

Multifold convolutions of binomial coefficients

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Abstract. A class of the multifold convolutions of binomial coefficients will be evaluated by employing a pair of Lambert series. The corresponding multisums on Abel coefficients will also be examined.

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1. Introduction and motivation

There exist numerous binomial identities scattered in the mathematical literature (see Gould [7], Graham et al. [8, Chapter 5] and Riordan [11]). In 2011, Chang and Xu [2, Theorem 1] gave a probabilistic proof of the following identity:

$$\sum_{k=0}^m \binom{2k}{k} \binom{2m-2k}{m-k} = 4^m. \quad (1)$$

Two years later, Duarte and Guedes de Oliveira [5, 6] not only reproved the above formula by the principle of inclusion and exclusion, but also extended it to

$$\sum_{k_1+k_2+\dots+k_\lambda=m} \prod_{i=1}^{\lambda} \binom{a_i+2k_i}{k_i} = 4^m \binom{m-1+\lambda/2}{m}, \quad (2)$$

where $(a_1, a_2, \dots, a_\lambda) \in \mathbb{R}^\lambda$ with their component sum being equal to zero. When this zero-sum restriction is removed, U. Abel et al. [1] evaluated further the corresponding multisum by making use of differential operators.

Inspired by this recent work of U. Abel et al. (2015), we shall present more general summation formulae in this paper. In the next section, several multifold convolutions on binomial coefficients will be evaluated by utilizing a pair of Lambert series. Then in Section 3, we shall establish the corresponding identities about the Abel coefficients. Finally, the paper will end up with the fourth section, where some alternating multisums will be examined.

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2. Multifold convolutions of binomial coefficients

Let \mathbb{N} and \mathbb{C} be the sets of natural numbers and complex numbers, respectively, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For the vectors $\mathbf{a} = (a_1, a_2, \dots, a_\lambda) \in \mathbb{C}^\lambda$, $\mathbf{c} = (c_1, c_2, \dots, c_\mu) \in \mathbb{C}^\mu$ and $\mathbf{n} = (n_1, n_2, \dots, n_{\lambda+\mu}) \in \mathbb{N}_0^{\lambda+\mu}$, their component sums are denoted by $|\mathbf{a}|$, $|\mathbf{c}|$ and $|\mathbf{n}|$, respectively. Our main result can be stated in the following theorem.

Theorem 1 (Convolution formulae). *For $\lambda, \mu \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $a_i, c_j, \beta \in \mathbb{C}$, define the multifold convolution of binomial coefficients*

$$S_m(\lambda, \mu | \beta, \mathbf{a}, \mathbf{c}) := \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + \beta n_i}{n_i} \prod_{j=1}^{\mu} \frac{c_j}{c_j + \beta n_{\lambda+j}} \binom{c_j + \beta n_{\lambda+j}}{n_{\lambda+j}}.$$

Then the following summation formulae hold:

- (a) $S_m(\lambda, \mu | \beta, \mathbf{a}, \mathbf{c}) = \sum_{k=0}^m \binom{-\lambda}{k} \frac{|\mathbf{a}| + |\mathbf{c}| + \beta k - k}{|\mathbf{a}| + |\mathbf{c}| + \beta m - k} \binom{|\mathbf{a}| + |\mathbf{c}| + \beta m - k}{m - k} (-\beta)^k,$
- (b) $S_m(\lambda, \mu | \beta, \mathbf{a}, \mathbf{c}) = \sum_{k=0}^m \binom{-\lambda}{k} \frac{|\mathbf{a}| + |\mathbf{c}| + \lambda + \beta k}{|\mathbf{a}| + |\mathbf{c}| + \lambda + \beta m} \binom{|\mathbf{a}| + |\mathbf{c}| + \lambda + \beta m}{m - k} (1 - \beta)^k,$
- (c) $S_m(\lambda, \mu | \beta, \mathbf{a}, \mathbf{c}) = \sum_{k=0}^m \binom{1 - \lambda}{k} \binom{|\mathbf{a}| + |\mathbf{c}| + \beta m - k}{m - k} (-\beta)^k,$
- (d) $S_m(\lambda, \mu | \beta, \mathbf{a}, \mathbf{c}) = \sum_{k=0}^m \binom{1 - \lambda}{k} \binom{|\mathbf{a}| + |\mathbf{c}| + \beta m + \lambda - 1}{m - k} (1 - \beta)^k.$

Proof. In classical analysis, the following binomial series due to Lambert [9] (see also [3, 4, 7], [8, §5.4] and [11, §5.4] are well known:

$$\left. \begin{aligned} \sum_{n=0}^{\infty} \frac{\alpha}{\alpha + n\beta} \binom{\alpha + n\beta}{n} x^n &= (1 + y)^\alpha \\ \sum_{n=0}^{\infty} \binom{\alpha + n\beta}{n} x^n &= \frac{(1 + y)^{\alpha+1}}{1 + y - \beta y} \end{aligned} \right\} \text{ where } x = y/(1 + y)^\beta. \quad (3)$$

Let $[x^n]\phi(x)$ be the coefficient of x^n in the formal series $\phi(x)$. According to the above generating functions, we have the expression

$$S_m(\lambda, \mu | \beta, \mathbf{a}, \mathbf{c}) = [x^m] \prod_{i=1}^{\lambda} \frac{(1 + y)^{1+a_i}}{1 + y - \beta y} \prod_{j=1}^{\mu} (1 + y)^{c_j} = [x^m] \frac{(1 + y)^{|\mathbf{a}|+|\mathbf{c}|+\lambda}}{(1 + y - \beta y)^\lambda}. \quad (4)$$

Then the four formulae in Theorem 1 can be shown by making use of (3) and expanding, through the binomial series, the above generating function on the right-hand side of (4) in four different ways.

Firstly, the formula (a) follows by expressing (4) as

$$\begin{aligned} S_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) &= [x^m](1+y)^{|\mathbf{a}|+|\mathbf{c}|} \left(1 - \frac{\beta y}{1+y}\right)^{-\lambda} \\ &= [x^m](1+y)^{|\mathbf{a}|+|\mathbf{c}|} \sum_{k=0}^{\infty} \binom{-\lambda}{k} \frac{(-\beta y)^k}{(1+y)^k} \\ &= \sum_{k=0}^m (-\beta)^k \binom{-\lambda}{k} [x^{m-k}](1+y)^{|\mathbf{a}|+|\mathbf{c}|+\beta k-k}. \end{aligned}$$

Secondly, the formula (b) follows by reformulating (4) as

$$\begin{aligned} S_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) &= (1+y)^{|\mathbf{a}|+|\mathbf{c}|+\lambda} \{1 + (1-\beta)y\}^{-\lambda} \\ &= [x^m](1+y)^{|\mathbf{a}|+|\mathbf{c}|+\lambda} \sum_{k=0}^{\infty} (1-\beta)^k \binom{-\lambda}{k} y^k \\ &= [x^{m-k}] \sum_{k=0}^{\infty} (1-\beta)^k \binom{-\lambda}{k} (1+y)^{|\mathbf{a}|+|\mathbf{c}|+\lambda+\beta k}. \end{aligned}$$

Thirdly, the formula (c) follows by manipulating (4) as

$$\begin{aligned} S_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) &= [x^m] \frac{(1+y)^{|\mathbf{a}|+|\mathbf{c}|+1}}{1+y-\beta y} \left(1 - \frac{\beta y}{1+y}\right)^{1-\lambda} \\ &= [x^m] \frac{(1+y)^{|\mathbf{a}|+|\mathbf{c}|+1}}{1+y-\beta y} \sum_{k=0}^{\infty} (-\beta)^k \binom{1-\lambda}{k} \frac{y^k}{(1+y)^k} \\ &= [x^{m-k}] \sum_{k=0}^{\infty} (-\beta)^k \binom{1-\lambda}{k} \frac{(1+y)^{|\mathbf{a}|+|\mathbf{c}|+(\beta-1)k+1}}{1+y-\beta y}. \end{aligned}$$

Finally, the formula (d) follows by rewriting (4) as

$$\begin{aligned} S_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) &= [x^m] \frac{(1+y)^{|\mathbf{a}|+|\mathbf{c}|+\lambda}}{1+y-\beta y} [1 + (1-\beta)y]^{1-\lambda} \\ &= [x^m] \frac{(1+y)^{|\mathbf{a}|+|\mathbf{c}|+\lambda}}{1+y-\beta y} \sum_{k=0}^{\infty} (1-\beta)^k \binom{1-\lambda}{k} y^k \\ &= [x^{m-k}] \sum_{k=0}^{\infty} (1-\beta)^k \binom{1-\lambda}{k} \frac{(1+y)^{|\mathbf{a}|+|\mathbf{c}|+\beta k+\lambda}}{1+y-\beta y}. \end{aligned}$$

This completes the proof of Theorem 1. □

When $\lambda = 0$ (which implies that $|\mathbf{a}| = 0$) in Theorem 1, all four formulae there reduce to the following same closed formula.

Proposition 2. *Let $\mu \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $c_j, \beta \in \mathbb{C}$. It holds:*

$$\sum_{|\mathbf{n}|=m} \prod_{j=1}^{\mu} \frac{c_j}{c_j + \beta n_j} \binom{c_j + \beta n_j}{n_j} = \frac{|\mathbf{c}|}{|\mathbf{c}| + \beta m} \binom{|\mathbf{c}| + \beta m}{m}.$$

Instead, when $\mu = 0$ (which implies that $|\mathbf{c}| = 0$) in Theorem 1, we get the four reduced formulae below, where the last one (d) is equivalent to U. Abel et al. [1, Remark 2].

Proposition 3. *Let $\lambda \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $a_i, \beta \in \mathbb{C}$. It holds:*

$$\begin{aligned}
 (a) \quad & \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + \beta n_i}{n_i} = \sum_{k=0}^m \binom{-\lambda}{k} \frac{|\mathbf{a}| + \beta k - k}{|\mathbf{a}| + \beta m - k} \binom{|\mathbf{a}| + \beta m - k}{m - k} (-\beta)^k, \\
 (b) \quad & \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + \beta n_i}{n_i} = \sum_{k=0}^m \binom{-\lambda}{k} \frac{|\mathbf{a}| + \lambda + \beta k}{|\mathbf{a}| + \lambda + \beta m} \binom{|\mathbf{a}| + \lambda + \beta m}{m - k} (1 - \beta)^k, \\
 (c) \quad & \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + \beta n_i}{n_i} = \sum_{k=0}^m \binom{1 - \lambda}{k} \binom{|\mathbf{a}| + \beta m - k}{m - k} (-\beta)^k, \\
 (d) \quad & \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + \beta n_i}{n_i} = \sum_{k=0}^m \binom{1 - \lambda}{k} \binom{|\mathbf{a}| + \beta m + \lambda - 1}{m - k} (1 - \beta)^k.
 \end{aligned}$$

In particular, when $\beta = 2$, apart from the identities displayed in Theorem 1 and Propositions 2 and 3, we can derive further simpler formulae for the corresponding multifold convolutions. This is mainly due to the following crucial fact about the two variables x and y :

$$x = y/(1 + y)^2 \implies \frac{1 - y}{1 + y} = \sqrt{1 - 4x}.$$

Proposition 4. *Let $\lambda, \mu \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $a_i, c_j \in \mathbb{C}$. It holds:*

$$\begin{aligned}
 (a) \quad & \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + 2n_i}{n_i} \prod_{j=1}^{\mu} \frac{c_j}{c_j + 2n_{i+j}} \binom{c_j + 2n_{i+j}}{n_{i+j}} \\
 & = \sum_{k=0}^m \frac{|\mathbf{a}| + |\mathbf{c}|}{|\mathbf{a}| + |\mathbf{c}| + 2k} \binom{|\mathbf{a}| + |\mathbf{c}| + 2k}{k} (-4)^{m-k} \binom{-\lambda/2}{m - k}, \\
 (b) \quad & \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + 2n_i}{n_i} \prod_{j=1}^{\mu} \frac{c_j}{c_j + 2n_{i+j}} \binom{c_j + 2n_{i+j}}{n_{i+j}} \\
 & = \sum_{k=0}^m \binom{|\mathbf{a}| + |\mathbf{c}| + 2k}{k} (-4)^{m-k} \binom{\frac{1-\lambda}{2}}{m - k}.
 \end{aligned}$$

Proof. Letting $\beta = 2$ in (4), we can readily confirm these two formulae by manip-

ulating the generating function in the following two different manners:

$$\begin{aligned} S_m(\lambda, \mu|2, \mathbf{a}, \mathbf{c}) &= [x^m] \left(\frac{1+y}{1-y} \right)^\lambda (1+y)^{|\mathbf{a}|+|\mathbf{c}|} = [x^m] \frac{(1+y)^{|\mathbf{a}|+|\mathbf{c}|}}{\sqrt{(1-4x)^\lambda}} \\ &= [x^m] (1+y)^{|\mathbf{a}|+|\mathbf{c}|} \sum_{k=0}^m \binom{-\lambda/2}{k} (-4x)^k \\ &= \sum_{k=0}^m \binom{-\lambda/2}{k} (-4)^k [x^{m-k}] (1+y)^{|\mathbf{a}|+|\mathbf{c}|} \end{aligned}$$

and

$$\begin{aligned} S_m(\lambda, \mu|2, \mathbf{a}, \mathbf{c}) &= [x^m] \left(\frac{1+y}{1-y} \right)^{\lambda-1} \frac{(1+y)^{1+|\mathbf{a}|+|\mathbf{c}|}}{1-y} \\ &= [x^m] \frac{(1+y)^{1+|\mathbf{a}|+|\mathbf{c}|}}{(1-y)\sqrt{(1-4x)^{\lambda-1}}} \\ &= [x^m] \frac{(1+y)^{1+|\mathbf{a}|+|\mathbf{c}|}}{1-y} \sum_{k=0}^m \binom{\frac{1-\lambda}{2}}{k} (-4x)^k \\ &= \sum_{k=0}^m \binom{\frac{1-\lambda}{2}}{k} (-4)^k [x^{m-k}] \frac{(1+y)^{1+|\mathbf{a}|+|\mathbf{c}|}}{1-y}. \end{aligned}$$

□

For $\mu = 0$ (which implies that $|\mathbf{c}| = 0$), Proposition 4 gives the following reduced formulae.

Corollary 5. *Let $\lambda \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $a_i \in \mathbb{C}$. It holds:*

$$\begin{aligned} (a) \quad \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + 2n_i}{n_i} &= \sum_{k=0}^m (-4)^{m-k} \binom{-\lambda/2}{m-k} \binom{|\mathbf{a}| + 2k}{k} \frac{|\mathbf{a}|}{|\mathbf{a}| + 2k}, \\ (b) \quad \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{a_i + 2n_i}{n_i} &= \sum_{k=0}^m (-4)^{m-k} \binom{\frac{1-\lambda}{2}}{m-k} \binom{|\mathbf{a}| + 2k}{k}. \end{aligned}$$

Furthermore, when $|\mathbf{a}| \in \mathbb{Z} \setminus \mathbb{N}$, the above convolution admits yet a third expression.

Corollary 6. *Let $\lambda \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $a_i \in \mathbb{C}$ with $|\mathbf{a}| \in \mathbb{N}_0$. It holds:*

$$\sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \binom{2n_i - a_i}{n_i} = 2^{2m-|\mathbf{a}|} \sum_{k=0}^{|\mathbf{a}|} \binom{|\mathbf{a}|}{k} \binom{m-1+(\lambda+k-|\mathbf{a}|)/2}{m}.$$

U. Abel et al. [1, Theorem 1] found this formula by making use of differential operators. When $|\mathbf{a}| = 0$, the above formula recovers (2) due to Duarte and Guedes

de Oliveira [5, Theorem 2], as mentioned in the introduction. Here we present another proof by the generating function approach:

$$\begin{aligned}
 S_m(\lambda, 0|2, -\mathbf{a}, \mathbf{0}) &= [x^m] \frac{(1+y)^{\lambda-|\mathbf{a}|}}{(1-y)^\lambda} = [x^m] \left(\frac{1+y}{1-y}\right)^{\lambda-|\mathbf{a}|} (1-y)^{-|\mathbf{a}|} \\
 &= 2^{-|\mathbf{a}|} [x^m] \left(\frac{1+y}{1-y}\right)^{\lambda-|\mathbf{a}|} \left(1 + \frac{1+y}{1-y}\right)^{|\mathbf{a}|} \\
 &= 2^{-|\mathbf{a}|} \sum_{k=0}^{|\mathbf{a}|} \binom{|\mathbf{a}|}{k} [x^m] \left(\frac{1+y}{1-y}\right)^{\lambda-|\mathbf{a}|+k} \\
 &= 2^{-|\mathbf{a}|} \sum_{k=0}^{|\mathbf{a}|} \binom{|\mathbf{a}|}{k} [x^m] (1-4x)^{(|\mathbf{a}|-\lambda-k)/2} \\
 &= 2^{-|\mathbf{a}|} \sum_{k=0}^{|\mathbf{a}|} \binom{|\mathbf{a}|}{k} \binom{(|\mathbf{a}|-\lambda-k)/2}{m} (-4)^m.
 \end{aligned}$$

This is equivalent to the expression in Corollary 6. □

3. Multifold convolutions of Abel coefficients

Theorem 7. For $\lambda, \mu \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $a_i, c_j, \beta \in \mathbb{C}$, define the multifold convolution of Abel coefficients

$$T_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) := \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \frac{(a_i + \beta n_i)^{n_i}}{n_i!} \prod_{j=1}^{\mu} \frac{c_j}{c_j + \beta n_{\lambda+j}} \frac{(c_j + \beta n_{\lambda+j})^{n_{\lambda+j}}}{n_{\lambda+j}!}.$$

Then the following summation formulae hold:

$$\begin{aligned}
 (a) \quad T_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) &= \sum_{k=0}^m \binom{-\lambda}{k} (-\beta)^k \frac{\beta k + |\mathbf{a}| + |\mathbf{c}|}{\beta m + |\mathbf{a}| + |\mathbf{c}|} \frac{(\beta m + |\mathbf{a}| + |\mathbf{c}|)^{m-k}}{(m-k)!}, \\
 (b) \quad T_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) &= \sum_{k=0}^m \binom{1-\lambda}{k} (-\beta)^k \frac{(\beta m + |\mathbf{a}| + |\mathbf{c}|)^{m-k}}{(m-k)!}.
 \end{aligned}$$

Proof. Analogously, by making use of the following series for the Abel coefficients (see [3] and [11, §4.5] for an example)

$$\left. \begin{aligned}
 \sum_{n=0}^{\infty} \frac{\alpha}{\alpha + n\beta} \frac{(\alpha + n\beta)^n}{n!} x^n &= e^{y\alpha} \\
 \sum_{n=0}^{\infty} \frac{(\alpha + n\beta)^n}{n!} x^n &= \frac{e^{y\alpha}}{1 - y\beta}
 \end{aligned} \right\} \text{ where } x = ye^{-y\beta}, \tag{5}$$

we have the expression

$$T_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) = [x^m] \prod_{i=1}^{\lambda} \frac{e^{ya_i}}{1 - \beta y} \prod_{j=1}^{\mu} e^{yc_j} = [x^m] \frac{e^{y(|\mathbf{a}|+|\mathbf{c}|)}}{(1 - \beta y)^\lambda}. \tag{6}$$

The formulae in Theorem 7 can be proved by combining (5) with (6) as follows.

Formula (a) follows by rewriting (6) as

$$\begin{aligned} T_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) &= [x^m] \frac{e^{y(|\mathbf{a}|+|\mathbf{c}|)}}{(1-\beta y)^\lambda} \\ &= [x^m] e^{y(|\mathbf{a}|+|\mathbf{c}|)} \sum_{k=0}^{\infty} (-\beta)^k \binom{-\lambda}{k} y^k \\ &= [x^{m-k}] \sum_{k=0}^{\infty} (-\beta)^k \binom{-\lambda}{k} e^{y(|\mathbf{a}|+|\mathbf{c}|+k\beta)}. \end{aligned}$$

Formula (b) follows by expressing (6) alternatively as

$$\begin{aligned} T_m(\lambda, \mu|\beta, \mathbf{a}, \mathbf{c}) &= [x^m] \frac{e^{y(|\mathbf{a}|+|\mathbf{c}|)}}{1-\beta y} (1-\beta y)^{1-\lambda} \\ &= [x^m] \frac{e^{y(|\mathbf{a}|+|\mathbf{c}|)}}{1-\beta y} \sum_{k=0}^{\infty} (-\beta)^k \binom{1-\lambda}{k} y^k \\ &= [x^{m-k}] \sum_{k=0}^{\infty} (-\beta)^k \binom{1-\lambda}{k} \frac{e^{y(|\mathbf{a}|+|\mathbf{c}|+k\beta)}}{1-\beta y}. \end{aligned}$$

This complete the proof of Theorem 7. □

When $\lambda = 0$ (which implies that $|\mathbf{a}| = 0$) in Theorem 7, the two formulae there reduce to the following same closed formula.

Proposition 8. *Let $\mu \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $c_j, \beta \in \mathbb{C}$. It holds:*

$$\sum_{|\mathbf{n}|=m} \prod_{j=1}^{\mu} \frac{c_j}{c_j + \beta n_j} \frac{(c_j + \beta n_j)^{n_j}}{n_j!} = \frac{|\mathbf{c}|}{\beta m + |\mathbf{c}|} \frac{(\beta m + |\mathbf{c}|)^m}{m!}.$$

Instead, when $\mu = 0$ (which implies that $|\mathbf{c}| = 0$) in Theorem 7, we have the two reduced formulae below.

Proposition 9. *Let $\lambda \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $a_i, \beta \in \mathbb{C}$. It holds:*

$$\begin{aligned} (a) \quad \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \frac{(a_i + \beta n_i)^{n_i}}{n_i!} &= \sum_{k=0}^m \binom{-\lambda}{k} (-\beta)^k \frac{\beta k + |\mathbf{a}|}{\beta m + |\mathbf{a}|} \frac{(\beta m + |\mathbf{a}|)^{m-k}}{(m-k)!} \\ (b) \quad \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} \frac{(a_i + \beta n_i)^{n_i}}{n_i!} &= \sum_{k=0}^m \binom{1-\lambda}{k} (-\beta)^k \frac{(\beta m + |\mathbf{a}|)^{m-k}}{(m-k)!}. \end{aligned}$$

4. Alternating multisums of binomial coefficients

For binomial convolution (1), there is the following alternating counterpart:

$$\sum_{k=0}^m (-1)^k \binom{2k}{k} \binom{2(m-k)}{m-k} = \begin{cases} 2^m \binom{m}{m/2}, & m \text{ even;} \\ 0, & m \text{ odd.} \end{cases} \tag{7}$$

Recently, Spivey [10] gave a combinatorial proof by applying an involution to certain colored permutations. We offer a generating function proof by employing the following series for central binomial coefficients:

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}. \tag{8}$$

In fact, it is routine to check that

$$\sum_{k=0}^m (-1)^k \binom{2k}{k} \binom{2(m-k)}{m-k} = [x^m] \frac{1}{\sqrt{1-16x^2}} = [x^m] \sum_{k=0}^{\infty} (-16)^k \binom{-\frac{1}{2}}{k} x^{2k},$$

which becomes the right member of (7) in view of the binomial relation

$$(-16)^{\frac{m}{2}} \binom{-\frac{1}{2}}{\frac{m}{2}} = 2^m \binom{m}{\frac{m}{2}}.$$

In general, define the alternating multisums of binomial coefficients

$$\Omega_m(\lambda, \mu) = \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} (-1)^{n_i} \binom{2n_i}{n_i} \prod_{j=1}^{\mu} \binom{2n_{\lambda+j}}{n_{\lambda+j}}.$$

By means of (8), we can first evaluate $\Omega_m(\lambda, \lambda)$ as follows:

$$\begin{aligned} \Omega_m(\lambda, \lambda) &= [x^m] \left(\frac{1}{\sqrt{1-16x^2}} \right)^{\lambda} \\ &= [x^m] \sum_{k=0}^{m/2} (-16)^k \binom{-\lambda/2}{k} x^{2k}, \end{aligned}$$

which results in the following closed formula.

Lemma 10. *Let $\lambda \in \mathbb{N}$ and $m \in \mathbb{N}_0$. It holds:*

$$\Omega_m(\lambda, \lambda) = \sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} (-1)^{n_i} \binom{2n_i}{n_i} \binom{2n_{\lambda+i}}{n_{\lambda+i}} = \begin{cases} (-16)^{\frac{m}{2}} \binom{-\frac{\lambda}{2}}{\frac{m}{2}}, & m \text{ even;} \\ 0, & m \text{ odd.} \end{cases}$$

We can further evaluate the multisum $\Omega_m(\lambda, \mu)$ even for $\lambda \neq \mu$. Because of the reciprocity $\Omega_m(\lambda, \mu) = (-1)^m \Omega_m(\mu, \lambda)$, without loss of generality, we can assume that $\lambda \geq \mu$. By applying (8), we get the generating function

$$\begin{aligned} \Omega_m(\lambda, \mu) &= [x^m] \left(\frac{1}{\sqrt{1+4x}} \right)^{\lambda} \left(\frac{1}{\sqrt{1-4x}} \right)^{\mu} \\ &= [x^m] (1-16x^2)^{-\frac{\mu}{2}} (1+4x)^{\frac{\mu-\lambda}{2}}. \end{aligned}$$

Recalling Lemma 10, we can express $\Omega_m(\lambda, \mu)$ as the following binomial convolution.

Theorem 11. *Let $\lambda \in \mathbb{N}$ and $m \in \mathbb{N}_0$. It holds:*

$$\sum_{|\mathbf{n}|=m} \prod_{i=1}^{\lambda} (-1)^{n_i} \binom{2n_i}{n_i} \prod_{j=1}^{\mu} \binom{2n_{\lambda+j}}{n_{\lambda+j}} = 4^m \sum_{0 \leq k \leq m/2} (-1)^k \binom{-\frac{\mu}{2}}{k} \binom{\frac{\mu-\lambda}{2}}{m-2k}.$$

When $\lambda = \mu$, this returns obviously to the formula displayed in Lemma 10.

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