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A remark on the radial solutions of a modified Schrödinger system by the dual approach

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Abstract. By using some reorganized ideas combined with a successive approximation technique we establish conditions for the existence of positive entire radially symmetric solutions for a modified Schrödinger system

$$\begin{cases} \Delta u_1 + \Delta(|u_1|^{2\gamma_1})\,|u_1|^{2\gamma_1-2}\,u_1 = a_1(|x|)\Psi_1\,(u_1)\,F_1(u_2) \text{ in } \mathbb{R}^N, \\ \Delta u_2 + \Delta(|u_2|^{2\gamma_2})\,|u_2|^{2\gamma_2-2}\,u_2 = a_2(|x|)\Psi_2\,(u_2)\,F_2(u_1) \text{ in } \mathbb{R}^N, \end{cases}$$

where $\gamma_1, \gamma_2 \in \left(\frac{1}{2}, \infty\right)$, $N \geq 3$, and the functions a_1, a_2, Ψ_1, Ψ_2, F_1, F_2 are suitably chosen. Our results improve and extend some previous works and have applications in several areas of mathematics and various applied sciences including the study of nonreactive scattering of atoms and molecules.
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1. Introduction

This paper deals with the study of positive ground state solutions of the following coupled nonlinear system of differential equations:

$$\begin{cases} \Delta u_1 + \Delta(|u_1|^{2\gamma_1})\,|u_1|^{2\gamma_1-2}\,u_1 = a_1(|x|)\Psi_1\,(u_1)\,F_1(u_2) \text{ in } \mathbb{R}^N(N \geq 3), \\ \Delta u_2 + \Delta(|u_2|^{2\gamma_2})\,|u_2|^{2\gamma_2-2}\,u_2 = a_2(|x|)\Psi_2\,(u_2)\,F_2(u_1) \text{ in } \mathbb{R}^N(N \geq 3), \end{cases} \tag{1}$$

where $\gamma_1, \gamma_2 \in \left(\frac{1}{2}, \infty\right)$, $\Psi_i\,(u_i) = \sqrt{1 + 2\gamma_i\,|u_i|^{2(2\gamma_i-1)}}$ $(i = 1, 2)$, and the functions a_1, a_2, F_1, F_2 are suitably chosen.

System (1) will be considered together with one of the following conditions:

- both components (u_1, u_2) are bounded, that is,

$$\lim_{|x|\to\infty} u_1\,(|x|) < \infty, \quad \lim_{|x|\to\infty} u_2\,(|x|) < \infty, \tag{2}$$

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- both components (u_1, u_2) are large, that is,

$$\lim_{|x|\to\infty} u_1(|x|) = \infty, \quad \lim_{|x|\to\infty} u_2(|x|) = \infty, \tag{3}$$

- one of the components is bounded, while the other is large, that is,

$$\lim_{|x|\to\infty} u_1(|x|) < \infty, \quad \lim_{|x|\to\infty} u_2(|x|) = \infty, \tag{4}$$

  or

$$\lim_{|x|\to\infty} u_1(|x|) = \infty, \quad \lim_{|x|\to\infty} u_2(|x|) < \infty. \tag{5}$$

Definition 1. *A function $(u_1, u_2) \in C^1([0,\infty)) \times C^1([0,\infty))$ satisfying system (1) is called an entire bounded solution if condition (2) holds; it is called an entire large solution if condition (3) holds; it is called a semifinite entire large solution when (4) or (5) holds.*

As one knows, the issue regarding the existence of solutions to problem (1) is very delicate and interesting to many mathematicians over the last few decades. We refer to [14] for more details on the results. Usually, we still call problem (1) a modified Schrödinger system.

The motivation for working on system (1) goes back to a recent paper [6], where we give an affirmative answer for the following open problem proposed by Zhang, Liu, Wu and Cui [14, p. 1105]:

Open problem. Can we establish existence results on system (1) with functions F_1, F_2 whose growth rate is faster than power functions?

Surprisingly, we could not find any answer to the question of establishing necessary and sufficient conditions on a_1 and a_2 such that system (1) has a nonnegative entire large radial solution, by looking through the literature on this topic including [6]. The purpose of this paper is to give a positive answer to this question.

Problems such as (1) can arise more naturally in mathematical physics such as superfluid film equation, in the theory of Heisenberg ferromagnets and magnons, in dissipative quantum mechanics, in condensed matter theory among other applications – for more details, see e.g. Colin [1, 2], Grosse and Martin [7], Luthey [9] and Smooke [12].

The approach to prove our main results are, up to details, adaptations of the original works of [3, 4, 5, 6] and most of them refer the reader for certain parts of the reasoning to the original paper [4], where systems of the type (1) are studied, but that does not contain the quasilinear and nonconvex diffusion term $\Delta(|u_i|^{2\gamma_i})\,|u_i|^{2\gamma_i-2}\,u_i$, called in the literature non-square diffusion for $\gamma_i \neq 1$ and square diffusion for $\gamma_i = 1$.

Finally, we would like to mention that the method presented here also yields much more precise information on the behavior of solutions that other works.

The paper is structured as follows. In the forthcoming section reformulation of system (1) is given. Next, Section 3 contains the settings and notations of this paper. In Section 4, we formulate our main results regarding system (1). In the main body of this section, we shall devote to proving our theorems.

2. Reformulation of the system by the dual approach

To go into detail, let f_i $(i = 1, 2)$ be a solution to the following ordinary differential equations;

$$\begin{cases} f_i'(t) = \frac{1}{\sqrt{1+2\gamma_i |f_i(t)|^{2(2\gamma_i - 1)}}} \text{ on } t \in [0, \infty), \\ f_i(t) = -f_i(-t) \text{ on } t \in (-\infty, 0), \\ f_i(0) = 0. \end{cases} \tag{6}$$

Let us give some properties of f_i that we need to prove our main results. The readers can also find its proof in works of Santos and Zhou [11] and Vieira [13], however, for the convenience of the reader we will write its proof.

Lemma 1. *The function f_i $(i = 1, 2)$ and its derivative satisfy the following properties:*

(*i*) *f_i is uniquely determined, of class $C^1(\mathbb{R}, \mathbb{R})$ and invertible on all $\mathbb{R}$;*

(*ii*) *$0 < f_i'(t) \leq 1$, $\forall t \in \mathbb{R}$;*

(*iii*) *$\frac{f_i(t)}{2\gamma_i} \leq t f_i'(t) \leq f_i(t)$, $\forall t \in \mathbb{R}_+$;*

(*iv*) *$|f_i(t)| \leq |t|$, $\forall t \in \mathbb{R}$;*

(*v*) *$f_i(st) \leq s f_i(t)$, $\forall t \in \mathbb{R}_+$ and $s \geq 1$;*

(*vi*) *$f_i(t)/t \to 1$, when $t \to 0$;*

(*vii*) *$|f_i(t)| / |t|^{1/2\gamma_i} \to (2\gamma_i)^{\frac{1}{4\gamma_i}}$, when $|t| \to \infty$;*

(*viii*) *setting $\sigma_i = 1$ if $|t| \leq 1$ and $\sigma_i = \frac{1}{2\gamma_i}$ if $|t| \geq 1$, there exists a positive constant $\theta_i > 0$ such that $|f_i(t)| \geq \theta_i |t|^{\sigma_i}$;*

(*ix*) *$|f_i(t)| \leq (\sqrt{2\gamma_i}|t|)^{\frac{1}{2\gamma_i}}$, $\forall t \in \mathbb{R}$.*

Proof. Hereafter, once that f_i is an odd function, we can assume without loss of generality, that $t \geq 0$.

(i) It follows from theorem of the existence and uniqueness for the initial value problem in ordinary differential equations that (6) has a unique solution, of class $C^1(\mathbb{R}, \mathbb{R})$, namely, $f_i(t) \in C^1(\mathbb{R}, \mathbb{R})$. Besides this, $f_i'(t) > 0$ for all $t \in \mathbb{R}$ implies that $f_i(t)$ is invertible.

(ii) From the fact that

$$\frac{1}{\sqrt{1 + 2\gamma_i |f_i(t)|^{2(2\gamma_i - 1)}}} \in (0, 1],$$

there follows the conclusion $f_i'(t) \in (0, 1]$.

(iii) We claim that

$$L_i^1(t) := f_i(t)\sqrt{1 + 2\gamma_i |f_i(t)|^{2(2\gamma_i - 1)}} - t \geq 0, \forall t \in \mathbb{R}_+.$$

Indeed,

$$L_i^1(0) = 0 \text{ and } \left(L_i^1(t)\right)' = \frac{2\gamma_i(2\gamma_i - 1)|f_i(t)|^{2(2\gamma_i-1)}}{1 + 2\gamma_i |f_i(t)|^{2(2\gamma_i-1)}} \geq 0, \forall t \in \mathbb{R}_+.$$

Thus, $L_i(t) \geq L_i(0) = 0, \forall t \in \mathbb{R}_+$, from where it follows that

$$\frac{t}{\sqrt{1 + 2\gamma_i |f_i(t)|^{2(2\gamma_i-1)}}} = tf_i'(t) \leq f_i(t), \forall t \in \mathbb{R}_+.$$

The second inequality is also shown in a similar way.

(iv) This follows by observing that

$$L_i^2(t) := f_i(t) - t \geq 0, \forall t \in \mathbb{R}_+.$$

Indeed,

$$f_i(0) = 0 \text{ and } f_i'(t) \leq 1 \text{ for all } t \geq 0,$$

from where there follows the inequality

$$\left(L_i^2(t)\right)' := f_i'(t) - 1 \geq 0, \forall t \in \mathbb{R}_+.$$

This $L_i^2(t) \geq L_i^2(0) = 0\ \forall t \in \mathbb{R}_+$, together with the fact that $f_i(t)$ is an odd function leads to a conclusion.

(v) We notice that, since $f_i''(t) \leq 0$ in $[0, \infty)$, we have that $f_i'(t)$ is non-increasing in this interval. For any $t \geq 0$ fixed we consider the function

$$L_i^3(s) := f_i(st) - sf_i(t) \text{ defined for } s \geq 1.$$

As a consequence

$$\left(L_i^3(s)\right)' = tf_i'(st) - f_i(t) \leq tf_i'(t) - f_i(t) \leq 0,$$

by iii). Since $L_i^3(1) = 0$, we can consider that $L_i^3(s) \leq 0$ for any $s \geq 1$, that is, $f_i(st) \leq sf_i(t)$, $\forall t \in \mathbb{R}_+$ and $s \geq 1$. Thus the proof is complete.

(vi) To deduce vi), we start observing that

$$f_i'(t) = \frac{1}{\sqrt{1 + 2\gamma_i |f_i(t)|^{2(2\gamma_i-1)}}},$$

can be written in the following way

$$\sqrt{1 + 2\gamma_i |f_i(t)|^{2(2\gamma_i-1)}}\, df_i = dt. \tag{7}$$

Integrating in (7) we obtain

$$\int_0^t ds = \int_0^{f_i(t)} \sqrt{1 + 2\gamma_i |y|^{2(2\gamma_i-1)}}\, dy.$$

By L'Hospital's rule, we obtain this information

$$\lim_{t\to 0+}\frac{f_i(t)}{t}=\lim_{t\to 0+}\frac{f_i(t)}{\int_0^{f_i(t)}\sqrt{1+2\gamma_i|y|^{2(2\gamma_i-1)}}dy}$$
$$=\lim_{t\to 0+}\frac{f_i'(t)}{\sqrt{1+2\gamma_i|f_i(t)|^{2(2\gamma_i-1)}}f_i'(t)}=1.$$

On the other hand, $f_i(t)=-f_i(-t)$ on $t\in(-\infty,0)$ implies that

$$\lim_{t\to 0}\frac{f_i(t)}{t}=1.$$

(vii) Since

$$\left(\frac{f_i(t)}{t^{1/2\gamma_i}}\right)'=\frac{2\gamma_i f_i'(t)-f_i(t)}{2\gamma_i t t^{1/2\gamma_i}},$$

it follows from iii) that the map $t\in(0,\infty)\to\frac{f_i(t)}{t^{1/2\gamma_i}}$, $t>0$ is non-decreasing and then $\lim_{t\to\infty}\frac{f_i(t)}{t^{1/2\gamma_i}}$ exists. Repeating the same type of arguments explored in the proof of vi), we have

$$\lim_{t\to\infty}\frac{f_i(t)}{t^{1/2\gamma_i}}=\lim_{t\to\infty}(\frac{f_i^{2\gamma_i}(t)}{t})^{1/2\gamma_i}=(\lim_{t\to\infty}\frac{f_i^{2\gamma_i}(t)}{t})^{1/2\gamma_i}$$
$$=(\lim_{t\to\infty}2\gamma_i f_i^{2\gamma_i-1}(t)f_i'(t))^{1/2\gamma_i}$$
$$=(\lim_{t\to\infty}\frac{2\gamma_i f_i^{2\gamma_i-1}(t)}{\sqrt{1+2\gamma_i|f_i(t)|^{2(2\gamma_i-1)}}})^{1/2\gamma_i}$$
$$=(\lim_{t\to\infty}\frac{2\gamma_i f_i^{2\gamma_i-1}(t)}{|f_i(t)|^{2\gamma_i-1}\sqrt{\frac{1}{|f_i(t)|^{2(2\gamma_i-1)}}+2\gamma_i}})^{1/2\gamma_i}$$
$$=(\lim_{t\to\infty}\frac{2\gamma_i}{\sqrt{\frac{1}{|f_i(t)|^{2(2\gamma_i-1)}}+2\gamma_i}})^{1/2\gamma_i}=(\frac{2\gamma_i}{\sqrt{2\gamma_i}})^{1/2\gamma_i}$$
$$=[(2\gamma_i)^{\frac{1}{2}}]^{1/2\gamma_i}=(2\gamma_i)^{\frac{1}{4\gamma_i}},$$

by L'Hospital's rule.

(viii) This is a consequence of limits vi) and vii).

(ix) Integrating

$$f_i'(t)\sqrt{1+2\gamma_i f_i^{2(2\gamma_i-1)}(t)}=1$$

we obtain

$$\int_0^t f_i'(s)\sqrt{1+2\gamma_i f_i^{2(2\gamma_i-1)}(s)}ds=t.$$

After changing variables $y = f_i(s)$, we find

$$t = \int_0^{f_i(t)} \sqrt{1 + 2\gamma_i y^{2(2\gamma_i - 1)}} dy \geq (2\gamma_i)^{\frac{1}{2}} \int_0^{f_i(t)} y^{2\gamma_i - 1} dy$$
$$= \frac{(2\gamma_i)^{\frac{1}{2}}}{2\gamma_i} f_i^{2\gamma_i}(t) = \frac{1}{\sqrt{2\gamma_i}} f_i^{2\gamma_i}(t),$$

which proves ix), and the proof of Lemma 1 is complete. □

Next, we consider the following semilinear elliptic problem:

$$\begin{cases} \Delta u(|x|) = a_1(|x|) F_1(f_2(v(|x|))) \text{ in } \mathbb{R}^N, \\ \Delta v(|x|) = a_2(|x|) F_2(f_1(u(|x|))) \text{ in } \mathbb{R}^N, \end{cases} \tag{8}$$

we call it the dual problem of (1). From the monotonicity of f_i, we know that f_i has an inverse f_i^{-1}. Moreover, the inverse function of f_i is given by

$$f_i^{-1}(t) = \int_0^t \sqrt{1 + 2\gamma_i y^{2(2\gamma_i - 1)}} dy \text{ for all } t \geq 0.$$

As to asymptotic behaviours on the unique solution f_i $(i = 1, 2)$ of (6), we have the following:

Lemma 2. *Fixing the change of variables*

$$w_1 = f_1^{-1}(u_1) \text{ and } w_2 = f_2^{-1}(u_2),$$

(u_1, u_2) *is a positive solution of (1) if and only if* (w_1, w_2) *is a positive solution of (8).*

Proof. From $w_i = f_i^{-1}(u_i)$ $(i = 1, 2)$ we have

$$\nabla w_i = \left(f_i^{-1}\right)'(u_i) \nabla u_i$$

and

$$\Delta w_i = (f_i^{-1})''(u_i) |\nabla u_i|^2 + (f_i^{-1})'(u_i) \Delta u_i.$$

Moreover, direct calculations yield

$$(f_i^{-1})'(t) = \frac{1}{f_i'\left(f_i^{-1}(t)\right)} = \sqrt{1 + 2\gamma_i \left|f_i\left(f_i^{-1}(t)\right)\right|^{2(2\gamma_i - 1)}} = \sqrt{1 + 2\gamma_i |t|^{2(2\gamma_i - 1)}}$$

and

$$(f_i^{-1})''(t) = \frac{\gamma_i(4\gamma_i - 2)|t|^{2(2\gamma_i - 1) - 2} t}{\sqrt{1 + 2\gamma_i |t|^{2(2\gamma_i - 1)}}}.$$

Using these information, we have

$$\Delta w_i = \frac{\gamma_i(4\gamma_i - 2)|u_i|^{2(2\gamma_i - 2)} u_i}{\sqrt{1 + 2\gamma_i |u_i|^{2(2\gamma_i - 1)}}} |\nabla u_i|^2 + \sqrt{1 + 2\gamma_i |u_i|^{2(2\gamma_i - 1)}} \Delta u_i$$
$$= \frac{1}{\sqrt{1 + 2\gamma_i |u_i|^{2(2\gamma_i - 1)}}} [\gamma_i(4\gamma_i - 2)|u_i|^{2(2\gamma_i - 2)} u_i |\nabla u_i|^2 + (1 + 2\gamma_i |u_i|^{2(2\gamma_i - 1)}) \Delta u_i].$$

On the other hand, it follows that

$$\begin{aligned}\Delta(|u_i|^{2\gamma_i}) &= \mathrm{div}\, 2\gamma_i\, |u_i|^{2\gamma_i-2}\, u_i \nabla u_i) \\ &= 2\gamma_i\, |u_i|^{2\gamma_i-2}\, u_i \Delta u_i + \nabla u_i \cdot \nabla (2\gamma_i\, |u_i|^{2\gamma_i-2}\, u_i) \\ &= 2\gamma_i\, |u_i|^{2\gamma_i-2}\, u_i \Delta u_i + \nabla u_i \cdot (2\gamma_i (2\gamma_i - 1)\, |u_i|^{2\gamma_i-2}\, \nabla u_i) \\ &= 2\gamma_i\, |u_i|^{2\gamma_i-2}\, u_i \Delta u_i + 2\gamma_i (2\gamma_i - 1)\, |u_i|^{2\gamma_i-2}\, |\nabla u_i|^2 ,\end{aligned}$$

and as a consequence,

$$\begin{aligned}\Delta(|u_i|^{2\gamma_i})\, |u_i|^{2\gamma_i-2}\, u_i &= 2\gamma_i\, |u_i|^{4\gamma_i-4}\, u_i^2 \Delta u_i + 2\gamma_i (2\gamma_i - 1)\, |u_i|^{4\gamma_i-4}\, u_i\, |\nabla u_i|^2 \\ &= 2\gamma_i\, |u_i|^{2(2\gamma_i-1)}\, \Delta u_i + 2\gamma_i (2\gamma_i - 1)\, |u_i|^{2(2\gamma_i-2)}\, u_i\, |\nabla u_i|^2 .\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\Delta w_1 &= \frac{1}{\sqrt{1 + 2\gamma_1\, |u_1|^{2(2\gamma_1-1)}}} [\Delta u_1 + \Delta \left(|u_1|^{2\gamma_1}\right) |u_1|^{2\gamma_1-2}\, u_1] \\ &= a_1\,(|x|)\, F_1\,(f_2\,(w_2)) ,\end{aligned}$$

and

$$\begin{aligned}\Delta w_2 &= \frac{1}{\sqrt{1 + 2\gamma_2\, |u_2|^{2(2\gamma_2-1)}}} [\Delta u_2 + \Delta \left(|u_2|^{2\gamma_2}\right) |u_2|^{2\gamma_2-2}\, u_2] \\ &= a_2\,(|x|)\, F_2\,(f_1\,(w_1)) ,\end{aligned}$$

which shows that (w_1, w_2) is a solution to (8). Notice that the above process is invertible, so the quasilinear Schrödinger elliptic system (1) and the semilinear elliptic system (8) are equivalent, concluding the proof of the lemma. □

3. Mathematical statement and notations

In order to establish the existence of solutions to system (8) it is helpful to keep in mind the following mathematical settings of the paper:

$$\begin{aligned}&\alpha, \beta \in (0, \infty)\,, M_1 \geq \max\{1, \frac{\beta}{F_2\,(\alpha)}\},\ M_2 \geq \max\{1, \frac{\alpha}{F_1\,(\beta)}\}, \\ &\sigma_1 = 1 \text{ if } \alpha \in (0, 1] \text{ and } \sigma_1 = \frac{1}{2\gamma_1} \text{ if } \alpha \in (1, \infty), \\ &\sigma_2 = 1 \text{ if } \beta \in (0, 1] \text{ and } \sigma_2 = \frac{1}{2\gamma_2} \text{ if } \beta \in (1, \infty), \\ &\theta_1 \leq \frac{f_1\,(\alpha)}{\alpha^{\sigma_1}} \text{ and } \theta_2 \leq \frac{f_2\,(\beta)}{\beta^{\sigma_2}}, \\ &m_1 \in (0, \min\{\beta, F_2\,(\theta_1 \alpha^{\sigma_1})\}) \text{ and } m_2 \in (0, \min\{\alpha, F_1\,(\theta_2 \beta^{\sigma_2})\}) .\end{aligned}$$

In the statement of the theorems, it will be convenient to use the following notations:

$$
\begin{aligned}
r &:= |x| \text{ is the Euclidean norm in } \mathbb{R}^N,\\
\mathcal{A}_{a_i}(r) &:= \int_0^r s^{1-N}\int_0^s z^{N-1}a_i(z)dzds,\ i=1,2,\\
\mathcal{A}_{a_i}(\infty) &:= \lim_{r\to\infty}\mathcal{A}_{a_i}(r),\ C_i^+ = \mathcal{A}_{a_i}(\infty),\ (i=1,2),\\
\overline{A}_{1,2}(r) &:= \overline{C}_1\int_0^r y^{1-N}\int_0^y t^{N-1}a_1(t)F_1((1+\mathcal{A}_{a_2}(t))^{\frac{1}{2\gamma_2}})dtdy,\\
\overline{A}_{2,1}(r) &:= \overline{C}_2\int_0^r y^{1-N}\int_0^y t^{N-1}a_2(t)F_2((1+\mathcal{A}_{a_1}(t))^{\frac{1}{2\gamma_1}})dtdy,\\
\underline{A}_{1,2}(r) &:= \underline{C}_1\int_0^r y^{1-N}\int_0^y t^{N-1}a_1(t)F_1\left((1+\mathcal{A}_{a_2}(t))^{\sigma_2}\right)dtdy,\\
\underline{A}_{2,1}(r) &:= \underline{C}_2\int_0^r y^{1-N}\int_0^y t^{N-1}a_2(t)F_2\left((1+\mathcal{A}_{a_1}(t))^{\sigma_1}\right)dtdy,\\
\overline{A}_{1,2}(\infty) &:= \lim_{r\to\infty}\overline{A}_{1,2}(r),\ \ \overline{A}_{2,1}(\infty) := \lim_{r\to\infty}\overline{A}_{2,1}(r),\ \overline{C}_1,\overline{C}_2\in(0,\infty),\\
\underline{A}_{1,2}(\infty) &:= \lim_{r\to\infty}\underline{A}_{1,2}(r),\ \underline{A}_{2,1}(\infty) := \lim_{r\to\infty}\underline{A}_{2,1}(r),\ \underline{C}_1,\underline{C}_2\in(0,\infty).
\end{aligned}
$$

Next, we define the following functions and limits:

$$
\begin{aligned}
F_{1,2}(r) &:= \int_\alpha^r \frac{1}{F_1((M_1\sqrt{2\gamma_2})^{\frac{1}{2\gamma_2}}F_2^{\frac{1}{2\gamma_2}}(t))}dt,\ F_{1,2}(\infty) := \lim_{s\to\infty}F_{1,2}(s),\\
F_{2,1}(r) &:= \int_\beta^r \frac{1}{F_2((M_2\sqrt{2\gamma_1})^{\frac{1}{2\gamma_1}}F_1^{\frac{1}{2\gamma_1}}(t))}dt,\ F_{2,1}(\infty) := \lim_{s\to\infty}F_{2,1}(s),\\
\widetilde{F}_{1,2}(r) &:= \int_\alpha^r \frac{1}{F_1([\sqrt{2\gamma_2}\left(\beta+F_2(t)C_2^+\right)]^{\frac{1}{2\gamma_2}})}dt,\ \widetilde{F}_{1,2}(\infty) := \lim_{s\to\infty}\widetilde{F}_{1,2}(s),\\
\widetilde{F}_{2,1}(r) &:= \int_\beta^r \frac{1}{F_2([\sqrt{2\gamma_1}\left(\alpha+F_1(t)C_1^+\right)]^{\frac{1}{2\gamma_1}})}dt,\ \widetilde{F}_{2,1}(\infty) := \lim_{s\to\infty}\widetilde{F}_{2,1}(s).
\end{aligned}
$$

Note that

$$F'_{1,2}(r) = \frac{1}{F_1((M_1\sqrt{2\gamma_2})^{\frac{1}{2\gamma_2}}F_2^{\frac{1}{2\gamma_2}}(r))} > 0 \text{ for } r>\alpha,$$

and

$$F'_{2,1}(r) = \frac{1}{F_2((M_2\sqrt{2\gamma_1})^{\frac{1}{2\gamma_1}}F_1^{\frac{1}{2\gamma_1}}(r))} > 0 \text{ for } r>\beta.$$

Then $F_{1,2}$ has the inverse function $F_{1,2}^{-1}$ on $[0, F_{1,2}(\infty))$ and $F_{2,1}$ has the inverse function $F_{2,1}^{-1}$ on $[0, F_{2,1}(\infty))$. The same can be done for $\widetilde{F}_{1,2}$ and $\widetilde{F}_{2,1}$.

To begin, we first state some assumptions on a_1, a_2, F_1 and F_2 that will be used throughout the remainder of this paper:

(A) a_1, $a_2 : [0,\infty) \to [0,\infty)$ are continuous functions;

(C1) F_1, $F_2 : [0,\infty) \to [0,\infty)$ are continuous, non-decreasing and $F_1(s) \cdot F_2(s) > 0$ whenever $s > 0$;

(C2) there are parameters $\overline{C}_1, \overline{C}_2 \in (0,\infty)$ such that

$$F_1(t_1 w_1) \le \overline{C}_1 F_1(t_1) F_1(w_1), \forall t_1 \ge (M_1\sqrt{2\gamma_2})^{\frac{1}{2\gamma_2}} F_2^{\frac{1}{2\gamma_2}}(\alpha), \forall w_1 \ge 1, \quad (9)$$

$$F_2(t_2 w_2) \le \overline{C}_2 F_2(t_2) F_2(w_2), \forall t_2 \ge (M_2\sqrt{2\gamma_1})^{\frac{1}{2\gamma_1}} F_1^{\frac{1}{2\gamma_1}}(\beta), \forall w_2 \ge 1, \quad (10)$$

(C3) there are parameters $\underline{C}_1, \underline{C}_2 \in (0,\infty)$ such that

$$F_1(\theta_2 m_1^{\sigma_2} w_1) \ge \underline{C}_1 F_1(w_1), \forall\, w_1 \ge 1, \quad (11)$$

$$F_2(\theta_1 m_2^{\sigma_1} w_2) \ge \underline{C}_2 F_2(w_2), \forall\, w_2 \ge 1. \quad (12)$$

It is easily seen that the following functions satisfy hypotheses (C1)-(C3):

$$F_1(u_2) = u_2^{\vartheta_1} \text{ and } F_2(u_1) = u_1^{\vartheta_2} \text{ with } \vartheta_1,\, \vartheta_2 \in (0,\infty).$$

It is important to note that:

Remark 1. *Assumptions (C2) and (C3) are further discussed in Krasnosel'skiĭ and Rutickiĭ [8] (see also Rao and Ren [10]).*

4. The main results

Our first substantial result is now the following.

Theorem 1. *Assume that (A) and $F_{1,2}(\infty) = F_{2,1}(\infty) = \infty$ hold. Furthermore, suppose that F_1 and F_2 satisfy the hypotheses (C1) and (C2). Then system (8) has one positive radial solution*

$$(u,v) \in C^1([0,\infty)) \times C^1([0,\infty)) \text{ with } u(0) = \alpha \text{ and } v(0) = \beta.$$

If in addition, F_1 and F_2 satisfy the hypothesis (C3), $\underline{A}_{1,2}(\infty) = \infty$ and $\underline{A}_{2,1}(\infty) = \infty$, then $\lim_{r\to\infty} u(r) = \infty$ and $\lim_{r\to\infty} v(r) = \infty$. Conversely, if (C1), (C2), (C3) hold true, and (u,v) is a nonnegative entire large solution to (8) such that $u(0) = \alpha$ and $v(0) = \beta$, then a_1 and a_2 satisfy $\underline{A}_{1,2}(\infty) = \overline{A}_{1,2}(\infty) = \infty$ and $\underline{A}_{2,1}(\infty) = \overline{A}_{2,1}(\infty) = \infty$.

Our Theorem 1 significantly improves the well-known results about the large solutions to (8) and therefore gives an answer to our main goal.

Proof. It is the purpose of the present theorem to demonstrate that radially symmetric settings (8) always admit a radially symmetric solution. On the other hand, if we look for a radially symmetric solution, system (8) becomes ordinary differential equations system

$$\begin{cases} \left(r^{N-1}u'(r)\right)' = r^{N-1}a_1(r)F_1(f_2(v(r))) \text{ on } [0,\infty), \\ \left(r^{N-1}v'(r)\right)' = r^{N-1}a_2(r)F_2(f_1(u(r))) \text{ on } [0,\infty), \end{cases} \quad (13)$$

subject to the initial conditions $u(0)=\alpha$, $v(0)=\beta$, $u'(0)=0$ and $v'(0)=0$. The approach that we will use for solving this problem (13) is the following. Integrating (13) from 0 to r, we obtain

$$\begin{cases} u'(r)=\frac{1}{r^{N-1}}\int_0^r s^{N-1}a_1(s)F_1(f_2(v(s)))\,ds, \text{ on } [0,\infty), \\ v'(r)=\frac{1}{r^{N-1}}\int_0^r s^{N-1}a_2(s)F_2(f_1(u(s)))\,ds, \text{ on } [0,\infty). \end{cases} \tag{14}$$

From integral equations (14), it is easy to see that $u(r)$ is an increasing function on $[0,\infty)$ of the radial variable r, and the same conclusion holds for $v(r)$. Thus, for radial solutions of system (13) we seek for solutions of the system of integral equations

$$\begin{cases} u(r)=\alpha+\int_0^r t^{1-N}\int_0^t s^{N-1}a_1(s)F_1(f_2(v(s)))\,dsdt, & r\geq 0, \\ v(r)=\beta+\int_0^r t^{1-N}\int_0^t s^{N-1}a_2(s)F_2(f_1(u(s)))\,dsdt, & r\geq 0. \end{cases} \tag{15}$$

To solve (15), we inductively define a sequence of collections of functions $\{u_m\}_{m\geq 0}$ and $\{v_m\}_{m\geq 0}$ on $[0,\infty)$ as follows:

$$\begin{cases} u_0(r)=\alpha,\ v_0(r)=\beta, \\ u_m(r)=\alpha+\int_0^r t^{1-N}\int_0^t s^{N-1}a_1(s)F_1(f_2(v_{m-1}(s)))\,dsdt, & r\geq 0, \\ v_m(r)=\beta+\int_0^r t^{1-N}\int_0^t s^{N-1}a_2(s)F_2(f_1(u_{m-1}(s)))\,dsdt, & r\geq 0. \end{cases}$$

In the first stage, for all $r\geq 0$ and $m\in\mathbb{N}$ it holds that

$$u_m(r)\geq\alpha, v_m(r)\geq\beta \text{ and } v_0\leq v_1.$$

In the second stage, our assumptions yield

$$u_1(r)\leq u_2(r), \text{ for all } r\geq 0,$$

so

$$v_1(r)\leq v_2(r), \text{ for all } r\geq 0.$$

Then a simple induction completes the proof of

$$\{u_m\}_m \text{ and } \{v_m\}_m \text{ are increasing on } [0,\infty).$$

Next, we shall show that the monotonically increasing sequences

$$\{u_m\}_m \text{ and } \{v_m\}_m$$

are bounded above whenever r is bounded, and hence converge on $\mathbb{R}^N$.

Thanks to item iv) of Lemma 1, for any $m \in \mathbb{N}$, one has

$$
\begin{aligned}
v_m(r) &= \beta + \int_0^r t^{1-N} \int_0^t s^{N-1} a_2(s) F_2\left(f_1\left(u_{m-1}(s)\right)\right) dsdt \\
&\leq \beta + \int_0^r F_2\left(f_1\left(u_m(t)\right)\right) t^{1-N} \int_0^t z^{N-1} a_2(z) dzdt \\
&\leq \beta + \int_0^r F_2\left(u_m(t)\right) t^{1-N} \int_0^t z^{N-1} a_2(z) dzdt \\
&\leq \beta + F_2\left(u_m(r)\right) \int_0^r t^{1-N} \int_0^t z^{N-1} a_2(z) dzdt \qquad (16) \\
&\leq F_2\left(u_m(r)\right) \left(\frac{\beta}{F_2\left(u_m(r)\right)} + \mathcal{A}_{a_2}(r)\right) \\
&\leq F_2\left(u_m(r)\right) \left(\frac{\beta}{F_2(\alpha)} + \mathcal{A}_{a_2}(r)\right) \\
&\leq M_1 F_2\left(u_m(r)\right) \left(1 + \mathcal{A}_{a_2}(r)\right),
\end{aligned}
$$

where $M_1 \geq \max\{1, \frac{\beta}{F_2(\alpha)}\}$.

One shows, like in the proof of (16), that

$$
\begin{aligned}
u_m(r) &= \alpha + \int_0^r t^{1-N} \int_0^t s^{N-1} a_1(s) F_1\left(f_2\left(v_{m-1}(s)\right)\right) dsdt \\
&\leq \alpha + \int_0^r t^{1-N} \int_0^t s^{N-1} a_1(s) F_1\left(v_m(s)\right) dsdt \qquad (17) \\
&\leq M_2 F_1\left(v_m(r)\right) \left(1 + \mathcal{A}_{a_1}(r)\right),
\end{aligned}
$$

where $M_2 \geq \max\{1, \frac{\alpha}{F_1(\beta)}\}$.

Moreover, using (9), (16) and item ix) of Lemma 1, by an elementary computation it follows that

$$
\begin{aligned}
u'_m(r) &\leq r^{1-N} \int_0^r s^{N-1} a_1(s) F_1\left(f_2\left(v_m(s)\right)\right) ds \\
&\leq r^{1-N} \int_0^r s^{N-1} a_1(s) F_1\left(\left(\sqrt{2\gamma_2} v_m(s)\right)^{\frac{1}{2\gamma_2}}\right) ds \\
&\leq r^{1-N} \int_0^r s^{N-1} a_1(s) F_1\left(\left(\sqrt{2\gamma_2} M_1 F_2\left(u_m(r)\right)\left(1 + \mathcal{A}_{a_2}(r)\right)\right)^{\frac{1}{2\gamma_2}}\right) ds \qquad (18) \\
&\leq \overline{C}_1 F_1\left(\left(M_1\sqrt{2\gamma_2}\right)^{\frac{1}{2\gamma_2}} F_2^{\frac{1}{2\gamma_2}}\left(u_m(r)\right)\right) r^{1-N} \int_0^r s^{N-1} a_1(s) F_1\left(\left(1 + \mathcal{A}_{a_2}(s)\right)^{\frac{1}{2\gamma_2}}\right) ds \\
&\leq F_1\left(\left(M_1\sqrt{2\gamma_2}\right)^{\frac{1}{2\gamma_2}} F_2^{\frac{1}{2\gamma_2}}\left(u_m(r)\right)\right) \overline{A}'_{1,2}(r).
\end{aligned}
$$

Proceeding as above, but now with the second inequality (17), one can show that

$$
\begin{aligned}
v'_m(r) &= r^{1-N} \int_0^r s^{N-1} a_2(s) F_2\left(f_1\left(u_{m-1}(s)\right)\right) ds \qquad (19) \\
&\leq F_2\left(\left(M_2\sqrt{2\gamma_1}\right)^{\frac{1}{2\gamma_1}} F_1^{\frac{1}{2\gamma_1}}\left(v_m(r)\right)\right) \overline{A}'_{2,1}(r).
\end{aligned}
$$

Consequently, dividing inequality (18) by $F_1((M_1\sqrt{2\gamma_2})^{\frac{1}{2\gamma_2}}(F_2^{\frac{1}{2\gamma_2}}(u_m(r)))$, we deduce that

$$\frac{(u_m(r))'}{F_1((M_1\sqrt{2\gamma_2})^{\frac{1}{2\gamma_2}}(F_2^{\frac{1}{2\gamma_2}}(u_m(r)))} \leq \overline{A}'_{1,2}(r). \tag{20}$$

One shows similarly that

$$\frac{(v_m(r))'}{F_2((M_2\sqrt{2\gamma_1})^{\frac{1}{2\gamma_1}}F_1^{\frac{1}{2\gamma_1}}(v_m(r)))} \leq \overline{A}'_{2,1}(r). \tag{21}$$

Integrating inequalities (20) and (21) from 0 to r yields that

$$\int_\alpha^{u_m(r)} \frac{1}{F_1((M_1\sqrt{2\gamma_2})^{\frac{1}{2\gamma_2}}F_2^{\frac{1}{2\gamma_2}}(t))}dt \leq \overline{A}_{1,2}(r), \tag{22}$$

$$\int_\beta^{v_m(r)} \frac{1}{F_2((M_2\sqrt{2\gamma_1})^{\frac{1}{2\gamma_1}}F_1^{\frac{1}{2\gamma_1}}(t))}dt \leq \overline{A}_{2,1}(r). \tag{23}$$

Also, going back to the setting of $F_{1,2}$ and $F_{2,1}$ we rewrite (22) and (23) as

$$F_{1,2}(u_m(r)) \leq \overline{A}_{1,2}(r) \text{ and } F_{2,1}(v_m(r)) \leq \overline{A}_{2,1}(r). \tag{24}$$

Since $F_{1,2}$ (resp. $F_{2,1}$) is a bijection with the inverse function $F_{1,2}^{-1}$ (resp. $F_{2,1}^{-1}$) strictly increasing on $[0,\infty)$, inequalities (24) can be reformulated as

$$u_m(r) \leq F_{1,2}^{-1}\left(\overline{A}_{1,2}(r)\right) \text{ and } v_m(r) \leq F_{2,1}^{-1}\left(\overline{A}_{2,1}(r)\right). \tag{25}$$

Inequalities in (25) imply that

$$\{u_m(r)\}_{m\geq 0} \text{ and } \{v_m(r)\}_{m\geq 0}$$

are bounded whenever r is bounded. Indeed, we prove that the sequences

$$\{u_m(r)\}_{m\geq 0} \text{ and } \{v_m(r)\}_{m\geq 0},$$

are bounded on $[0, c_0]$ for arbitrary $c_0 > 0$. To do this, we take

$$K_1 = F_{1,2}^{-1}\left(\overline{A}_{1,2}(c_0)\right) \text{ and } K_2 = F_{2,1}^{-1}\left(\overline{A}_{2,1}(c_0)\right),$$

and since

$$(u_m(r))' \geq 0 \text{ and } (v_m(r))' \geq 0,$$

it follows that

$$u_m(r) \leq u_m(c_0) \leq K_1 \text{ and } v_m(r) \leq v_m(c_0) \leq K_2, \tag{26}$$

from which, since $c_0 > 0$ was arbitrary, we conclude that $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are bounded from above on bounded sets. That is the conclusion we are looking for. Consequently, because of (26) and taking into account the monotonicity of the sequences, we can define

$$u\left(r\right)=\lim_{m\to\infty}u_{m}\left(r\right)\text{ and }v\left(r\right)=\lim_{m\to\infty}v_{m}\left(r\right)$$

as a radially symmetric solution to problem (13). Moreover, Dini's theorem shows that the sequence $\{(u_m(r),v_m(r))\}_{m\geq 0}$ converges uniformly to (u,v) on any compact subinterval of $[0,\infty)$.

It is clear that $(u\left(r\right),v\left(r\right))$ is continuous. By standard elliptic regularity theory it can be shown that $(u,v)\in C^1\left[0,\infty\right)\times C^1\left[0,\infty\right)$. Going back to system (8), the radial solutions of (13) are solutions of the ordinary differential equations system (8).

We conclude that radial solutions of (8) with $u\left(0\right)=\alpha$, $v\left(0\right)=\beta$ satisfy

$$u(r)=\alpha+\int_0^r t^{1-N}\int_0^t s^{N-1}a_1(s)F_1\left(f_2\left(v\left(s\right)\right)\right)dsdt, r\geq 0, \tag{27}$$

$$v(r)=\beta+\int_0^r t^{1-N}\int_0^t s^{N-1}a_2(s)F_2\left(f_1\left(u\left(s\right)\right)\right)dsdt,\ r\geq 0. \tag{28}$$

From (27)-(28) and item viii) of Lemma 1 we obtain

$$\begin{aligned} v(r)&=\beta+\int_0^r t^{1-N}\int_0^t s^{N-1}a_2(s)F_2\left(f_1\left(u\left(s\right)\right)\right)dsdt\\ &\geq\beta+\int_0^r z^{1-N}\int_0^z s^{N-1}a_2(s)F_2\left(f_1\left(\alpha\right)\right)dsdz\\ &\geq\beta+F_2\left(\theta_1\alpha^{\sigma_1}\right)\mathcal{A}_{a_2}\left(r\right)\\ &\geq m_1(1+\mathcal{A}_{a_2}\left(r\right)). \end{aligned}$$

With the previous argument, one also shows that

$$\begin{aligned} u(r)&=\alpha+\int_0^r t^{1-N}\int_0^t s^{N-1}a_1(s)F_1\left(f_2\left(v\left(s\right)\right)\right)dsdt\\ &\geq m_2(1+\mathcal{A}_{a_1}\left(r\right)). \end{aligned}$$

Using item viii) of Lemma 1, we observe that

$$\begin{aligned} u\left(r\right)&=\alpha+\int_0^r t^{1-N}\int_0^t s^{N-1}a_1(s)F_1\left(f_2\left(v\left(s\right)\right)\right)dsdt\\ &\geq\alpha+\int_0^r y^{1-N}\int_0^y t^{N-1}a_1(t)F_1\left(f_2\left(m_1(1+\mathcal{A}_{a_2}\left(r\right))\right)\right)dtdy\\ &\geq\alpha+\int_0^r y^{1-N}\int_0^y t^{N-1}a_1(t)F_1\left(\theta_2 m_1^{\sigma_2}(1+\mathcal{A}_{a_2}\left(r\right))^{\sigma_2}\right)dtdy\\ &\geq\alpha+\int_0^r y^{1-N}\int_0^y t^{N-1}a_1(t)\underline{C}_1F_1\left((1+\mathcal{A}_{a_2}\left(t\right))^{\sigma_2}\right)dtdy\\ &=\alpha+\underline{A}_{1,2}\left(r\right). \end{aligned} \tag{29}$$

By the same proof, we can also obtain

$$\begin{aligned} v(r) &= \beta + \int_0^r t^{1-N} \int_0^t s^{N-1} a_2(s) F_2 \left(f_1 \left(u\left(s\right)\right)\right) ds dt \\ &\geq \beta + \underline{A}_{2,1}\left(r\right). \end{aligned}$$

Hence, we can pass to the limit as $r \to \infty$ in (29) and in the above inequality to conclude that

$$\lim_{r\to\infty} u\left(r\right) = \lim_{r\to\infty} v\left(r\right) = \infty,$$

which yields the result for $\underline{A}_{1,2}\left(\infty\right) = \underline{A}_{2,1}\left(\infty\right) = \infty$. In order to prove the converse, let (u, v) be an entire large radial solution to (8) such that $(u, v) = (\alpha, \beta)$. Then, u and v satisfy

$$\begin{aligned} u(r) &= \alpha + \int_0^r t^{1-N} \int_0^t s^{N-1} a_1(s) F_1 \left(f_2\left(v\left(s\right)\right)\right) ds dt, \ r \geq 0, \\ v(r) &= \beta + \int_0^r t^{1-N} \int_0^t s^{N-1} a_2(s) F_2 \left(f_1\left(u\left(s\right)\right)\right) ds dt, \ r \geq 0. \end{aligned}$$

Before ending the proof, let us point that

$$F_{1,2}\left(u\left(r\right)\right) \leq \overline{A}_{1,2}\left(r\right) \text{ and } F_{2,1}\left(v\left(r\right)\right) \leq \overline{A}_{2,1}\left(r\right). \tag{30}$$

By passing to the limit as $r \to \infty$ in (30) we find that a_1 and a_2 satisfy $\overline{A}_{1,2}\left(\infty\right) = \overline{A}_{2,1}\left(\infty\right) = \infty$, since (u, v) is large and $F_{1,2}\left(\infty\right) = F_{2,1}\left(\infty\right) = \infty$. Hence, the theorem holds true. □

Our second substantial result is related to the entire bounded radial solutions to system (1).

Theorem 2. *Suppose that (A), (C1) and* $F_{1,2}\left(\infty\right) = F_{2,1}\left(\infty\right) = \infty$, $C_1^+, C_2^+ \in (0, \infty)$ *hold. Then, the system (8) has one positive radial solution*

$$(u, v) \in C^1\left([0, \infty)\right) \times C^1\left([0, \infty)\right) \ \textit{with } u\left(0\right) = \alpha \textit{ and } v\left(0\right) = \beta,$$

such that $\lim_{r\to\infty} u\left(r\right) < \infty$ *and* $\lim_{r\to\infty} v\left(r\right) < \infty$.

Proof. Following the approach in Theorem 1 we write (8) in the following form:

$$\begin{cases} u(r) = \alpha + \int_0^r t^{1-N} \int_0^t s^{N-1} a_1(s) F_1\left(f_2\left(v\left(s\right)\right)\right) ds dt, & r \geq 0, \\ v(r) = \beta + \int_0^r t^{1-N} \int_0^t s^{N-1} a_2(s) F_2\left(f_1\left(u\left(s\right)\right)\right) ds dt, & r \geq 0, \end{cases}$$

with boundary conditions $u(0) = \alpha$, $v(0) = \beta$, $u'(0) = 0$ and $v'(0) = 0$.

At this point, we are going to start the iteration procedure by considering the sequence $\{u_m\}_{m\geq 0}$ and $\{v_m\}_{m\geq 0}$ on $[0, \infty)$ as follows:

$$\begin{cases} u_0(r) = \alpha, \ v_0(r) = \beta, \\ u_m(r) = \alpha + \int_0^r t^{1-N} \int_0^t s^{N-1} a_1(s) F_1\left(f_2\left(v_{m-1}\left(s\right)\right)\right) ds dt, & r \geq 0, \\ v(r) = \beta + \int_0^r t^{1-N} \int_0^t s^{N-1} a_2(s) F_2\left(f_1\left(u_m\left(s\right)\right)\right) ds dt, & r \geq 0. \end{cases}$$

Obviously, for all $r \geq 0$ and $m \in \mathbb{N}$ it holds that $\{u_m\}_{m\geq 0}$ and $\{v_m\}_{m\geq 0}$ are increasing on $[0, \infty)$.

Next, we establish bounds for the increasing sequences $\{u_m\}_m$ and $\{v_m\}_m$. We can now use the same strategy of the previous theorem to obtain the following inequalities:

$$\begin{aligned} v_m(r) &= \beta + \int_0^r t^{1-N} \int_0^t s^{N-1} a_2(s) F_2 \left(f_1 \left(u_{m-1}(s)\right)\right) ds dt \\ &\leq \beta + \int_0^r F_2 \left(f_1 \left(u_m(t)\right)\right) t^{1-N} \int_0^t z^{N-1} a_2(z) dz dt \\ &\leq \beta + \int_0^r F_2 \left(u_m(t)\right) t^{1-N} \int_0^t z^{N-1} a_2(z) dz dt \\ &\leq \beta + F_2 \left(u_m(r)\right) C_2^+ \end{aligned}$$

and

$$\begin{aligned} u_m(r) &= \alpha + \int_0^r t^{1-N} \int_0^t s^{N-1} a_1(s) F_1 \left(f_2 \left(v_{m-1}(s)\right)\right) ds dt \\ &\leq \alpha + \int_0^r t^{1-N} \int_0^t s^{N-1} a_1(s) F_1 \left(v_m(s)\right) ds dt \qquad (31) \\ &\leq \alpha + F_1 \left(v_m(r)\right) C_1^+. \end{aligned}$$

Moreover, using (16) and item ix) of Lemma 1, by an elementary computation it follows that

$$\begin{aligned} u_m'(r) &\leq r^{1-N} \int_0^r s^{N-1} a_1(s) F_1 \left(f_2 \left(v_m(s)\right)\right) ds \\ &\leq r^{1-N} \int_0^r s^{N-1} a_1(s) F_1((\sqrt{2\gamma_2} v_m(s))^{\frac{1}{2\gamma_2}}) ds \\ &\leq r^{1-N} \int_0^r s^{N-1} a_1(s) F_1((\sqrt{2\gamma_2} \left(\beta + F_2 \left(u_m(s)\right) C_2^+\right))^{\frac{1}{2\gamma_2}}) ds \qquad (32) \\ &\leq F_1((\sqrt{2\gamma_2} \left(\beta + F_2 \left(u_m(r)\right) C_2^+\right))^{\frac{1}{2\gamma_2}}) r^{1-N} \int_0^r s^{N-1} a_1(s) ds. \end{aligned}$$

Arguing as above, but now with the second inequality (31), one can show that

$$\begin{aligned} v_m'(r) &= r^{1-N} \int_0^r s^{N-1} a_2(s) F_2 \left(f_1 \left(u_{m-1}(s)\right)\right) ds \qquad (33) \\ &\leq F_2((\sqrt{2\gamma_1} \left(\alpha + F_1 \left(v_m(r)\right) C_1^+\right))^{\frac{1}{2\gamma_1}}) r^{1-N} \int_0^r s^{N-1} a_2(s) ds. \end{aligned}$$

In the proof that follows, it will be convenient to write previous relations (32) and (33) as follows

$$\frac{\left(u_m(r)\right)'}{F_1((\sqrt{2\gamma_2} \left(\beta + F_2 \left(u_m(r)\right) C_2^+\right))^{\frac{1}{2\gamma_2}})} \leq \mathcal{A}_{a_1}'(r), \qquad (34)$$

$$\frac{\left(v_m(r)\right)'}{F_2((\sqrt{2\gamma_1} \left(\alpha + F_1 \left(v_m(r)\right) C_1^+\right))^{\frac{1}{2\gamma_1}})} \leq \mathcal{A}_{a_2}'(r). \qquad (35)$$

After integration of (34)-(35) from 0 to r, we find the estimates

$$\int_{\alpha}^{u_m(r)} \frac{1}{F_1((\sqrt{2\gamma_2}(\beta+F_2(t)C_2^+))^{\frac{1}{2\gamma_2}})} dt \leq \mathcal{A}_{a_1}(r),$$
$$\int_{\beta}^{v_m(r)} \frac{1}{F_2((\sqrt{2\gamma_1}(\alpha+F_1(t)C_1^+))^{\frac{1}{2\gamma_1}})} dt \leq \mathcal{A}_{a_2}(r),$$

which are equivalent to

$$\widetilde{F}_{1,2}(u_m(r)) \leq \mathcal{A}_{a_1}(r) \text{ and } \widetilde{F}_{2,1}(v_m(r)) \leq \mathcal{A}_{a_2}(r). \tag{36}$$

With the above inequalities in hand, the proof is now an obvious modification of the proof of Theorem 1. Indeed, since $\widetilde{F}_{1,2}$ (resp. $\widetilde{F}_{2,1}$) is a bijection with the inverse function $\widetilde{F}_{1,2}^{-1}$ (resp. $\widetilde{F}_{2,1}^{-1}$) strictly increasing on $[0,\infty)$, inequalities (36) can be reformulated as

$$u_m(r) \leq \widetilde{F}_{1,2}^{-1}(\mathcal{A}_{a_1}(r)) \text{ and } v_m(r) \leq \widetilde{F}_{2,1}^{-1}(\mathcal{A}_{a_2}(r)). \tag{37}$$

So, we have found upper bounds for $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ depending only on r. Finally, if $C_1^+ < \infty$ and $C_2^+ < \infty$, then using the same arguments as in (27) and (28), we can see from (37) that

$$u(r) \leq \widetilde{F}_{1,2}^{-1}\left(C_1^+\right) < \infty \text{ and } v(r) \leq \widetilde{F}_{2,1}^{-1}\left(C_2^+\right) < \infty \text{ for all } r \geq 0.$$

Hence (u,v) is bounded. This concludes the proof. □

Our next task is to give an existence result for (8) such that one of the components is bounded, while the other is large.

Theorem 3. *Assume that (A), (C1),* $\widetilde{F}_{1,2}(\infty) = F_{2,1}(\infty) = \infty$, $C_1^+ \in (0,\infty)$ *and* $\underline{A}_{2,1}(\infty) = \infty$ *hold. Furthermore, if* f_2 *satisfies hypotheses (10), then the system (8) has one positive radial solution*

$$(u,v) \in C^1([0,\infty)) \times C^1([0,\infty)) \text{ with } u(0) = \alpha \text{ and } v(0) = \beta,$$

such that $\lim_{r\to\infty} u(r) < \infty$ *and* $\lim_{r\to\infty} v(r) = \infty$.

Proof. Since the proof of Theorem 4 borrows the ideas from the proofs of Theorem 1 and Theorem 2, we have included just the estimates

$$u(r) \leq \widetilde{F}_{1,2}^{-1}\left(C_1^+\right) < \infty \text{ and } v(r) \geq b + \underline{A}_{2,1}(r).$$

Assuming that

$$C_1^+ < \infty \text{ and } \underline{A}_{2,1}(\infty) = \infty$$

holds, we have

$$\lim_{r\to\infty} u(r) < \infty \text{ and } \lim_{r\to\infty} v(r) = \infty.$$

This completes the proof. □

It remains to proceed to

Theorem 4. *Assume that (A), (C1),* $\widetilde{F}_{2,1}(\infty) = F_{1,2}(\infty) = \infty$, $\overline{A}_{1,2}(\infty) = \infty$ *and* $C_2^+ \in (0, \infty)$ *hold. Furthermore, if* f_1 *satisfies hypotheses (9), then the system (8) has one positive radial solution*

$$(u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } u(0) = \alpha \text{ and } v(0) = \beta,$$

such that $\lim_{r\to\infty} u(r) = \infty$ *and* $\lim_{r\to\infty} v(r) < \infty$.

Proof. Since the proof is an obvious modification of the proof of the preceding theorems 1 and 2, we only mention one detail. We first establish the existence and later we have that

$$u(r) \geq \alpha + \underline{A}_{1,2}(r) \text{ and } v(r) \leq \widetilde{F}_{2,1}^{-1}\left(C_2^+\right). \tag{38}$$

Passing to the limit as r goes to infinity in (38), we obtain the desired conclusion. □

The following theorems are an immediate consequence of the above results.

Theorem 5. *Assume that conditions (A) and (C1) hold. If* $C_1^+ < \widetilde{F}_{1,2}(\infty) < \infty$ *and* $C_2^+ < \widetilde{F}_{2,1}(\infty) < \infty$ *are satisfied, then system (8) has one positive bounded radial solution*

$$(u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } u(0) = \alpha \text{ and } v(0) = \beta,$$

such that

$$\begin{cases} \alpha + \underline{A}_{1,2}(r) \leq u(r) \leq \widetilde{F}_{1,2}^{-1}\left(C_1^+\right), \\ \beta + \underline{A}_{2,1}(r) \leq v(r) \leq \widetilde{F}_{2,1}^{-1}\left(C_2^+\right). \end{cases}$$

Theorem 6. *Assume that conditions (A) and (C1) hold. If (9), (11),* $F_{1,2}(\infty) = \underline{A}_{1,2}(\infty) = \infty$ *and* $C_2^+ < \widetilde{F}_{2,1}(\infty) < \infty$ *are satisfied, then system (8) has one positive radial solution*

$$(u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } u(0) = \alpha \text{ and } v(0) = \beta,$$

such that $\lim_{r\to\infty} u(r) = \infty$ *and* $\lim_{r\to\infty} v(r) < \infty$.

Theorem 7. *Assume that condition (A) holds. If (10), (12),* $C_1^+ < \widetilde{F}_{1,2}(\infty) < \infty$ *and* $\underline{A}_{2,1}(\infty) = F_{2,1}(\infty) = \infty$ *are satisfied, then system (8) has one positive radial solution*

$$(u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } u(0) = \alpha \text{ and } v(0) = \beta,$$

such that $\lim_{r\to\infty} u(r) < \infty$ *and* $\lim_{r\to\infty} v(r) = \infty$.

4.1. Proof of theorems 5 − 7

Proof of Theorem 5. Using the idea in the proof of (36) and the conditions of the theorem it follows that

$$\begin{aligned} &\widetilde{F}_{1,2}(u_m(r)) \leq \mathcal{A}_{a_1}(r) \leq C_1^+ < \widetilde{F}_{1,2}(\infty) < \infty, \\ &\widetilde{F}_{2,1}(v_m(r)) \leq \mathcal{A}_{a_2}(r) \leq C_1^+ < \widetilde{F}_{2,1}(\infty) < \infty. \end{aligned}$$

On the other hand, since $\widetilde{F}_{1,2}^{-1}$ and $\widetilde{F}_{2,1}^{-1}$ are strictly increasing on $[0,\infty)$, we find out that

$$u_m(r) \leq \widetilde{F}_{1,2}^{-1}\left(C_1^+\right) < \infty \text{ and } v_m(r) \leq \widetilde{F}_{2,1}^{-1}\left(C_2^+\right) < \infty,$$

and then the non-decreasing sequences $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are bounded from above for all $r \geq 0$ and all m. Putting these two facts together yields

$$(u_m(r), v_m(r)) \to (u(r), v(r)) \text{ as } m \to \infty$$

and the limit functions u and v are positive entire bounded radial solutions to system (8). The proof is complete. □

We are then lead to the proof of theorems 6 and 7, which are not difficult to establish, and so, we omit their proofs.

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