MATHEMATICAL COMMUNICATIONS
Math. Commun. **24**(2019), 265–278

Local convergence analysis of two competing two-step iterative methods free of derivatives for solving equations and systems of equations

Ioannis K. Argyros[1,*] and Santhosh George[2]

[1] *Department of Mathematical Sciences, Cameron University, Lawton, OK 73 505, USA*
[2] *Department of Mathematical and Computational Sciences, National Institute of Technology, Karnataka, Surathkal 575 025, India*

Received April 14, 2018; accepted May 2, 2019

Abstract. We present the local convergence analysis of two-step iterative methods free of derivatives for solving equations and systems of equations under similar hypotheses based on Lipschitz-type conditions. The methods are in particular useful for solving equations or systems involving non-differentiable terms. A comparison is also provided using suitable numerical examples.
AMS subject classifications: 47H09, 47H10, 65G99, 65H10, 49M15
Key words: two-step secant method, two-step Kurchatov method, local convergence, divided differences, Fréchet-derivative, radius of convergence, Lipschitz conditions

1. Introduction

Numerous problems in mathematics, computational sciences, engineering and related sciences using mathematical modeling [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 11, 12, 13, 16, 15, 17] can be reduced to locating a solution x^* of the nonlinear equation in the form

$$F(x) = 0,$$

where X, Y are Banach spaces, D is nonempty, open, convex, and $F : D \subseteq X \longrightarrow Y$ is Fréchet-differentiable.

Analytic solutions or closed form solutions are hard or impossible to find in general. That explains why researchers utilize iterative methods to generate a sequence approximating x^*.

In this study, we present the local convergence of two-step secant method (TSSM) and the two-step Kurchatov-type method (TSKM) defined, respectively, for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} x_{n+1} &= x_n - [x_n, y_n; F]^{-1} F(x_n) \qquad (1)\\ y_{n+1} &= x_{n+1} - [x_{n+1}, y_n; F]^{-1} F(x_{n+1})\\ x_{n+1} &= x_n - [2y_n - x_n, y_n; F]^{-1} F(x_n) \qquad (2)\\ y_{n+1} &= x_{n+1} - [2y_n - x_n, x_n; F] F(x_{n+1}), \end{aligned}$$

*Corresponding author. *Email addresses:* iargyros@cameron.edu (I. K. Argyros), sgeorge@nitk.edu.in (S. George)

where $x_0, y_0 \in D$ are initial points and $[.,.;F] : D \times D \longrightarrow \mathcal{L}(X,Y)$ is a divided difference of order one [16, 15] for F on D satisfying

$$[x,y;F](x-y) = F(x) - F(y) \text{ for each } x, y \text{ with } x \neq y,$$

and $F'(x) = [x,x;F]$, if F is Fréchet-differentiable. TSSM uses two inverses and three function evaluations per complete step, whereas TSKM uses one inverse and four function evaluations.

The rest of the paper is structured as follows: Section 2 and Section 3 contain the local convergence of TSSM and TSKM, respectively, under similar Lipschitz-type hypotheses. The numerical examples in Section 4 conclude this paper.

2. Local convergence I

We present the local convergence analysis of TSSM based on scalar parameters and functions. Let $\alpha \geq 0, \beta \geq 0$ and $b > 0$ with $\alpha + \beta \neq 0$. Define parameters ρ_0, ρ_1 and functions f and h_f on the interval $[0, \rho_0)$ by

$$\rho_0 = \frac{1}{\alpha+\beta}, \quad \rho_1 = \frac{1}{\alpha+\beta+b},$$
$$f(t) = (b + \frac{\alpha b t}{1-(\alpha+b)t} + \beta)t$$

and

$$h_f(t) = f(t) - 1.$$

We have that $h_f(0) = -1$ and $h_f(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_0^-$. The intermediate value theorem assures that equation $h_f(t) = 0$ has solutions on the interval $(0, \rho_0)$. Denote by ρ^* the smallest such solution. Notice that $h_f(\rho_1) = 0$, so $\rho^* \leq \rho_1$. Then, we have that for each $t \in [0, \rho^*)$

$$0 \leq \frac{bt}{1-(\alpha+\beta)t} < 1$$

and

$$0 \leq f(t) < 1.$$

Let $U(z,\lambda)$ and $\bar{U}(z,\lambda)$ denote the open and closed balls in X, respectively, where $z \in X$ is the center and $\lambda > 0$ is the radius. The local convergence analysis of TSSM is also based on the hypotheses (H):

(h_1) $F : D \subset X \longrightarrow Y$ is a continuously Fréchet differentiable operator and $[.,.;F] : D \times D \longrightarrow L(X,Y)$ is a divided difference of order one.

(h_2) There exist parameters $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta \neq 0$, $x^* \in D$ such that

$$F(x^*) = 0, \; F'(x^*)^{-1} \in L(Y,X)$$

and for each $x, y \in D$

$$\|F'(x^*)^{-1}([x,y;F] - F'(x^*))\| \leq \alpha\|x - x^*\| + \beta\|y - x^*\|.$$

Set $D_0 = D \cap U(x^*, \rho_0)$, where ρ_0 was defined previously.

(h_3) There exists $b > 0$ such that for each $x, y \in D_0$

$$\|F'(x^*)^{-1}([x, y; F] - [x, x^*; F])\| \le b\|y - x^*\|.$$

(h_4) $\bar{U}(x^*, \rho^*) \subset D$, where ρ^* was defined previously.

(h_5) There exists $R^* \ge \rho^*$ such that

$$R^* < \frac{1}{\beta}, \ \beta \neq 0.$$

Set $D_1 = D \cap \bar{U}(x^*, R^*)$.

Theorem 1. *Suppose that the hypotheses (H) hold. Then, sequences* $\{x_n\}$, $\{y_n\}$ *starting from* $x_0, y_0 \in U(x^*, \rho^*) - \{x^*\}$ *and generated by TSSM are well defined in* $U(x^*, \rho^*)$ *for each* $n = 0, 1, 2\ldots,$ *remain in* $U(x^*, \rho^*)$ *and converge to* x^*. *Moreover, the following estimates hold for each* $n = 0, 1, 2, \ldots$

$$\|x_{n+1} - x^*\| \le \frac{b\|y_n - x^*\|}{1 - (\alpha\|x_n - x^*\| + \beta\|y_n - x^*\|)}\|x_n - x^*\| \le \|x_n - x^*\| < \rho^* \quad (3)$$

and

$$\|y_{n+1} - x^*\| \le \frac{b\|y_n - x^*\|}{1 - (\alpha\|x_{n+1} - x^*\| + \beta\|y_n - x^*\|)}\|x_{n+1} - x^*\|. \quad (4)$$

Furthermore, the limit point x^* *is the only solution to equation* $F(x) = 0$ *in* D_1, *where* D_1 *is defined in* (h_5).

Proof. Let $x, y \in U(x^*, \rho^*)$. Using (h_2), we have in turn that

$$\begin{aligned}\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| &\le \alpha\|x - x^*\| + \beta\|y - x^*\| \\ &< (\alpha + \beta)\rho^* < 1.\end{aligned} \quad (5)$$

In view of (5) and the Banach lemma on invertible operators [5, 6, 7, 13], $[x, y; F]^{-1} \in L(Y, X)$ and

$$\|[x, y; F]^{-1}F'(x^*)\| \le \frac{1}{1 - (\alpha\|x - x^*\| + \beta\|y - x^*\|)}. \quad (6)$$

In particular, $[x_0, y_0; F]^{-1} \in L(Y, X)$, since $x_0, y_0 \in U(x^*, \rho^*)$. By the first substep of TSSM, we can write

$$\begin{aligned}x_1 - x^* &= x_0 - x^* - [x_0, y_0; F]^{-1}F(x_0) \\ &= [x_0, y_0; F]^{-1}([x_0, y_0; F] - [x_0, x^*; F])(x_0 - x^*).\end{aligned} \quad (7)$$

By (h_3), (6) for $x = x_0, y = y_0$ and (7), we get in turn

$$\begin{aligned}\|x_1 - x^*\| &\le \|[x_0, y_0; F]^{-1}F'(x^*)\|\|F'(x^*)^{-1}([x_0, y_0; F] - [x_0, x^*; F])(x_0 - x^*)\| \\ &\le \frac{b\|y_0 - x^*\|}{1 - (\alpha\|x_0 - x^*\| + \beta\|y_0 - x^*\|)}\|x_0 - x^*\| \\ &\le \|x_0 - x^*\| < \rho^*,\end{aligned}$$

so (3) holds for $n = 0$ and $x_1 \in U(x^*, \rho^*)$ and $[x_1, y_0; F]^{-1} \in L(Y, X)$. We also have by (6) that

$$\|[x_1, y_0; F]^{-1} F'(x^*)\| \leq \frac{1}{1 - (\alpha\|x_1 - x^*\| + \beta\|y_0 - x^*\|)}.$$

Moreover, by the second substep of TSSM, we can write that

$$\begin{aligned} y_1 - x^* &= x_1 - x^* - [x_1, y_0; F]^{-1} F(x_1) \\ &= [x_1, y_0; F]^{-1}([x_1, y_0; F] - [x_1, x^*; F])(x_1 - x^*), \end{aligned}$$

so

$$\begin{aligned} \|y_1 - x^*\| &\leq \frac{b\|y_0 - x^*\|\|x_1 - x^*\|}{1 - (\alpha\|x_1 - x^*\| + \beta\|y_0 - x^*\|)} \\ &\leq \frac{b\rho^*}{1 - (\alpha + \beta)\rho^*}\|x_1 - x^*\| < \rho^*, \end{aligned}$$

which shows (4) for $n = 0$ and $y_1 \in U(x^*, \rho^*)$. The induction for (3) and (4) is completed analogously if x_0, y_0, x_1, y_1 are replaced by $x_m, y_m, x_{m+1}, y_{m+1}$ in the preceding estimates, respectively. Then, from the estimates

$$\|x_{m+1} - x^*\| \leq \mu_1\|x_m - x^*\| < \rho^*$$

and

$$\|y_{m+1} - x^*\| \leq \mu_2\|x_{m+1} - x^*\| < \rho^*,$$

where $\mu_1 = \frac{b\rho^*}{1-(\alpha+\beta)\rho^*} \in [0, 1)$ and $\mu_2 = f(\rho^*) \in [0, 1)$, we deduce that $\lim_{m\longrightarrow+\infty} x_m$ $= \lim_{m\longrightarrow+\infty} y_m = x^*$, $x_{m+1} \in U(x^*, \rho^*)$ and $y_{m+1} \in U(x^*, \rho^*)$. The uniqueness part is shown by letting $T = [x^*, y^*; F]$ for some $y^* \in D_1$ with $F(y^*) = 0$. Using (h_2) and (h_5), we obtain in turn that

$$\|F'(x^{-1}([x^*, y^*; F] - F'(x^*))\| \leq \beta\|y^* - x^*\| \leq \beta R < 1,$$

so $T^{-1} \in L(Y, X)$. Finally, from the identity

$$0 = F(x^*) - F(y^*) = [x^*, y^*; F](x^* - y^*),$$

we conclude that $x^* = y^*$. □

3. Local convergence II

In this section, the local convergence of TSKM is presented in the way analogous to that shown in Section 2 for TSSM. Let $a \geq 0, b_1 \geq 0, p \geq 0, q \geq 0, a + b_1 \neq 0$ and $c > 0$ be given parameters. Define parameters r_0, r_1, functions g_1 and h_{g_1} on interval $[0, r_0)$ by

$$r_0 = \frac{2}{a + \sqrt{a^2 + 16c}}, \quad r_1 = \frac{2}{a + b_)1 + \sqrt{(x + b_1)^2 + 32c}}$$

$$g_1(t) = \frac{b_1 + 4ct}{1 - (a + 4ct)t}t$$

and

$$h_{g_1}(t) = g_1(t) - 1.$$

Notice that $h_{g_1}(r_1) = 0$ and r_1 is the only solution to equation $h_{g_1}(t) = 0$ in $(0, r_0)$. Moreover, define functions g_2 and h_{g_2} of the interval $[0, r_0)$ by

$$g_2(t) = \frac{p[\frac{(b_1+4ct)t}{1-(a+4ct)t} + 1] + q + 4ct}{1 - (a + 4ct)t} t$$

and

$$h_{g_2}(t) = g_2(t) - 1.$$

We get $h_{g_2}(0) = -1 < 0$ and $h_{g_2}(t) \longrightarrow +\infty$ as $t \longrightarrow r_0^-$. Denote by r_2 the smallest solution to equation $h_{g_2}(t) = 0$ in $(0, r_1)$.

Define the radius of convergence r^* by

$$r^* = \min\{r_1, r_2\}. \tag{8}$$

Then, we have that for each $t \in [0, r^*)$,

$$0 \le g_i(t) < 1, \ i = 1, 2.$$

The local convergence analysis of TSKM is based on hypotheses (A):

1. (a_1)=(h_1)

(a_2) There exist $a \ge 0, c \ge 0, x^* \in D$ such that $F(x^*) = 0, F'(x^*)^{-1} \in L(Y, X)$ for each $x, y \in D$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le a\|x - x^*\|$$

and

$$\|F'(x^*)^{-1}([2y - x, x; F] - F'(y))\| \le c\|y - x\|^2$$

Set $D_2 = D \cap \bar{U}(x^*, r_0)$, where r_0 was defined previously.

(a_3) There exists $b \ge 0, p \ge 0, q \ge 0$ such that for each $x, y \in D_2$

$$\|F'(x^*)^{-1}([x, y; F] - [x, x^*; F])\| \le b\|y - x^*\|$$

and

$$\|F'(x^*)^{-1}([x, x^*; F] - F'(y))\| \le p\|x - y\| + q\|y - x^*\|.$$

(a_4) $\bar{U}(x^*, 3r^*) \subseteq D$, where r^* was defined previously.

(a_5) There exists $R_1^* \ge R^*$ such that

$$R_1^* < \frac{2}{a}, a \ne 0.$$

Set $D_3 = D \cap \bar{U}(x^*, R_1^*)$.

Theorem 2. *Suppose that the hypotheses (A) hold. Then, sequences $\{x_n\}, \{y_n\}$ starting from $x_0, y_0 \in U(x^*, r^*) - \{x^*\}$ and generated by TSKM are well defined in $U(x^*, r^*)$ for each $n = 0, 1, 2, \ldots,$ remain in $U(x^*, r^*)$, and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2 \ldots$*

$$\|x_{n+1} - x^*\| \leq \frac{b\|y_n - x^*\| + c\|y_n - x_n\|^2}{1 - (a\|x_n - x^*\| + c\|y_n - x_n\|^2)} \|x_n - x^*\| \leq \|x_n - x^*\| < r^* \quad (9)$$

and

$$\|y_{n+1} - x^*\| \leq \frac{p\|x_{n+1} - y_n\| + q\|y_n - x^*\| + c\|y_n - x_n\|^2}{1 - (a\|x_n - x^*\| + c\|y_n - x_n\|^2)} \|x_{n+1} - x^*\|. \quad (10)$$

Furthermore, the limit point x^* is the only solution to equation $F(x) = 0$ in D_3.

Proof. Let $x, y \in U(x^*, r^*)$ and set $Q = [2y - x, x; F]$. Using $(a_2\ )$ and (8), we have in turn that

$$\begin{aligned}
&\|F'(x^*)^{-1}(F'(x^*) - Q)\| \\
&\quad = \|F'(x^*)^{-1}(F'(x^*) - F'(y)) + (F'(y) - [2y - x, x; F])\| \\
&\quad \leq \|F'(x^*)^{-1}(F'(y) - F'(x^*))\| + \|F'(x^*)^{-1}([2y - x, x; F] - F'(y))\| \\
&\quad \leq a\|y - x^*\| + c\|y - x\|^2 \\
&\quad \leq ar^* + c(\|y - x^*\| + \|x^* - x\|)^2 \\
&\quad \leq ar^* + 4c(r^*)^2 < 1,
\end{aligned}$$

so $Q^{-1} \in L(Y, X)$,

$$\|Q^{-1}F'(x^*)\| \leq \frac{1}{1 - (a\|y - x^*\| + c\|x - y\|^2)} \quad (11)$$

and $[2y_0 - x_0, x_0; F]^{-1} \in L(Y, X)$ for $x = x_0$ and $y = y_0$. Hence, x_1 and y_1 are well defined by the first and the second substep of TSKM. Notice that condition $(a_4\ )$ guarantees that for $x, y \in U(x^*, r^*)$ we have $2y - x \in U(x^*, r^*) \subseteq D$. By $(a_2\ )$ and $(a_3\ )$, we get in turn the estimate

$$\begin{aligned}
&\|F'(x^*)^{-1}(Q - [x_0, x^*; F])\| \\
&\quad \leq \|F'(x^*)^{-1}(([y_0, x^*; F] - F'(y_0)) + (F'(y_0) - [2y_0 - x_0, x_0; F]))\| \\
&\quad \leq \|F'(x^*)^{-1}([y_0, x^*; F] - F,(y_0))\| + \|F'(x^*)^{-1}(F'(y_0) - [2y_0 - x_0, x_0; F])\| \\
&\quad \leq b\|y_0 - x^*\| + c\|y_0 - x_0\|^2. \quad (12)
\end{aligned}$$

In view of the first substep of TSKM, (8), (11) and (12), we obtain in turn from

$$\begin{aligned}
x_1 - x_0 &= x_0 - x^* - Q^{-1}F(x_0) \\
&= Q^{-1}(Q - [x_0, x^*; F])(x_0 - x^*),
\end{aligned}$$

so

$$\begin{aligned}
\|x_1 - x_0\| &\leq \mu_3\|x_0 - x^*\| \\
&\leq \|x_0 - x^*\| < r^*,
\end{aligned}$$

where $\mu_3 = \frac{b\|y_0-x^*\|+c\|x_0-y_0\|^2}{1-(a\|y_0-x^*\|+c\|x_0-y_0\|^2)} \in [0,1)$, which shows (9) for $n = 0$ and $x_1 \in U(x^*, r^*)$. Similarly, from the second substep of TSKM, we can also write

$$\begin{aligned}y_1 - x^* &= x_1 - x^* - Q^{-1}F(x_1)\\ &= Q^{-1}(([2y_0 - x_0, x_0; F] - F'(y_0)) + (F'(y_0) - [x_1, x^*; F]))(x_1 - x^*),\end{aligned}$$

so

$$\begin{aligned}&\|y_1 - x^*\|\\ &\quad\leq \frac{\|F'(x^*)^{-1}([2y_0 - x_0, x_0; F] - F'(y_0))\| + \|F'(x^*)^{-1}(F'(y_0) - [x_1, x^*; F])\|}{1 - (a\|x_0 - x^*\| + c\|y_0 - x_0\|^2)}\\ &\quad\quad \times \|x_1 - x^*\|\\ &\quad\leq \frac{p\|x_1 - y_0\| + q\|y_0 - x^*\| + c\|y_0 - x_0\|^2}{1 - (a\|x_0 - x^*\| + c\|y_0 - x_0\|^2)}\|x_1 - x^*\|\\ &\quad\leq g_2(\|x_0 - x^*\|)\|x_1 - x^*\| \leq \|x_1 - x^*\| < r^*,\end{aligned}$$

which shows (10) for $n = 0$ and $y_1 \in U(x^*, r^*)$. Then, from the estimates

$$\|x_{m+1} - x^*\| \leq \mu_3\|x_n - x^*\| < r^*,$$

and

$$\|y_{n+1} - x^*\| \leq \mu_4\|x_{m+1} - x^*\| < r^*,$$

where $\mu_4 = g_2(\|x_0 - x^*\|) \in [0, 1)$, we obtain $\lim_{m\longrightarrow+\infty} x_m = \lim_{m\longrightarrow+\infty} y_m = x^*$ and $x_{m+1}, y_{m+1} \in U(x^*, r^*)$. As in Theorem 1, but using (a_2) and (a_5) for $P = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$, we obtain

$$\begin{aligned}\|F'(x^*)^{-1}(P - F'(x^*))\| &\leq \int_0^1 \theta\|y^* - x^*\|d\theta\\ &\leq \frac{a}{2}\|y^* - x^*\| \leq \frac{a}{2}R_1^* < 1,\end{aligned}$$

so $P^{-1} \in L(Y, X)$. Then, from the identity

$$0 = F(y^*) - F(x^*) = P(y^* - x^*),$$

we derive that $x^* = y^*$. □

Remark 1. *Condition (a_4) can be weakened if replaced by*
(a_4)' $\bar{U}(x^*, r^*) \subseteq D$ and for each $x, y \in D$

$$2y - x \in D. \tag{13}$$

Condition (13) certainly holds if $D = X$ (see also [1, 2, 3, 4, 5, 6, 7]).

4. Numerical examples

Let $X = Y = \mathbb{R}^k, k$ be a positive integer equipped with the standard difference [13], and for

$$x_m = (x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(k)})$$
$$y_m = (y_m^{(1)}, y_m^{(2)}, \ldots, y_m^{(k)}),$$

there exists $i = 1, 2, \ldots, k$ such that $x_m^{(i)} = y_m^{(i)}$. Then, we cannot use TSSM or TSKM in the form (1) and (2). Assuming that $x_0^{(i)} \neq y_0^{(i)}, y_0^{(i)} \neq x_1^{(i)}$ for each $i = 1, 2, \ldots, k, [x_0, y_0; F]^{-1}$ and $[x_1, y_0; F]^{-1} \in L(Y, X)$, we can use a mehod similar to the TSSM method defined for each $n = 0, 1, 2, \ldots,$ by

$$\begin{aligned} x_{n+1} &= x_n - [v_j, w_j; F]^{-1} F(x_n) \\ y_{n+1} &= x_{n+1} - [z_{j+1}, w_j; F]^{-1} F(x_{n+1}), \end{aligned} \tag{14}$$

where $j = 0, 1, 2, \ldots, n$ is the smallest index for which $v_j^{(i)} \neq w_j^{(i)}$ and $z_{j+1}^{(i)} \neq w_j^{(i)}$. Then, method (14) is always well defined and can be used to solve equations containing non-differentiable terms. Similarly, assume that $[2y_0 - x_0, x_0; F]^{-1}$ and $[2x_1 - y_0, y_0; F]^{-1} \in L(Y, X), x_0^{(i)} \neq y_0^{(i)}$ and $y_0^{(i)} \neq x_1^{(i)}$ for each $i = 1, 2, \ldots, k$. Then, the method corresponding to TSKM is defined by

$$\begin{aligned} x_{n+1} &= x_n - [2w_j - v_j, v_j; F]^{-1} F(x_n) \\ y_{n+1} &= x_{n+1} - [2w_j - v_j, v_j; F]^{-1} F(x_{n+1}). \end{aligned} \tag{15}$$

Clearly, methods (14) and (15) generalize methods (1) and (2) since they coincide with those for $j = n$, respectively.

Next, we shall show the convergence of method (14) under similar conditions. Let us consider hypotheses (H'):

1. (h_1')=(h_1)
2. (h_2')= (h_2)

(h_3') There exists $\gamma \geq 0, \delta \geq 0$ such that for each $x, y, z \in D_0$

$$\|F'(x^*)^{-1}([x, y; F] - [z, x^*; F])\| \leq \gamma \|x - z\| + \delta \|y - x^*\|.$$

(h_4') $\bar{U}(x^*, \bar{\rho}^*) \subset D$, where $\bar{\rho}^* = \frac{1}{\alpha+\beta+2\gamma+\delta}$.

(h_5') There exists $\bar{R}^* \geq \bar{\rho}^*$ such that

$$\bar{R}^* < \frac{1}{\beta}, \ \beta \neq 0.$$

Let $D_5 = D \cap \bar{U}(x^*, \bar{R}^*)$.

Theorem 3. *Suppose that the hypotheses (H') hold. Then, sequences* $\{x_n\}, \{y_n\}$ *starting from* $x_0, y_0 \in U(x^*, \bar{\rho}^*) - \{x^*\}$ *and generated by method (14) are well defined in* $U(x^*, \bar{\rho}^*)$*, remain in* $U(x^*, \bar{\rho}^*)$ *for each* $n = 0, 1, 2, \dots,$ *and converge to* x^**. Moreover, the following estimates hold:*

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \frac{\gamma\|v_j - x_n\| + \delta\|w_j - x^*\|}{1 - (\alpha\|v_j - x^*\| + \beta\|w_j - x^*\|)}\|x_n - x^*\| \\ &\le \frac{\gamma(\|v_j - x^*\| + \|x_n - x^*\|) + \delta\|w_j - x^*\|}{1 - (\alpha\|v_j - x^*\| + \beta\|w_j - x^*\|)}\|x_n - x^*\| \\ &\le \frac{(2\gamma + \delta)\bar{\rho}^*}{1 - (\alpha + \beta)\bar{\rho}^*}\|x_n - x^*\| \le \|x_n - x^*\| < \bar{\rho}^* \end{aligned} \tag{16}$$

and

$$\begin{aligned} \|y_{n+1} - x^*\| &\le \frac{\gamma\|z_{j+1} - x_{n+1}\| + \delta\|w_j - x^*\|}{1 - (\alpha\|z_{j+1} - x^*\| + \beta\|w_j - x^*\|)}\|x_{n+1} - x^*\| \\ &\le \frac{\gamma(\|z_{j+1} - x^*\| + \|x_{n+1} - x^*\|) + \delta\|w_j - x^*\|}{1 - (\alpha\|z_{j+1} - x^*\| + \beta\|w_j - x^*\|)}\|x_{n+1} - x^*\| \\ &\le \frac{(2\gamma + \delta)\bar{\rho}^*}{1 - (\alpha + \beta)\bar{\rho}^*}\|x_{n+1} - x^*\| \le \|x_{n+1} - x^*\| < \bar{\rho}^*. \end{aligned} \tag{17}$$

Furthermore, the limit point x^* *is the only solution to equation* $F(x) = 0$ *in* D_5*.*

Proof. Use the proof of Theorem 1, the identities

$$\begin{aligned} x_{n+1} - x^* &= ([v_j, w_j; F]^{-1}F'(x^*)) \\ &\quad \times (F'(x^*)^{-1}([v_j, w_j; F] - [x_n, x^*; F]))(x_n - x^*) \end{aligned}$$

and

$$\begin{aligned} y_{n+1} - x^* &= ([z_{j+1}, v_j; F]^{-1}F'(x^*)) \\ &\quad \times (F'(x^*)^{-1}([z_{j+1}, w_j; F] - [x_{n+1}, x^*; F]))(x_{n+1} - x^*) \end{aligned}$$

to arrive at estimates (16) and (17), respectively. □

The hypotheses (A') are:

1. (a_1')=(a_1)
2. (a_2')= (h_2)
3. (a_3')= (h_3)

(a_4') $\bar{U}(x^*, \bar{r}^*) \subset D$, where $\bar{r}^* = \frac{1}{3\alpha+\beta+4\gamma+\delta}$.

(a_5') There exists $\bar{R}_1^* \ge \bar{r}^*$ such that

$$\bar{R}_1^* < \frac{1}{\beta},\ \beta \ne 0.$$

Let $D_6 = D \cap \bar{U}(x^*, \bar{R}_1^*)$.

Theorem 4. *Suppose that the hypotheses (A') hold. Then, sequences $\{x_n\}, \{y_n\}$ starting from $x_0, y_0 \in U(x^*, \bar{r}^*) - \{x^*\}$ and generated by method (15) are well defined in $U(x^*, \bar{r}^*)$, remain in $U(x^*, \bar{r}^*)$ for each $n = 0, 1, 2, \ldots,$ and converge to x^*. Moreover, the following estimates hold:*

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \frac{\gamma\|2w_j - v_j - x_n\| + \delta\|v_j - x^*\|}{1 - (\alpha\|2w_j - v_j - x^*\| + \beta\|v_j - x^*\|)}\|x_n - x^*\| \\ &\leq \frac{\gamma(2\|w_j - x^*\| + \|v_j - x^*\| + \|x_n - x^*\|) + \delta\|v_j - x^*\|}{1 - (\alpha(2\|w_j - x^*\| + \|v_j - x^*\|) + \beta\|v_j - x^*\|)}\|x_n - x^*\| \\ &\leq \frac{(4\gamma + \delta)\bar{r}^*}{1 - (3\alpha + \beta)\bar{r}^*}\|x_n - x^*\| \leq \|x_n - x^*\| < \bar{r}^*, \qquad (18)\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1} - x^*\| &\leq \frac{\gamma\|2w_j - v_j - x_{n+1}\| + \delta\|v_j - x^*\|}{1 - (\alpha\|2w_j - v_j - x^*\| + \beta\|v_j - x^*\|)}\|x_{n+1} - x^*\| \\ &\leq \frac{\gamma(2\|w_j - x^*\| + \|v_j - x^*\| + \|x_{n+1} - x^*\|) + \delta\|v_j - x^*\|}{1 - (\alpha(2\|w_j - x^*\| + \|v_j - x^*\|) + \beta\|v_j - x^*\|)}\|x_{n+1} - x^*\| \\ &\leq \frac{(4\gamma + \delta)\bar{r}^*}{1 - (3\alpha + \beta)\bar{r}^*}\|x_{n+1} - x^*\| \leq \|x_{n+1} - x^*\| < \bar{r}^*. \qquad (19)\end{aligned}$$

Furthermore, the limit point x^* is the only solution to equation $F(x) = 0$ in D_6.

Proof. Use the proof of Theorem 2, the identities

$$\begin{aligned}x_{n+1} - x^* &= ([2w_j - v_j, w_j; F]^{-1}F'(x^*)) \\ &\quad \times (F'(x^*)^{-1}([2w_j - v_j, v_j; F] - [x_n, x^*; F]))(x_n - x^*)\end{aligned}$$

and

$$\begin{aligned}y_{n+1} - x^* &= ([2w_j - v_j, v_j; F]^{-1}F'(x^*)) \\ &\quad \times (F'(x^*)^{-1}([2w_j - v_j, v_j; F] - [x_{n+1}, x^*; F]))(x_{n+1} - x^*)\end{aligned}$$

to arrive at estimates (18) and (19), respectively. □

Example 1. *Let us consider the system for $h = (h_1, h_2)^T$*

$$\begin{aligned}f_1(h) &= 3h_1^2h_2 + h_2^2 - 1 + |h_1 - 1| = 0 \\ f_2(h) &= h_1^4 + h_1h_2^3 - 1 + |h_2| = 0\end{aligned}$$

which can be written as $F(h) = 0$, where $F = (f_1, f_2)^T$. Using the divided difference, $([a, b; F]_{ij})_{i,j=1}^2 \in L(\mathbb{R}^2, \mathbb{R}^2)$ [13], for $x_{-1} = (1, 0)^T, x_0 = (5, 5)^T$, we obtain by (2) Hence, the solution p is given by $p = (0.894655373334687, 0.3278626421746298)^T$. Notice that mapping F is not differentiable, so the earlier results mentioned in the introduction of this study cannot be used.

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	5
1	1	0	5
2	0.909090909090909	0.363636363636364	3.0636E-01
3	0.894886945874111	0.329098638203090	3.453E-02
4	0.894655531991499	0.327827544745569	1.271E-03
5	0.894655373334793	0.327826521746906	1.022E-06
6	0.8946655373334687	0.327826521746298	6.089E-13
7	0.8946655373334687	0.327826421746298	2.710E-20

Table 1:

Example 2. *We consider the boundary problem appearing in many studies of applied sciences [6] given by*

$$\varphi'' + \varphi^{1+\lambda} + \varphi^2 = 0, \ \ \lambda \in [0,1] \qquad (20)$$
$$\varphi(0) = \varphi(1) = 0.$$

Let $h = \frac{1}{l}$, *where* l *is a positive integer and set* $s_i = ih, i = 1, 2, \ldots, l-1$. *The boundary conditions are then given by* $\varphi_0 = \varphi_n = 0$. *We shall replace the second derivative* φ'' *by the popular divided difference*

$$\varphi''(t) \approx \frac{[\varphi(t+h) - 2\varphi(t) + \varphi(t-h)]}{h^2} \qquad (21)$$
$$\varphi''(s_i) = \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2}, \ i = 1, 2, \ldots l-1.$$

Using (20) and (21), we obtain the system of equations defined by

$$2\varphi_1 - h^2\varphi_1^{1+\lambda} - h^2\varphi_1^2 - \varphi_2 = 0$$
$$-\varphi_{i-1} + 2\varphi_i - h^2\varphi_i^{1+\lambda} - h^2\varphi_i^2 - \varphi_{i+1} = 0$$
$$-\varphi_{l-2} + 2\varphi_{l-1} - h^2\varphi_{l-1}^{1+\lambda} - h^2\varphi_{l-1}^2 = 0.$$

Define operator $F : \mathbb{R}^{l-1} \longrightarrow \mathbb{R}^{l-1}$ *by*

$$F(\varphi) = M(x) - h^2 f(\varphi),$$

where

$$M = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 \end{bmatrix}$$

and

$$f(\varphi) = [\varphi_1^{1+\lambda} + \varphi_1^2, \varphi_2^{1+\lambda} + \varphi_2, \ldots, \varphi_{l-1}^{1+\lambda} + \varphi_{l-1}^2]^T.$$

Then, the Fréchet-derivative F' of operator F is given by

$$F'(\varphi) = M - (1+\lambda)h^2 \begin{bmatrix} \varphi_1^\lambda & 0 & 0 \dots & 0 \\ 0 & \varphi_2^\lambda & 0 \dots & 0 \\ \vdots & \vdots & \vdots \ \vdots & \vdots \\ 0 & 0 & 0 \dots & \varphi_{l-1}^\lambda \end{bmatrix} - 2h^2 \begin{bmatrix} \varphi_1 & 0 & 0 \dots & 0 \\ 0 & \varphi_2 & 0 \dots & 0 \\ \vdots & \vdots & \vdots \ \vdots & \vdots \\ 0 & 0 & 0 \dots & \varphi_{l-1} \end{bmatrix}. \quad (22)$$

We shall use a special case of method (2) given by

$$\begin{aligned} \psi_n^{(1)} &= \psi_n - F'(\psi_n)^{-1}F(\psi_n) \\ \psi_n^{(2)} &= \psi_n^{(1)} - F'(\psi_n)^{-1}F(\psi_n^{(1)}) \\ &\vdots \\ \psi_n^{(k)} &= \psi_n^{(k-1)} - F'(\psi_n)^{-1}F(\psi_n^{(k-1)}) \\ \psi_{n+1} &= \psi_n^{(k)}. \end{aligned} \quad (23)$$

Let $\lambda = \frac{1}{2}, k = 3$ and $l = 10$. In this way, we obtain a 9×9 system. A good initial approximation is $10 \sin \pi t$ since a solution to (20) vanishes at the end points and is positive at the interior. This approximation gives the vector

$$\xi = \begin{bmatrix} 3.0901699423 \\ 5.877852523 \\ 8.090169944 \\ 9.510565163 \\ 10 \\ 9.510565163 \\ 8.090169944 \\ 5.877852523 \\ 3.090169923 \end{bmatrix},$$

which by using (23) leads to

$$\psi_0 = \begin{bmatrix} 2.396257294 \\ 4.698040582 \\ 6.677432200 \\ 8.038726637 \\ 8.526409945 \\ 8.038726637 \\ 6.6774432200 \\ 4.698040582 \\ 2.396257294 \end{bmatrix}.$$

Using vector ψ_0 as the initial vector in (23), we get the solution ψ^* given by

$$\psi^* = \psi_6 = \begin{bmatrix} 2.394640795 \\ 4.694882371 \\ 6.672977547 \\ 8.033409359 \\ 8.520791424 \\ 8.033409359 \\ 6.672977547 \\ 4.694882371 \\ 2.394640795 \end{bmatrix}.$$

Notice that the operator F' given in (22) is not Lipschitz.

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