2 be such that $\psi_1(x_{k_1}) = \min\limits_j \psi_1(x_j)$ and $\psi_2(x_{k_2}) = \min\limits_j \psi_2(x_j)$. Without loss of generality, we assume that $\psi_1(x_{k_1}) \le \psi_2(x_{k_2})$ and suppose $\psi_1(x_{k_1}) < 0$. Then, $k_1 \notin \{0, N\}$, $\psi_1(x_{k_1+1}) - \psi_1(x_{k_1}) \ge 0$ and $\psi_1(x_{k_1}) - \psi(x_{k-1}) \le 0$, implies that $(L^N\vec{\psi})_1(x_{k_1}) > 0$, a contradiction. Therefore, $\psi_1(x_{k_1}) \ge 0$ and hence, $\vec{\psi}(x_j) \ge \vec{0}$ for $0 \le j \le N$. □

An immediate consequence of the above discrete maximum principle is the following discrete stability result.

Lemma 5 (Discrete stability result)**.** *If $\vec{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j))^T$ is any vector valued mesh function defined on $\overline{\Omega}^N$, then for $i = 1, 2$ and $0 \le j \le N$,*

$$|\psi_i(x_j)| \le \max\Big\{\|\vec{\psi}(x_0)\|, \ \|\vec{\psi}(x_N)\|, \ \frac{1}{\beta}\ \|L^N\vec{\psi}\|_{\Omega^N}\Big\}.$$

5.1. Error estimate

Analogous by to the continuous case, the discrete solution $\vec{U}$ can be decomposed into $\vec{V}$ and $\vec{W}$ as defined below.

$$L^N\vec{V}(x_j) = \vec{f}(x_j), \text{ for } 0 < j < N, \ \vec{V}(x_0) = \vec{v}(x_0), \ \vec{V}(x_N) = \vec{v}(x_N), \tag{53}$$
$$L^N\vec{W}(x_j) = \vec{0}, \text{ for } 0 < j < N, \ \vec{W}(x_0) = \vec{w}(x_0), \ \vec{W}(x_N) = \vec{w}(x_N). \tag{54}$$

Lemma 6. *Let $\vec{v}$ be the solution of (8) and $\vec{V}$ the solution of (53); then*

$$\|\vec{V} - \vec{v}\|_{\overline{\Omega}^N} \le CN^{-1}.$$

Proof. For $1 \le j \le N-1$,

$$\begin{aligned} L^N(\vec{V}-\vec{v})(x_j) &= \vec{f}(x_j) - L^N\vec{v}(x_j) \\ &= (L-L^N)\vec{v}(x_j) \\ &= (\frac{d^2}{dx^2}-\delta^2)E\vec{v}(x_j) + (\frac{d}{dx}-D^+)A(x_j)\vec{v}(x_j) \\ &= \begin{pmatrix} \varepsilon_1(\frac{d^2}{dx^2}-\delta^2)v_1(x_j) + a_1(x_j)(\frac{d}{dx}-D^+)v_1(x_j) \\ \varepsilon_2(\frac{d^2}{dx^2}-\delta^2)v_2(x_j) + a_2(x_j)(\frac{d}{dx}-D^+)v_2(x_j) \end{pmatrix}. \end{aligned}$$

By the standard local truncation used in the Taylor expansions,

$$|\varepsilon_1(\frac{d^2}{dx^2}-\delta^2)v_1(x_j) + a_1(x_j)(\frac{d}{dx}-D^+)v_1(x_j)| \le C(x_{j+1}-x_{j-1})(\varepsilon_1\|v_1^{(3)}\| + \|v_1^{(2)}\|),$$

$$|\varepsilon_2(\frac{d^2}{dx^2}-\delta^2)v_2(x_j) + a_2(x_j)(\frac{d}{dx}-D^+)v_2(x_j)| \le C(x_{j+1}-x_{j-1})(\varepsilon_2\|v_2^{(3)}\| + \|v_2^{(2)}\|).$$

Since $(x_{j+1}-x_{j-1}) \le CN^{-1}$, using (28) and (29),

$$\|L^N(\vec{V}-\vec{v})\|_{\Omega^N} \le CN^{-1}.$$

Using Lemma 5,

$$\|\vec{V}-\vec{v}\|_{\overline{\Omega}^N} \le CN^{-1}. \tag{55}$$

□

To estimate the error in the singular components , we consider the mesh functions $B_1^N(x_j)$ and $B_2^N(x_j)$ on $\overline{\Omega}^N$ defined by

$$B_1^N(x_j) = \prod_{i=1}^{j}(1+\frac{\alpha h_i}{2\varepsilon_1})^{-1} \text{ and } B_2^N(x_j) = \prod_{i=1}^{j}(1+\frac{\alpha h_i}{2\varepsilon_2})^{-1}$$

with $B_1^N(x_0) = B_2^N(x_0) = 1$.

It is to be observed that B_1^N and B_2^N are monotonically decreasing.

Lemma 7. *The layer components W_1 and W_2 satisfy the following bounds on $\overline{\Omega}^N$.*

$$|W_1(x_j)| \le CB_2^N(x_j) \text{ and } |W_2(x_j)| \le CB_2^N(x_j).$$

Proof. Consider the following vector valued mesh functions on $\overline{\Omega}^N$:

$$\vec{\psi}^{\pm}(x_j) = C\big(B_2^N(x_j), B_2^N(x_j)\big)^T \pm \vec{W}(x_j).$$

Then for sufficiently large C, $\vec{\psi}^{\pm}(x_0) \ge \vec{0}$, $\vec{\psi}^{\pm}(x_N) \ge \vec{0}$ and

$$L^N\vec{\psi}^{\pm}(x_j) = CL^N\begin{pmatrix} B_2^N(x_j) \\ B_2^N(x_j) \end{pmatrix} \le \vec{0}.$$

Using the discrete maximum principle, we have $\vec{\psi}^{\pm}(x_j) \ge \vec{0}$ on $\overline{\Omega}^N$, which implies that

$$|W_1(x_j)| \le CB_2^N(x_j) \text{ and } |W_2(x_j)| \le CB_2^N(x_j).$$

□

Lemma 8. *Let $\vec{w}$ be the solution of (9) and $\vec{W}$ the solution of (54); then*

$$\|\vec{W} - \vec{w}\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N.$$

Proof. By the standard local truncation used in the Taylor expansions,

$$|\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)w_1(x_j) + a_1(x_j)(\frac{d}{dx} - D^+)w_1(x_j)| \leq C(x_{j+1} - x_{j-1})(\varepsilon_1\|w_1^{(3)}\| + \|w_1^{(2)}\|)$$
$$|\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)w_2(x_j) + a_2(x_j)(\frac{d}{dx} - D^+)w_2(x_j)| \leq C(x_{j+1} - x_{j-1})(\varepsilon_2\|w_2^{(3)}\| + \|w_2^{(2)}\|),$$

where the norm is taken over the interval $[x_{j-1}, x_{j+1}]$.

For the case $\tau_2 = 1/2$ and $\tau_1 = 1/4$, the mesh is uniform, $h = N^{-1}$, $\varepsilon_1^{-1} \leq C \ln N$ and $\varepsilon_2^{-1} \leq C \ln N$ and thus we obtain

$$|L^N(\vec{W} - \vec{w})(x_j)| \leq \begin{pmatrix} CN^{-1}\big(\varepsilon_1^{-2}B_1(x_{j-1}) + \varepsilon_2^{-2}B_2(x_{j-1})\big) \\ CN^{-1}\big(\varepsilon_1^{-1}\varepsilon_2^{-1}B_1(x_{j-1}) + \varepsilon_2^{-2}B_2(x_{j-1})\big) \end{pmatrix}. \tag{56}$$

Consider the following barrier function $\vec{\phi}$ given by

$$\phi_1(x_j) = \frac{CN^{-1}}{\gamma(\alpha - \gamma)}\Big(\exp(2\gamma h/\varepsilon_1)\varepsilon_1^{-1}Y_j + \exp(2\gamma h/\varepsilon_2)\varepsilon_2^{-1}Z_j\Big)$$
$$\phi_2(x_j) = \frac{CN^{-1}}{\gamma(\alpha - \gamma)}\Big(\exp(2\gamma h/\varepsilon_2)\varepsilon_1^{-1}Z_j\Big),$$

where γ is a constant such that $0 < \gamma < \alpha$,

$$Y_j = \frac{\lambda^{N-j} - 1}{\lambda^N - 1} \text{ with } \lambda = 1 + \frac{\gamma h}{\varepsilon_1}$$

and

$$Z_j = \frac{\Lambda^{N-j} - 1}{\Lambda^N - 1} \text{ with } \Lambda = 1 + \frac{\gamma h}{\varepsilon_2}.$$

It is not hard to see that

$$0 \leq Y_j,\ Z_j \leq 1,$$
$$(\varepsilon_1\delta^2 + \gamma D^+)Y_j = 0,\ \ (\varepsilon_2\delta^2 + \gamma D^+)Z_j = 0,$$
$$D^+Y_j \leq -\frac{\gamma}{\varepsilon_1}\exp(-\gamma x_{j+1}/\varepsilon_1),\ \ D^+Z_j \leq -\frac{\gamma}{\varepsilon_2}\exp(-\gamma x_{j+1}/\varepsilon_2).$$

Hence,

$$(L^N\vec{\phi})(x_j) \leq \frac{CN^{-1}}{\gamma(\alpha - \gamma)}\begin{pmatrix} \varepsilon_1^{-1}\exp(2\gamma h/\varepsilon_1)D^+Y_j + \varepsilon_2^{-1}\exp(2\gamma h/\varepsilon_2)D^+Z_j \\ \varepsilon_1^{-1}\exp(2\gamma h/\varepsilon_2)(a_2 - \gamma)D^+Z_j \end{pmatrix}$$
$$\leq -CN^{-1}\begin{pmatrix} \varepsilon_1^{-2}B_1(x_{j-1}) + \varepsilon_2^{-2}B_2(x_{j-1}) \\ \varepsilon_1^{-1}\varepsilon_2^{-1}B_1(x_{j-1}) \end{pmatrix}. \tag{57}$$

Consider the discrete functions

$$\vec{\psi}^{\pm}(x_j) = \vec{\phi}(x_j) \pm (\vec{W} - \vec{w})(x_j), x_j \in \overline{\Omega}^N.$$

Then for sufficiently large C, using (56) and (57), $\vec{\psi}^{\pm}(x_0) > \vec{0}$, $\vec{\psi}^{\pm}(x_N) = \vec{0}$ and $L^N\vec{\psi}^{\pm}(x_j) \le \vec{0}$ on Ω^N.

Using the discrete maximum principle, $\vec{\psi}^{\pm}(x_j) \ge \vec{0}$ on $\overline{\Omega}^N$. Hence,

$$|(\vec{W} - \vec{w})(x_j)| \le \begin{pmatrix} CN^{-1}\big(\varepsilon_1^{-1} + \varepsilon_2^{-1}\big) \\ CN^{-1}\big(\varepsilon_1^{-1}\big) \end{pmatrix} \le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}$$

implies that

$$\|(\vec{W} - \vec{w})\|_{\overline{\Omega}^N} \le CN^{-1}\ln N. \tag{58}$$

For other choices of τ_1 and τ_2, the estimate of $\|(\vec{W} - \vec{w})\|_{\overline{\Omega}^N}$ is as follows.

Let $\overline{\Omega}_1^N = \{x_j\}_{j=0}^{N/2}$ and $\overline{\Omega}_2^N = \{x_j\}_{j=N/2}^{N}$; then for $x_j \in \overline{\Omega}_2^N$, using Lemma 7 and Theorem 2,

$$\begin{aligned}
|(W_1 - w_1)(x_j)| &\le |W_1(x_j)| + |w_1(x_j)| \le CB_2^N(x_j) + CB_2(x_j) \\
&\le CB_2^N(\tau_2) + CB_2(\tau_2). \\
B_2(\tau_2) &= \exp\big(\frac{-\alpha\tau_2}{\varepsilon_2}\big) \le \exp(-\ln N) \le N^{-1}. \\
B_2^N(\tau_2) &= \prod_{i=1}^{j}\big(1 + \frac{\alpha h_i}{2\varepsilon_2}\big)^{-1} \\
&= \Big(1 + \frac{\alpha H_1}{2\varepsilon_2}\Big)^{\frac{-N}{4}}\Big(1 + \frac{\alpha H_2}{2\varepsilon_2}\Big)^{\frac{-N}{4}} \\
&= \Big(1 + \frac{2\alpha\tau_1}{N\varepsilon_2}\Big)^{\frac{-N}{4}}\Big(1 + \frac{2\alpha(\tau_2 - \tau_1)}{N\varepsilon_2}\Big)^{\frac{-N}{4}} \\
&\le \Big(1 + \frac{2\alpha\tau_2}{N\varepsilon_2}\Big)^{\frac{-N}{4}} \\
B_2^N(\tau_2) &\le N^{-1}.
\end{aligned}$$

Hence, $|(W_1 - w_1)(x_j)| \le CN^{-1}$.

Similarly, it is true that $|(W_2 - w_2)(x_j)| \le CN^{-1}$ and hence,

$$\|(\vec{W} - \vec{w})\|_{\overline{\Omega}_2^N} \le CN^{-1}. \tag{59}$$

For $N/4 \le j < N/2$, if $\varepsilon_2/2 \le \varepsilon_1 \le \varepsilon_2$, then $\tau_2 \le (4\varepsilon_1/\alpha)\ln N$ implies that

$$|L^N(\vec{W} - \vec{w})(x_j)| \le CN^{-1}\ln N \begin{pmatrix} \varepsilon_1^{-1}B_1(x_{j-1}) + \varepsilon_2^{-1}B_2(x_{j-1}) \\ \varepsilon_2^{-1}B_2(x_{j-1}) \end{pmatrix}. \tag{60}$$

On the other hand, if $\varepsilon_2 > 2\varepsilon_1$, then using (36),

$$\begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)w_1(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)w_2(x_j)| \end{pmatrix} \le \begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)w_{11}(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)w_{21}(x_j)| \end{pmatrix} + \begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)w_{12}(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)w_{22}(x_j)| \end{pmatrix}.$$

Also, by the standard local truncation used in the Taylor expansions and using Lemma 3,

$$\begin{aligned}
\begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2}-\delta^2)w_{11}(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2}-\delta^2)w_{21}(x_j)| \end{pmatrix} &\le \begin{pmatrix} C\varepsilon_1(x_{j+1}-x_{j-1})\|w_{11}^{(3)}\| \\ C\varepsilon_2(x_{j+1}-x_{j-1})\|w_{21}^{(3)}\| \end{pmatrix} \\
&\le C\varepsilon_2^{-1}N^{-1}\ln N \begin{pmatrix} B_2(x_{j-1}) \\ B_2(x_{j-1}) \end{pmatrix}, \\
\begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2}-\delta^2)w_{12}(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2}-\delta^2)w_{22}(x_j)| \end{pmatrix} &\le C\begin{pmatrix} \varepsilon_1\|w_{12}''\|_{[x_{j-1},x_{j+1}]} \\ \varepsilon_2\|w_{22}''\|_{[x_{j-1},x_{j+1}]} \end{pmatrix} \\
&\le C\begin{pmatrix} \varepsilon_1^{-1}B_1(x_{j-1}) \\ \varepsilon_2^{-1}B_1(x_{j-1}) \end{pmatrix}.
\end{aligned}$$

Thus, for $N/4 \le j < N/2$,

$$\begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2}-\delta^2)w_1(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2}-\delta^2)w_2(x_j)| \end{pmatrix} \le \begin{pmatrix} C\varepsilon_2^{-1}N^{-1}\ln NB_2(x_{j-1}) + C\varepsilon_1^{-1}B_1(x_{j-1}) \\ C\varepsilon_2^{-1}N^{-1}\ln NB_2(x_{j-1}) + C\varepsilon_2^{-1}B_1(x_{j-1}) \end{pmatrix}. \tag{61}$$

Using the alternate decomposition of $w_1(x)$ given in (47) and the arguments similar to the above, it is not hard to verify that for $N/4 \le j < N/2$

$$\begin{pmatrix} |(\frac{d}{dx}-D^+)w_1(x_j)| \\ |(\frac{d}{dx}-D^+)w_2(x_j)| \end{pmatrix} \le \begin{pmatrix} C\varepsilon_2^{-1}N^{-1}\ln NB_2(x_{j-1}) + C\varepsilon_1^{-1}B_1(x_{j-1}) \\ C\varepsilon_2^{-1}N^{-1}\ln NB_2(x_{j-1}) + C\varepsilon_2^{-1}B_1(x_{j-1}) \end{pmatrix}. \tag{62}$$

Hence, for $N/4 \le j < N/2$, expressions (61) and (62) yield

$$|L^N(\vec{W}-\vec{w})(x_j)| \le \begin{pmatrix} C\varepsilon_2^{-1}N^{-1}\ln NB_2(x_{j-1}) + C\varepsilon_1^{-1}B_1(x_{j-1}) \\ C\varepsilon_2^{-1}N^{-1}\ln NB_2(x_{j-1}) + C\varepsilon_2^{-1}B_1(x_{j-1}) \end{pmatrix}. \tag{63}$$

For $0 < j < N/4$, $\tau_1 \le (\varepsilon_1/\alpha)\ln N$ and hence

$$|L^N(\vec{W}-\vec{w})(x_j)| \le CN^{-1}\ln N \begin{pmatrix} \varepsilon_1^{-1}B_1(x_{j-1}) + \varepsilon_2^{-1}B_2(x_{j-1}) \\ \varepsilon_2^{-1}B_2(x_{j-1}) \end{pmatrix}. \tag{64}$$

Consider the following barrier functions for $0 < j < N/4$

$$\phi_1(x_j) = CN^{-1}\ln N\big(\exp(2\alpha H_1/\varepsilon_1)B_1^N(x_j) + \exp(2\alpha H_1/\varepsilon_2)B_2^N(x_j)\big) \tag{65}$$
$$\phi_2(x_j) = CN^{-1}\ln N\exp(2\alpha H_1/\varepsilon_2)B_2^N(x_j) \tag{66}$$

and for $N/4 \le j \le N/2$

$$\phi_1(x_j) = CN^{-1}\ln N\exp(2\alpha H_2/\varepsilon_2)B_2^N(x_j) + CB_1^N(x_j) \tag{67}$$
$$\phi_2(x_j) = CN^{-1}\ln N\exp(2\alpha H_2/\varepsilon_2)B_2^N(x_j) + CN^{-1}\big((\tau_2 - x_j)\varepsilon_2^{-1} + 1\big). \tag{68}$$

Let $\vec{\phi} = (\phi_1, \phi_2)^T$ and consider the following vector valued mesh functions for $0 \le j \le N/2$

$$\vec{\psi}^{\pm}(x_j) = \vec{\phi}(x_j) \pm (\vec{W}-\vec{w})(x_j).$$

For sufficiently large C,

$$\vec{\psi}^{\pm}(x_0) \geq \vec{0},\ \vec{\psi}^{\pm}(x_{\frac{N}{2}}) \geq \vec{0} \text{ and } L^N\vec{\psi}^{\pm}(x_j) \leq \vec{0}, \text{for } 0 < j < N/2.$$

Then by Lemma 4 $\vec{\psi}^{\pm}(x_j) \geq \vec{0}$ for $0 \leq j \leq N/2$. Hence,

$$\|(\vec{W} - \vec{w})\|_{\overline{\Omega}_1^N} \leq CN^{-1}\ln N. \tag{69}$$

Therefore, for any choice of τ_1 and τ_2,

$$\|(\vec{W} - \vec{w})\|_{\overline{\Omega}^N} \leq CN^{-1}\ln N. \tag{70}$$

□

Theorem 3. *Let $\vec{u}$ be the solution of the problem (1)-(2) and $\vec{U}$ the solution of the problem (51)-(52); then*

$$\|(\vec{u} - \vec{U})\|_{\overline{\Omega}^N} \leq CN^{-1}\ln N.$$

Proof. The result follows by using the triangle inequality, (55) and (70). □

6. Numerical illustrations

Example 1. *Consider the boundary value problem for the system of convection-diffusion equations on (0,1)*

$$\varepsilon_1 u_1''(x) + (1+x^2)u_1'(x) - (4+\sin x)u_1(x) + 2u_2(x) = -e^x, \tag{71}$$

$$\varepsilon_2 u_2''(x) + (2+x)u_2'(x) + u_1(x) - (2+\cos x)u_2(x) = -x^2, \tag{72}$$

$$\textit{with } u_1(0) = 3,\ u_2(0) = 3,\ u_1(1) = 1,\ u_2(1) = 1. \tag{73}$$

The above problem is solved using the suggested numerical method and a plot of the approximate solution for $N = 1024, \varepsilon_1 = 5^{-4}, \varepsilon_2 = 2^{-7}$ is shown in Figure 1.

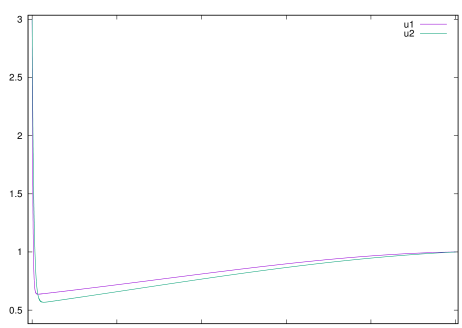


Figure 1: *Approximate solution of Example 1*

Parameter uniform error and order of convergence of the numerical method are shown in Table 1 which are computed using a two-mesh algorithm, a variant of the one suggested in [5].

ε_1	ε_2	Number of mesh elements N				
		128	256	512	1024	2048
5^{-4}	2^{-7}	$4.725E-02$	$2.887E-02$	$1.775E-02$	$1.019E-02$	$5.779E-03$
5^{-5}	2^{-8}	$4.789E-02$	$2.919E-02$	$1.792E-02$	$1.028E-02$	$5.827E-03$
5^{-6}	2^{-9}	$6.282E-02$	$4.456E-02$	$2.644E-02$	$1.535E-02$	$8.425E-03$
5^{-7}	2^{-10}	$7.146E-02$	$5.089E-02$	$3.212E-02$	$1.914E-02$	$1.095E-02$
5^{-8}	2^{-11}	$7.393E-02$	$5.243E-02$	$3.365E-02$	$2.009E-02$	$1.159E-02$
5^{-9}	2^{-12}	$7.470E-02$	$5.321E-02$	$3.437E-02$	$2.033E-02$	$1.174E-02$
5^{-10}	2^{-13}	$7.497E-02$	$5.355E-02$	$3.462E-02$	$2.040E-02$	$1.177E-02$
5^{-11}	2^{-14}	$7.508E-02$	$5.367E-02$	$3.471E-02$	$2.042E-02$	$1.179E-02$
5^{-12}	2^{-15}	$7.512E-02$	$5.372E-02$	$3.475E-02$	$2.042E-02$	$1.180E-02$
5^{-13}	2^{-16}	$7.513E-02$	$5.374E-02$	$3.477E-02$	$2.043E-02$	$1.181E-02$
5^{-14}	2^{-17}	$7.514E-02$	$5.375E-02$	$3.477E-02$	$2.043E-02$	$1.181E-02$
5^{-15}	2^{-18}	$7.514E-02$	$5.375E-02$	$3.478E-02$	$2.044E-02$	$1.181E-02$
5^{-16}	2^{-19}	$7.515E-02$	$5.376E-02$	$3.478E-02$	$2.044E-02$	$1.181E-02$
5^{-17}	2^{-20}	$7.515E-02$	$5.376E-02$	$3.478E-02$	$2.044E-02$	$1.181E-02$
5^{-18}	2^{-21}	$7.515E-02$	$5.376E-02$	$3.478E-02$	$2.044E-02$	$1.181E-02$
D^N		$7.515E-02$	$5.376E-02$	$3.478E-02$	$2.044E-02$	$1.181E-02$
p^N		$0.483E+00$	$0.628E+00$	$0.767E+00$	$0.791E+00$	
C_p^N		$2.755E+00$	$2.755E+00$	$2.491E+00$	$2.047E+00$	$1.654E+00$

Table 1: *Computed order of $(\varepsilon_1, \varepsilon_2)$-uniform convergence, $p^* = 0.4833$, computed $(\varepsilon_1, \varepsilon_2)$-uniform error constant, $C_{p^*}^N = 2.7546$*

From Table 1, it is to be noted that the error decreases as the number of mesh elements N increases. Also for each N, the error stabilizes as ε_1 and ε_2 tend to zero.

Example 2. *Consider the boundary value problem for the system of convection diffusion equations on (0,1)*

$$\varepsilon_1 u_1''(x) + u_1'(x) - 2u_1(x) + u_2(x) = -3(x-1), \tag{74}$$

$$\varepsilon_2 u_2''(x) + (1+x)u_2'(x) + xu_1(x) - (2x+1)u_2(x) = -2x, \tag{75}$$

$$\text{with } u_1(0) = 0,\ u_2(0) = 3,\ u_1(1) = 2,\ u_2(1) = 2. \tag{76}$$

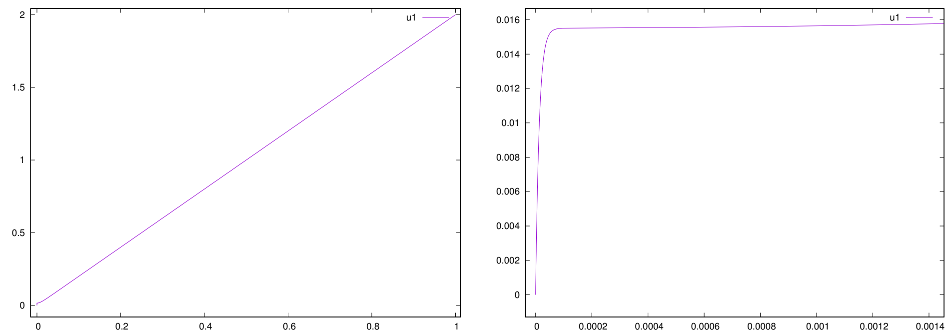
The reduced problem corresponding to (74) - (76) is

$$u_{01}'(x) - 2u_{01}(x) + u_{02}(x) = -3(x-1), \tag{77}$$

$$(1+x)u_{02}'(x) + xu_{01}(x) - (2x+1)u_{02}(x) = -2x, \tag{78}$$

$$\text{with } u_{01}(1) = 2,\ u_{02}(1) = 2. \tag{79}$$

A solution of the reduced problem is $(u_{01}(x), u_{02}(x))^T = (2x, x+1)^T$. Eventhough $u_{01}(x)$ coincides with $u_1(x)$ at the boundary points, $u_{02}(0) \neq u_2(0)$ implies that the ε_2-layer may occur at $x = 0$ in both solution components u_1 and u_2. For $N = 1024$, $\varepsilon_1 = 5^{-6}$, $\varepsilon_2 = 2^{-6}$, the plots of the approximate solution components of (74) - (76) shown in Figures 2 and 3 ensure the foresaid layer patterns.



(a) u_1 (b) u_1 near $x = 0$

Figure 2: *Approximation of solution component* u_1 *of Example 2*

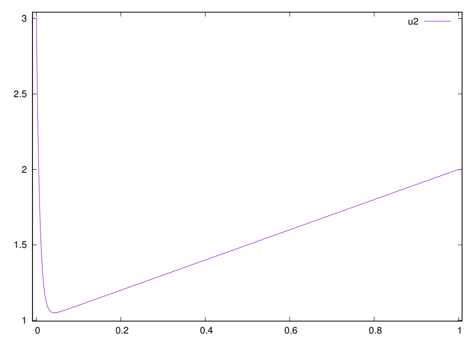


Figure 3: *Approximation of solution component* u_2 *of Example 2*

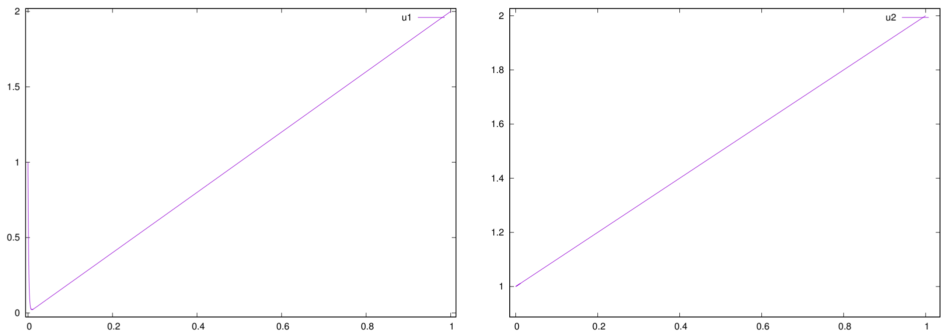
Example 3. *Consider the boundary value problem for the system of convection-diffusion equations on (0,1)*

$$\varepsilon_1 u_1''(x) + u_1'(x) - 2u_1(x) + u_2(x) = -3(x-1), \tag{80}$$

$$\varepsilon_2 u_2''(x) + (1+x)u_2'(x) + xu_1(x) - (2x+1)u_2(x) = -2x, \tag{81}$$

$$\textit{with}\; u_1(0) = 1,\; u_2(0) = 1,\; u_1(1) = 2,\; u_2(1) = 2. \tag{82}$$

A solution of the reduced problem is $(u_{01}(x), u_{02}(x))^T = (2x, x+1)^T$. Since $u_{01}(0) \neq u_1(0)$ and $u_{02}(0) = u_2(0)$, the ε_1-layer is expected near $x = 0$ only in the solution component u_1. For $N = 1024$, $\varepsilon_1 = 5^{-4}$, $\varepsilon_2 = 2^{-4}$, the plots of the approximate solution components of (80)–(82) shown in Figure 4 ensure the foresaid layer patterns.



(a) u_1 (b) u_2

Figure 4: *Approximation of solution components of Example 3*

Acknowledgement

The first author wishes to acknowledge financial support extended through the Junior Research Fellowship by the University Grants Commission, India, to carry out this research work. Also the first and the third authors thank the Department of Science & Technology, Government of India, for the support to the Department through the DST-FIST Scheme to set up the Computer Lab where the computations have been carried out.

References

[1] S. Bellow, E. O'Riordan, *A parameter robust numerical method for a system of two singularly perturbed convection-diffusion equations*, Appl. Numer. Math. **51**(2004), 171–186.

[2] Z. Cen, *Parameter-uniform finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations*, Int. J. Comput. Math. **82**(2005), 177–192.

[3] C. Clavero, J. L. Gracia, F. J. Lisbona, *An almost third order finite difference scheme for singularly perturbed reaction-diffusion system*, J. Comput. Appl. Math. **234**(2010), 2501–2515.

[4] E. P. Doolan, J. J. H. Miller, W. H. A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*, Boole Press, Dublin, 1980.

[5] P. A. Farrell, A. Hegarty, J. J. H. Miller, E O'Riordan, G. I. Shishkin, *Robust computational techniques for boundary layers*, Chapman and Hall/CRC Press, Boca Raton, 2000.

[6] T. Linss, *Analysis of an upwind finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations*, Computing **79**(2007), 23–32.

[7] T. Linss, N. Madden, *Layer-adapted meshes for a linear system of coupled singularly perturbed reaction-diffusion problems*, IMA J. Numer. Anal. **29**(2009), 109–125.

[8] N. Madden, M. Stynes, *A uniformly convergent numerical for a coupled system of two singularly perturbed linear reaction-diffusion problems*, IMA J. Numer. Anal. **23**(2003), 627–644.

[9] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, *Fitted numerical methods for singular perturbation problems*, World Scientific Publishing Co., Singapore, 1996.

[10] E. O'Riordan, M. Stynes, *Numerical analysis of a strongly coupled system of two singularly perturbed convection-diffusion problems*, Adv. Comput. Math. **30**(2009), 101–121.

[11] M. Paramasivam, S. Valarmathi, J. J. H. Miller, *Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear reaction-diffusion system*, Math. Commun. **15**(2010), 587–612.