Rank bounded Hibi subrings for planar distributive lattices

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Abstract. Let L be a distributive lattice and $R[L]$ the associated Hibi ring. We show that if L is planar, then any bounded Hibi subring of $R[L]$ has a quadratic Gröbner basis. We characterize all planar distributive lattices L for which any proper rank bounded Hibi subring of $R[L]$ has a linear resolution. Moreover, if $R[L]$ is linearly related for a lattice L, we find all rank bounded Hibi subrings of $R[L]$ which are linearly related, too.

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1. Introduction

Hibi rings and their defining ideals are attached in a natural way to finite distributive lattices. They were introduced by Hibi in [13].

Let L be a finite distributive lattice and P a subposet of L which consists of the join-irreducible elements of L. Then, by a famous theorem of Birkhoff [3], it follows that L is isomorphic to the lattice $\mathcal{I}(P)$ of the poset ideals of P. We recall that an element $a \in L$ is called join-irreducible if a is not the minimal element of L and if $a = b \vee c$, where $b, c \in L$, then $a = b$ or $a = c$; in other words, a does not admit a proper decomposition as a join of elements of L. Let us also recall the definition of a poset ideal. A subset $\alpha \subseteq P$ is called a poset ideal if it has the following property: for any $a \in \alpha$ and $b \in P$, if $b \leq a$, then $b \in \alpha$. For a comprehensive study of finite lattices we refer to [3, 18].

Let us assume that P consists of n elements, say $P = \{p_1, \ldots, p_n\}$. Let $A = K[t, x_1, \ldots, x_n]$ be a polynomial ring in $n + 1$ variables over a field K. The Hibi ring of $L = \mathcal{I}(P)$ is the K–subalgebra $R[L]$ of A generated over K by the monomials $u_\alpha = t \prod_{p_i \in \alpha} x_i$, with $\alpha \in L$. In [13], Hibi showed that $R[L]$ is an algebra with straightening laws on L over K (ASL, in brief). For an extensive survey on ASL, we refer the reader to [4]. Let $B = K[\{x_\alpha : \alpha \in L\}]$ be a polynomial ring in the indeterminates indexed by the elements of L. Then, the defining ideal $I_L \subset B$ of the toric ring $R[L]$ is generated by the straightening relations, namely

$$I_L = (x_\alpha x_\beta - x_{\alpha \cap \beta} x_{\alpha \cup \beta} : \alpha, \beta \in L, \alpha, \beta \text{ incomparable}).$$

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I_L is called the Hibi ideal or the join-meet ideal of L.

In the same paper [13], Hibi showed that $R[L]$ is a Cohen-Macaulay normal domain of $\dim R[L] = |P| + 1$. In the last decades, many authors have investigated various properties and invariants of Hibi rings; see, for example, [1, 7, 8, 9, 11, 12, 17]. Generalizations of Hibi rings are studied in papers [2, 10].

Much less is known about the so-called rank bounded Hibi subrings introduced in [1].

Let L be a finite distributive lattice and let p, q be integers such that $0 \leq p < q \leq \operatorname{rank} L$. The rank bounded Hibi subring $R[p, q; L] \subseteq R[L]$ is the K–subalgebra of $R[L]$ generated over K by all the monomials $u_\alpha = t \prod_{p_i \in \alpha} x_i$, with $\alpha \in L$ and $p \leq \operatorname{rank} \alpha \leq q$. In [1], it was shown that if a Hibi ring $R[L]$ possesses a quadratic Gröbner basis with respect to a rank lexicographic order, then all the rank bounded Hibi subrings have the same property.

In this paper, we study rank bounded Hibi subrings for planar distributive lattices. The paper is organized as follows. In Section 2, we show that any rank bounded Hibi subring of $R[L]$, where L is a planar distributive lattice, has a quadratic Gröbner basis. In order to prove this theorem, we interpret $R[p, q; L]$ as an edge ring of a suitable bipartite graph which has only cycles of length 4. As a consequence, we derive that $R[p, q; L]$ is a Cohen-Macaulay normal domain.

In Section 3, we study several homological properties of rank bounded Hibi subrings. In Theorem 2, we characterize the planar distributive lattices L with the property that every proper rank bounded Hibi subring of $R[L]$ has a linear resolution. In particular, we see that if $R[L]$ has a linear resolution, and then every rank bounded Hibi subring of $R[L]$ has a linear resolution. As it follows from Example 2, if $R[L]$ is linearly related, then it does not necessarily follow that any rank bounded Hibi subring of $R[L]$ inherits the same property. However, given a lattice L such that $R[L]$ is linearly related, we may find all rank bounded Hibi subrings of $R[L]$ which are linearly related too; see Theorem 5.

2. The Gröbner basis

Let $\mathbb{N}^2$ be a infinite distributive lattice with the partial order defined as follows: $(i, j) \leq (k, \ell)$ if $i \leq k$ and $j \leq \ell$. A planar distributive lattice L is a finite sublattice of $\mathbb{N}^2$ which contains $(0, 0)$ and has the following property: if $(i, j), (k, \ell) \in L$ and $(i, j) < (k, \ell)$, then there exists a chain of elements in L :

$$(i, j) = (i_0, j_0) < (i_1, j_1) < \cdots < (i_t, j_t) = (k, \ell)$$

such that $i_{s+1} + j_{s+1} = i_s + j_s + 1$ for all $0 \leq s \leq t - 1$. If $a, b \in \mathbb{N}^2$ with $a \leq b$, then the set $[a, b] = \{c \in \mathbb{N}^2 |\ a \leq c \leq b\}$ is an interval of $\mathbb{N}^2$. The interval $C = [a, b]$ with $b = a + (1, 1)$ is called a cell of $\mathbb{N}^2$. Any planar distributive lattice may be viewed as a convex polyomino, as observed in [7]. For more information about polyominoes and their ideals we refer to [16, 7].

In what follows, we consider only simple planar distributive lattices, that is, lattices L with the property that, for any $0 < \ell < \operatorname{rank} L$, there exist at least two elements of L of rank ℓ.

Definition 1. *Let L be a planar distributive lattice and let p, q be integers such that $0 \leq p < q \leq \operatorname{rank} L$. The K–subalgebra $R[p, q; L]$ of $R[L]$ generated by all monomials u_α with $p \leq \operatorname{rank} \alpha \leq q$ is called a rank bounded Hibi subring of $R[L]$.*

If $p = 0$, we call $R[p, q; L]$ a rank upper-bounded Hibi subring. Similarly, if $q = \operatorname{rank} L$, we call $R[p, q; L]$ a rank lower-bounded Hibi subring.

In this section, we show that any rank bounded Hibi subring has a quadratic Gröbner basis.

Let L be a planar distributive lattice. The elements of L are lattice points (i, j) in the plane with $0 \leq i \leq m$ and $0 \leq j \leq n$ for some positive integers m, n with $m + n = \operatorname{rank} L$. Then, the Hibi ring $R[L]$ may be viewed as the edge ring of a bipartite graph G_L which admits the vertex bipartition $\{s_1, \ldots, s_m\} \cup \{t_1, \ldots, t_n\}$ and whose edge set is $E(G_L) = \{\{s_i, t_j\} : (i, j) \in L\}$. Note that the generator $s_i t_j$ of the edge ring corresponds to an element in L whose rank is $i + j$. Let $0 \leq p < q \leq \operatorname{rank} L$ and $R[p, q; L]$ a rank bounded Hibi subring of $R[L]$. Then $R[p, q; L]$ coincides with the subring of the edge ring $K[\{s_i t_j : 0 \leq i \leq m, 0 \leq j \leq n\}]$ generated by all monomials $s_i t_j$ with $p \leq i + j \leq q$.

Example 1. *In Figure 1, we have a lattice of rank $5 + 4 = 9$ whose elements are the lattice points contained in the polygon bounded by the fat polygonal line. The fat points in the figure correspond to the generators of the subring of $R[L]$ bounded by $p = 3$ and $q = 7$. The bounded subring of $R[L]$ has 14 generators as an algebra over K.*
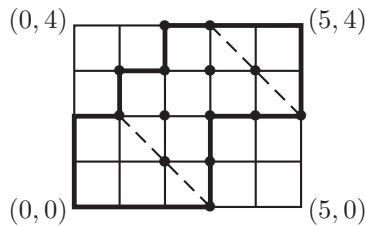

Figure 1: *Representation of a rank bounded Hibi subring*

Theorem 1. *Let L be a planar distributive lattice and p, q integers with $0 \leq p < q \leq \operatorname{rank} L$. The defining ideal of the rank bounded Hibi subring $R[p, q; L]$ has a quadratic Gröbner basis.*

Proof. As we have already observed, $R[p, q; L]$ may be identified with the subring $K[\{s_i t_j : (i, j) \in L, p \leq i + j \leq q\}]$ of the edge ring $R[L] = K[\{s_i t_j : (i, j) \in L\}]$. In [15], it was shown that an edge ring of a bipartite graph G has a quadratic Gröbner basis (with respect to a suitable monomial order) if and only if every cycle of G of length > 4 has a chord. We follow the ideas of the proof of [7, Theorem 2.1]. Let G be a bipartite graph with edges $\{s_i, t_j\}$, where $p \leq i + j \leq q$. An even cycle of length $2r$ in G is a sequence of edges of G which correspond to a sequence of lattice points in the plane, say $a_1, \ldots, a_{2r}$, with $a_{2k-1} = (i_k, j_k)$ and $a_{2k} = (i_{k+1}, j_k)$ for $k = 1, \ldots, r$, where $i_{r+1} = i_1$, $i_k \neq i_\ell$ and $j_k \neq j_\ell$ for $k \neq \ell, k, \ell \leq r$. By [7, Lemma

2.2], it follows that there exist integers c, d with $1 \leq c, d \leq r$, $d \neq c, c+1$ such that either $i_c < i_d < i_{c+1}$ or $i_{c+1} < i_d < i_c$. Let us choose $i_c < i_d < i_{c+1}$. The other case may be discussed similarly. Since $a_{2c-1} = (i_c, j_c)$ and $a_{2c} = (i_{c+1}, j_c)$ correspond to edges of G, we have: $p \leq i_c + j_c < i_{c+1} + j_c \leq q$. As $i_c < i_d < i_{c+1}$, it follows that $p \leq i_d + j_c \leq q$ which implies that (i_d, j_c) corresponds to a chord in our cycle of G. $\qquad \square$

3. Properties of rank bounded Hibi subrings

Theorem 1 and its proof have important consequences. In fact, in view of the theorem proved in [15], the proof of Theorem 1 shows that the binomials of the quadratic Gröbner basis of the defining ideal of $R[p, q; L]$ are differences of squarefree monomials of degree 2. This immediately implies the following.

Corollary 1. *Let L be a planar distributive lattice and p, q integers with $0 \leq p < q \leq \operatorname{rank} L$. Then $R[p, q; L]$ is a normal Cohen-Macaulay domain.*

Proof. By a theorem of Sturmfels [19], since the defining ideal of $R[p, q; L]$ has a squarefree initial ideal, it follows that $R[p, q; L]$ is a normal domain. The Cohen-Macaulay property follows from a classical theorem of Hochster [14]. $\qquad \square$

Remark 1. *In [1, Section 2], it was shown that if L is a chain ladder, then $R[L]$ and any bounded subring of $R[L]$ have a lexicographic quadratic Gröbner basis. Our Theorem 1 does not impose any additional condition on the planar distributive lattice L to derive that any rank bounded subring of $R[L]$ has a quadratic Gröbner basis.*

Remark 2. *Let L be a planar distributive lattice, p, q integers with $0 \leq p < q \leq \operatorname{rank} L$, and $T = K[\{y_{ij} : (i, j) \in L, p \leq i + j \leq q\}]$ the polynomial ring over K. By the proof of Theorem 1, the defining ideal of $R[p, q; L]$ is the binomial ideal of T generated by the quadratic binomials $y_{ij}y_{k\ell} - y_{i\ell}y_{kj}$, where $(i, j), (k, \ell) \in L$, $p \leq i + j, k + \ell \leq q$ and $p \leq i + \ell, j + k \leq q$.*

On the other hand, let us observe that one may consider the collection $\mathcal{P}$ of all the cells $[a, a + (1, 1)] \subset L$ with $a = (i, j) \in L, a + (1, 1) \in L$, and such that $p \leq i + j < i + j + 2 \leq q$. This is obviously a convex polyomino. Indeed, $\mathcal{P}$ is row convex and column convex. We give only the argument for row convexity since column convexity works similarly. Let $[a, a + (1, 1)], [b, b + (1, 1)] \in \mathcal{P}$, where $a = (i, j), b = (k, j), i < k$. Then, if $[c, c + (1, 1)] \in \mathbb{N}^2$ is a cell with $c = (\ell, j)$ where $i \leq \ell \leq k$, then we have

$$p \leq i + j \leq \ell + j < \ell + j + 2 \leq k + j + 2 \leq q.$$

This shows that $[c, c + (1, 1)] \in \mathcal{P}$.

Let $T' \subset T$ be a polynomial ring in the variables y_{ij}, where (i, j) is a vertex of $\mathcal{P}$. Then, according to the proof of [7, Theorem 2.1], the polyomino ideal $I_\mathcal{P}$ is generated by the quadratic binomials $y_{ij}y_{k\ell} - y_{i\ell}y_{kj}$, where $(i, j), (k, \ell) \in L$, $p \leq i + j, k + \ell \leq q$ and $p \leq i + \ell, j + k \leq q$. Thus, the defining ideal of $R[p, q; L]$ is nothing else but $I_\mathcal{P} T$.

This simple observation will be very useful in our further study.

Theorem 1 has another obvious consequence.

Corollary 2. *Let L be a planar distributive lattice and p, q integers with $0 \leq p < q \leq \operatorname{rank} L$. Then $R[p, q; L]$ is Koszul.*

Proof. It is a classical result that a standard graded K–algebra is Koszul if its defining ideal has a quadratic Gröbner basis. For a proof we refer to [6, Theorem 6.7]. $\square$

Remark 3. *Let $\mathcal{P}$ be the collection of all cells $[a, a + (1,1)] \subset L$ with $a = (i, j) \in L, a + (1, 1) \in L$, and such that $p \leq i + j < i + j + 2 \leq q$. Then*

$$\dim R[p, q; L] = (\text{the number of vertices } (i, j) \text{ in } L \text{ such that } p \leq i + j \leq q)$$
$$- \text{ the number of cells in } \mathcal{P}.$$

This fact is a direct consequence of [16, Corollary 2.3].

In what follows, we are interested in relating some homological properties of rank bounded Hibi subrings to the corresponding properties of the Hibi ring.

Let L be a planar distributive lattice and let $I_L \subset K[\{y_{ij} : (i, j) \in L\}]$ be the defining ideal of $R[L]$. For any rank bounded Hibi subring $R[p, q; L]$ of $R[L]$, we denote by $I_L^{p,q}$ the defining ideal of $R[p, q; L]$ contained in the polynomial ring $K[\{y_{ij} : (i, j) \in L, p \leq i + j \leq q\}]$.

Theorem 2. *Let L be a planar distributive lattice. Then the defining ideal of any proper rank bounded Hibi subring of $R[L]$ has a linear resolution if and only if one of the following conditions holds:*

(i) *I_L has a linear resolution, that is, $L = \mathcal{I}(P)$, where P is the direct sum of a chain and an isolated element.*

(ii) *$L = \mathcal{I}(P)$, where P is the poset $p_1 < p_2, q_1 < q_2, q_1 < p_2$.*

(iii) *$L = \mathcal{I}(P)$, where P is the poset $p_1 < p_2, q_1 < q_2, p_1 < q_2$.*

Proof. Let L be a planar distributive lattice. Suppose that the defining ideal of any proper rank bounded Hibi subring of $R[L]$ has a linear resolution. In Remark 3.3, it is shown that the defining ideal of any rank bounded Hibi subring is nothing else but the ideal $I_{\mathcal{P}}$, where $\mathcal{P}$ is a convex polyomino. We employ here [7, Theorem 4.1] which states that, for a convex polyomino $\mathcal{P}$, $I_{\mathcal{P}}$ has a linear resolution if and only if $\mathcal{P}$ consists of either a row of cells or a column of cells, that is, $\mathcal{P}$ is of one of the forms displayed in Figure 2:

Figure 2: $I_{\mathbb{P}}$ *linearly related*

Let $r = \operatorname{rank} L$, $q = r - 1$ and $p = 0$; then the upper bounded Hibi subring $R[p, q; L]$ has the defining ideal determined by a polyomino of one of the forms given

in Figure 2. Without loss of generality, we may assume that $I_L^{0,r-1} = I_{\mathcal{P}}$, where $\mathcal{P}$ is the polyomino consisting of the cells $[a, a + (1,1)]$, where $a \in \{(0,0), (1,0), \dots, (s-1,0)\}$ for some $s \geq 2$. Since rank $L = q + 1$, L contains exactly one of the cells $\alpha = [(s,0), (s,0) + (1,1)]$ and $\beta = [(s-1,1), (s-1,1) + (1,1)]$. If L contains the cell α, then L is of the form (i). Otherwise, L contains the cell β. If $s > 2$, then we may choose the Hibi subring $R[1, \text{rank } L; L]$ which does not have a linear resolution by [7, Theorem 4.1]. Hence $s = 2$ and $L = \mathcal{I}(P)$ is of one of the forms (ii) and (iii). $\square$
      The converse is obvious.

The above theorem shows that if I_L has a linear resolution, then $I_L^{p,q}$ has a linear resolution as well for any $0 \leq p < q \leq \text{rank } L$. We are now interested to see whether the property of I_L of being linearly related is inherited by all $I_L^{p,q}$. The following example shows that this is not the case. Before discussing this, let us recall some facts. The ideal I_L is linearly related if its relation module, $\text{Syz}_1(I_L)$ is generated only in degree 3. The planar lattices L whose ideal I_L is linearly related are characterized in [5, Theorem 3.12]:

Theorem 3 (see [5]). *Let L be a planar distributive lattice, $L \subseteq [(0,0),(m,n)]$ with $m, n \geq 2$. The ideal I_L is linearly related if and only if the following conditions hold:*

  (i) *At most one of the vertices $(m,0)$ and $(0,n)$ does not belong to L.*

  (ii) *The vertices $(1, n-1)$ and $(m-1, 1)$ belong to L.*

In [7, Theorem 3.1], the polyominoes whose associated binomial ideals are linearly related are characterized. We include here the complete statement for the convenience of the reader. Theorem 4 refers to Figure 3. We assume that $[(0,0),(m,n)]$ is the smallest interval with the property that $V(\mathcal{P}) \subseteq [(0,0),(m,n)]$. Here $V(\mathcal{P})$ denotes the set of all vertices of $\mathcal{P}$. The elements $(0,0), (m,0), (0,n)$ and (m,n) are the corners of $\mathcal{P}$. The corners $(0,0), (m,n)$ i.e., $(m,0), (0,n)$ are called opposite corners.
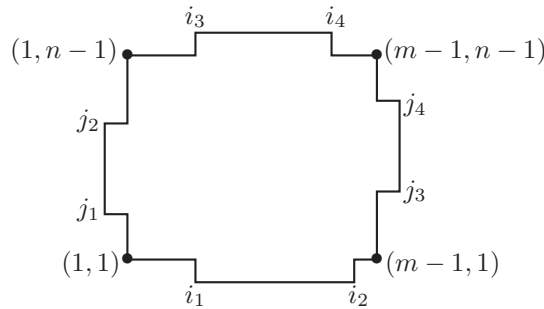

Figure 3: *Possible shapes for a linearly related polyomino*

In Figure 3, the number i_1 is also allowed to be 0, in which case also $j_1 = 0$. In this case, the polyomino contains the corner $(0,0)$. A similar convention applies to other corners. In Figure 3, all four corners $(0,0), (0,n), (m,0)$ and (m,n) are missing from the polyomino.

Theorem 4 (see [7]). *Let $\mathcal{P}$ be a convex polyomino. The following conditions are equivalent:*

(a) *$\mathcal{P}$ is linearly related;*

(b) *$I_{\mathcal{P}}$ admits no Koszul relation pairs;*

(c) *Let, as we may assume, $[(0,0),(m,n)]$ be the smallest interval with the property that $V(\mathcal{P}) \subseteq [(0,0),(m,n)]$. Then $\mathcal{P}$ has the shape as displayed in Figure 3, and one of the following conditions holds:*

    (i) *at most one of the corners does not belong to $V(\mathcal{P})$;*

    (ii) *two of the corners do not belong to $V(\mathcal{P})$, but they are not opposite to each other. In other words, the missing corners are not the corners $(0,0),(m,n)$ or the corners $(m,0),(0,n)$.*

    (iii) *three of the corners do not belong to $V(\mathcal{P})$. If the missing corners are $(m,0)$, $(0,n)$ and (m,n) (which one may assume without loss of generality), then referring to Figure 3, the following conditions must be satisfied: either $i_2 = m - 1$ and $j_4 \leq j_2$, or $j_2 = n - 1$ and $i_4 \leq i_2$.*

With all these tools at hand, we can move on to our study.

Example 2. *Let L be a planar distributive lattice whose poset of join-irreducible elements consists of two disjoint chains of lengths m and n, respectively. Then, as a planar lattice, L consists of all lattice points (i,j) with $0 \leq i \leq m$ and $0 \leq j \leq n$. In Figure 4, we displayed such a lattice for $m = 5$ and $n = 4$.*
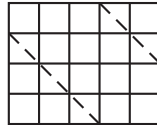

Figure 4: *Bounded subrings which are not linearly related*

If $m,n \geq 2$, then the defining ideal of any rank bounded Hibi subring $R[p,q;L]$ with $0 < p < q < \operatorname{rank} L = m+n$ is not linearly related. This is due to the fact that, for $0 < p < q < \operatorname{rank} L$, the ideal $I_L^{p,q}$ is actually the ideal of a polyomino $\mathcal{P}$ whose opposite corners are missing, thus $\mathcal{P}$ does not satisfy the conditions of Theorem 4.

On the other hand, if one considers the upper-bounded subring $R[0,q;L]$ with $q = m+n-1$, its defining ideal is linearly related since the corresponding polyomino satisfies condition (i) in Theorem 4.

The next theorem refers to Figure 5. In this figure, the number i_2 is allowed to be m, in which case also $j_2 = 0$. In this case, the polyomino contains the corner $(m,0)$. A similar convention applies to the corner$(0,n)$.

Theorem 5. *Let $m,n \geq 2$ and $L \subseteq [(0,0),(m,n)]$ be a planar distributive lattice with the property that I_L is linearly related. Let p,q be integers such that $0 \leq p < q \leq \operatorname{rank} L$. Then $I_L^{p,q}$ is linearly related as well if and only if one of the following conditions is satisfied:*
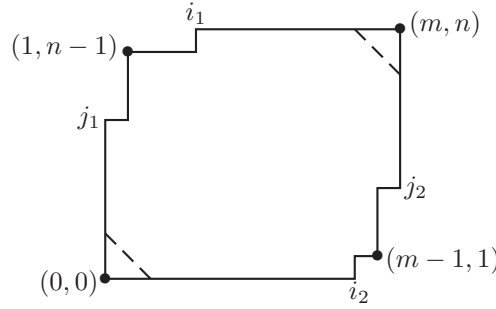
Figure 5:

(a) *Both of the corners* $(0, n)$ *and* $(m, 0)$ *belong to* L. *In this case,* (p, q) *may be any pair in the following set:*

$$\{(0, m + n - 2), (0, m + n - 1), (0, m + n), (1, m + n), (2, m + n)\}$$

(b) *Exactly one of the corners* $(0, n)$ *and* $(m, 0)$ *does not belong to* L. *If the missing corner is* $(0, n)$ *(we can state conditions analogy to this when the other corner is missing), then referring to Figure 5,* (p, q) *may be any pair in the following sets:*

(1) $\{(0, m + n)\}$

(2) $\{(1, m + n), (2, m + n), (0, m + n - 1), (0, m + n - 2)\}$

(3) $\{(1, m+n-1) \ if \ j_1 = n-1 \ or \ i_1 = 1\} \cup \{(1, m+n-1), (1, m+n-2) \ if \ j_1 < n - 1\} \cup \{(1, m + n - 1), (2, m + n - 1) \ if \ i_1 > 1\}$

Proof. Let L be a planar distributive lattice such that I_L is linearly related; then L is one of the following forms:
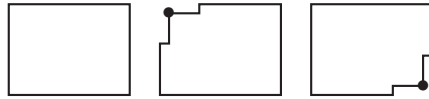

Figure 6:

Let $\mathcal{P}$ be the convex polyomino such that $I_\mathcal{P}$ is the polyomino ideal which corresponds to the defining ideal $I_L^{p,q}$ of the rank bounded Hibi subring $R[p, q; L]$. As Theorem 4 states all possible shapes of linearly related polyominoes, we can derive conditions for p and q by making use of this theorem. We know that $\mathcal{P}$ should contain all vertices $(1, 1)$, $(m - 1, 1)$, $(m - 1, n - 1)$ and $(1, n - 1)$ if $I_\mathcal{P}$ is linearly related.

Let L be a lattice of the form as displayed on the left-hand side of Figure 6. Then we do not have the choice to miss both of the corners $(0, 0)$ and (m, n) at the same time because they are opposite corners. If we miss $(0, 0)$, then $q = m + n$ and we can take $0 < p \le 2$ and if we miss (m, n), then we have $p = 0$ and can take $m + n - 2 \le q < m + n$.

Let L be a lattice of the form as displayed in the middle of Figure 6, Then we have a few choices of the corners of $\mathcal{P}$:

Case (1). We miss none of the corners $(0,0)$ and (m,n). In this case, $p = 0$ and $q = m + n$.

Case (2). We miss exactly one of the corners $(0,0)$ and (m,n). If the missing corner is $(0,0)$, then $q = m + n$ and we have two choices for p, either $p = 1$ or $p = 2$. We can not take $p > 2$ because then $(1,1)$ will no longer be a vertex of $\mathcal{P}$. In a similar way, if we miss only (m,n), then $p = 0$ and we have two choices for q, either $q = m + n - 2$ or $q = m + n - 1$. Again, we can not take $q < m + n - 2$ because then $\mathcal{P}$ will miss the vertex $(m - 1, n - 1)$ as well.

Case (3). We can miss even both of the corners $(0,0)$ and (m,n). But in this case, we must put some extra conditions. The first choice is to fix $p = 1$. Then referring to Figure 5, if $j_1 = n - 1$, then we can take $q = m + n - 1$ only. And if $j_1 < n-1$, then we have again two choices for q, either $q = m+n-1$ or $q = m+n-2$ in order to assure that $j_1 \leq q - m$ and that $(m - 1, n - 1)$ belongs to $V(\mathcal{P})$. The other choice is to fix $q = m + n - 1$. Then in a similar way, if $i_1 = 1$, we can take $p = 1$ only. And if $i_1 > 1$, then we have two choices for p, either $p = 1$ or $p = 2$ in order to assure that $i_1 \geq p$ and that $(1,1)$ is a vertex of $\mathcal{P}$. $\qquad\square$

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References

[1] A. ARAMOVA, J. HERZOG, T. HIBI, *Finite lattices and lexicographic Gröbner bases*, European J. Combin. **21**(2000), 431–439.

[2] M. BIGDELI, J. HERZOG, T. HIBI, A. SHIKAMA, *Isotonian Algebras*, to appear in Nagoya Math. J.

[3] G. BIRKHOFF, *Lattice Theory* (3rd ed.), Amer. Math. Soc. Colloq. Publ. No. 25. Providence, RI, 1967.

[4] D. EISENBUD, *Introduction to algebras with straightening laws*, in: *Ring theory and algebra III: Proceedings of the third Oklahoma Conference*, (B. R. McDonald), Marcel Dekker Inc., New York and Basel, 1980, 243–268.

[5] V. ENE, *Syzygies of Hibi rings*, Acta Math. Vietnam. **40**(2015), 403–446.

[6] V. ENE, J. HERZOG, *Gröbner bases in commutative algebra*, Graduate Studies in Mathematics **130**, AMS, Providence, RI, 2012.

[7] V. ENE, J. HERZOG, T. HIBI, *Linearly related polyominoes*, J. Algebraic Combin. **41**(2015), 949–968.

[8] V. ENE, J. HERZOG, T. HIBI, S. SAEEDI MADANI, *Pseudo-Gorenstein and level Hibi rings*, J. Algebra **431**(2015), 138–161.

[9] V. ENE, J. HERZOG, S. SAEEDI MADANI, *A note on the regularity of Hibi rings*, Manuscripta Math. **148**(2015), 501–506.

[10] V. Ene, J. Herzog, F. Mohammadi, *Monomial ideals and toric rings of Hibi type arising from a finite poset*, European J. Combin. **32**(2011), 404–421.

[11] V. Ene, A. A. Qureshi, A. Rauf, *Regularity of join-meet ideals of distributive lattices*, Electron. J. Combin. **20**(2013), p. 20.

[12] J. Herzog, T. Hibi, *Finite lattices and Gröbner bases*, Math. Nachr. **285**(2012), 1969–1973.

[13] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in: *Commutative Algebra and Combinatorics*, (M. Nagata and H. Matsumura, Eds.), Adv. Stud. Pure Math. **11**, North–Holland, Amsterdam, 1987, 93–109.

[14] M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. **96**(1972), 228–235.

[15] H. Ohsugi, T. Hibi, *Koszul bipartite graphs*, Adv. Appl. Math. **22**(1999), 25–28.

[16] A. Qureshi, *Ideals generated by 2-minors, collections of cells and stack polyominoes*, J. Algebra **357**(2012), 279–303.

[17] A. Qureshi, *Indispensable Hibi relations and Gröbner bases*, to appear in Algebra Colloq.

[18] R. P. Stanley, *Enumerative combinatorics*, Vol. 1, 2nd edition, Cambridge University Press, Cambridge, 1996.

[19] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Amer. Math. Soc., Providence, RI, 1995.