Economic Research-Ekonomska Istraživanja

ISSN: 1331-677X (Print) 1848-9664 (Online) Journal homepage: https://www.tandfonline.com/loi/rero20

The generalized dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information

Guiwu Wei

To cite this article: Guiwu Wei (2019) The generalized dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information, Economic Research-Ekonomska Istraživanja, 32:1, 1498-1520, DOI: 10.1080/1331677X.2019.1637765

To link to this article: https://doi.org/10.1080/1331677X.2019.1637765

ECONOMIC RESEARCH-EKONOMSKA ISTRAŽIVANJA
2019, VOL. 32, NO. 1, 1498–1520
https://doi.org/10.1080/1331677X.2019.1637765

Routledge
Taylor & Francis Group

OPEN ACCESS Check for updates

The generalized dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information

Guiwu Wei

School of Business, Sichuan Normal University, Chengdu, P. R. China

ABSTRACT

In this paper, we shall present some novel Dice similarity measures of hesitant fuzzy linguistic term sets and the generalized Dice similarity measures of hesitant fuzzy linguistic term sets and indicate that the Dice similarity measures and asymmetric measures (projection measures) are the special cases of the generalized Dice similarity measures in some parameter values. Then, we propose the generalized Dice similarity measures-based multiple attribute decision making models with hesitant fuzzy linguistic term sets. Finally, a practical example concerning the evaluation of the quality of movies is given to illustrate the applicability and advantage of the proposed generalized Dice similarity measures.

ARTICLE HISTORY

Received 24 May 2018
Accepted 2 October 2018

KEYWORDS

Multiple attribute decision making; generalized dice similarity measures; dice similarity measures; hesitant fuzzy linguistic term sets; asymmetric measures; projection measures; quality of movies

Introduction

Multiple attribute decision making problems under linguistic information processing environment is an interesting research topic having received more and more attention during the last several years (Beg & Rashid, 2015; Dutta & Guha, 2015; Herrera, Herrera-Viedma 2000a-b; Herrera, Martínez 2001a-b; Herrera, Martínez, & Sánchez, 2005; Hu, Rao, Zheng, & Huang, 2015; Martínez-López, Rodríguez, R, & Herrera, 2015; Rao, Zheng, Wang, & Xiao, 2016; Wu et al. 2015; Zhang & Chu, 2009; Zhang & Liu, 2010). Herrera and Martínez (2001a) show 2-tuple linguistic information processing manner can effectively avoid the loss and distortion of information. Herrera, Herrera-Viedma (2000a) developed 2-tuple arithmetic average (TAA) operator, 2-tuple weighted average (TWA) operator, 2-tuple ordered weighted average (TOWA) operator and extended 2-tuple weighted average (ET-WA) operator. Herrera-Viedma, Martinez, Mata, and Chiclana (2005) proposed the consensus support system with multi-granular linguistic information. Herrera et al. (2005) presented the group decision making model for managing non-homogeneous information. Herrera, Herrera-Viedma, and Martínez (2008) developed the fuzzy linguistic model

CONTACT Guiwu Wei weiguiwu@163.com

to solve the unbalanced linguistic term sets. Fan, Feng, Sun, and Ou (2009) evaluated the knowledge management capability of organizations by using a fuzzy linguistic method. Tai and Chen (2009) evaluated the intellectual capital with linguistic variables. Wang (2009) selected the agile manufacturing system with 2-tuple fuzzy linguistic information. Fan and Liu (2010) developed the multi-granularity uncertain linguistic group decision making model. Martínez & Herrera, (2012)gave an overview on the 2-tuple linguistic model for Computing with Words in Decision Making. Rodríguez & Martínez,(2013) overviewed the symbolic linguistic computing models that have been widely used in linguistic decision making to analyze if all of them can be considered inside of the computing with words paradigm. Liu, Lin, and Wu (2014) defined the dependent interval 2-tuple linguistic aggregation operators for multiple attribute group decision making. Xu, Ma, Tao, and Wang (2014) proposed some models to solve the unacceptable incomplete 2-tuple fuzzy linguistic preference relations. Estrella, Espinilla, Herrera, and Martínez (2014) proposed a fuzzy linguistic decision tools enhancement suite based on the 2-tuple linguistic model and extensions. Wang, Wang, Zhang, and Chen (2015) developed the multi-criteria group decision making method with interval 2-tuple linguistic information and Choquet integral aggregation operators. Dong and Herrera-Viedma (2015) proposed the consistency-driven automatic methodology in the linguistic GDM with preference relation. Dutta, Guha, and Mesiar (2015) solved the heterogeneous relationship among attributes in multi-expert decision making based on linguistic 2-tuples. Qin and Liu (2016) proposed the 2-tuple linguistic Muirhead mean operators for multiple attribute group decision making. Zhang, Xu, and Wang (2016) developed the consensus reaching model for 2-tuple linguistic multiple attribute group decision making with incomplete weight information. Zhou et al. (2017) studied the performance evaluation of experiment platforms with 2-tuple linguistic information. Yang (2017) proposed the model for evaluating the visual design quality with 2-tuple linguistic information. Yao and Khalid (2018) completed 2-tuple linguistic preference relations based on upper bound condition. Zhao et al. (2018) proposed a new emergency decision support methodology based on multi-source knowledge in 2-tuple linguistic model.

The similarity measure is one of the important and useful tools for degree of similarity between objects (Hung, 2012; Hung & Yang, 2004, Hung & Yang, 2007; Li, Olson, & Zheng, 2007; Liao, Xu, & Zeng, 2014; Liao & Xu, 2015; Liu, 2005; Rajarajeswari &Uma, 2013; Shi & Ye, 2013; Singh, 2014; Su, Xu, Liu, & Liu, 2015; Szmidt, 2014; Szmidt & Kacprzyk, 2000; Tian, 2013; Wei et al., 2017a, 2018a; Wei & Gao, 2018; Wei & Wei, 2018; Xu & Xia, 2010; Ye, 2011, 2016b, 2017). Functions expressing the degree of similarity of items or sets are used in physical anthropology, automatic classification, ecology, psychology, citation analysis, information retrieval, patterns recognition and numerical taxonomy (Ye, 2012a). The degree of similarity or dissimilarity between the objects under study plays an important role. In vector space, especially the Jaccard, Dice, and cosine similarity measures (Dice, 1945; Jaccard, 1901; Salton & McGill, 1987) are often used in information retrieval, citation analysis, and automatic classification. Therefore, Ye (2012a) proposed the Jaccard, Dice, and cosine similarity measures between trapezoidal intuitionistic fuzzy numbers (TIFNs) and applied them to group decision-making problems. Ye (2012b) proposed the multi-criteria decision making models by using the Dice similarity measure between expected intervals of trapezoidal fuzzy numbers. Ye (2012c) investigated the multicriteria decision-making method by using the Dice similarity measure based on the reduct intuitionistic fuzzy sets of IVIFSs. Ye (2014) developed the Dice measures for simplified neutrosophic sets. Ye (2016a) proposed the generalized Dice measures for multiple attribute decision making under simplified neutrosophic environments.

However, these Dice similarity measures do not deal with the similarity measures for hesitant fuzzy linguistic term sets (HFLTSs) directly. Therefore, it is necessary to extend the Dice measure to HFLTSs to handle patterns recognition, citation analysis, information retrieval and multiple attribute decision making problems to satisfy the requirements of decision makers' preference and flexible decision making. In order to do so, the main purposes of this paper are: 1) to propose two forms of the Dice measures of HFLTSs, 2) to present the generalized Dice measures of HFLTSs, and 3) to develop the generalized Dice measures-multiple attribute decision making (MADM) methods of HFLTSs. In the MADM process, the main advantage of the proposed methods is more general and more flexible than existed MADM methods with HFLTSs to satisfy the practical requirements.

In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to HFLTSs. In Section 3, we shall propose some Dice similarity measure and some weighted Dice similarity measure between HFLTSs. In Section 4, we propose the generalized Dice similarity measures-based MADM models with HFLTSs in Section 5, an illustrative example is given to demonstrate the efficiency of the similarity measures for concerning the evaluation of the quality of movies. Section 6 concludes the paper with some remarks.

Preliminaries

Let $S = \{s_i | i = -\tau, ..., -1, 0, 1, ..., \tau\}$ be a linguistic term set with odd cardinality. Any label,s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics (Herrera, Martínez & Sánchez, 2005; Herrera & Martínez, 2000a, 200 b; 2001a-b; Xu, 2004a, 2006):

1. The set is ordered:$s_i > s_j$, if $i > j$; (2) Max operator:$\max(s_i, s_j) = s_i$, if $s_i \geq s_j$; (3) Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$. For example, S can be defined as

$$S = \{s_{-3} = extremelypoor, s_{-2} = verypoor, s_{-1} = poor, s_0 = medium,$$
$$s_1 = good, s_2 = verygood, s_3 = extremelygood\}$$

Hesitant fuzzy sets, which permit the membership degree of an element to a reference set represented by several possible values, is a powerful structure in reflecting a decision maker's hesitance.

Definition 1 (Torra, 2010). Given a fixed set X, then a hesitant fuzzy sets (HFSs) on X is in terms of a function that when applied to X returns a subset of $[0, 1]$. A hesitant fuzzy set (HFS) can be expressed the HFS by the mathematical symbol:

$$H_E = \left\{ \langle x, h_E(x) \rangle | x \in X \right\}, \tag{1}$$

where $h_E(x)$ is a set of some values in$[0, 1]$, denoting the possible membership degree of the element $x \in X$ to the set E. For convenience, Xia and Xu (2011) call $h = h_E(x)$ a hesitant fuzzy element (HFE) and H_E the set of all HFEs.

Similar to the situations of HFSs where a decision maker may hesitate between several possible values as the membership degree when evaluating an alternative, in a qualitative circumstance, a decision maker may hesitate between several terms to assess a linguistic variable. Hence, motivated by the idea of HFSs, Rodríguez, Martínez, and Herrera (2012) introduced the hesitant fuzzy linguistic term set (HFLTS), whose envelope is an uncertain linguistic variable (Xu et al., 2014).

Definition 2 (Rodríguez et al., 2012). Let $S = \{s_i | i = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set, a hesitant fuzzy linguistic term sets (HFLTSs),H_S, is an ordered finite subset of the consecutive linguistic terms of S.

Let $S = \{s_i | i = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. The HFLTSH_S for a linguistic variable $\nu \in S$ can then be represented mathematically as $H_S(\nu)$. For the convenience of statement, we call $\eta = \{H_S(\nu) | \nu \in S\}$ a set of HFLTSs. The aim of introducing HFLTS is to improve the elicitation of linguistic information, mainly when decision makers hesitate between several values in assessing linguistic variables. Linguistic information, which is more similar to the decision makers' expressions, is semantically represented by HFLTS and generated by a context-free grammar (Rodríguez et al., 2012).

Definition 3 (Rodríguez et al., 2012). For three HFLTSsH_S, H_S^1 and H_S^2, the following operations are defined:

1. Lower bound: $h_S^- = \min(s_i) = s_j, s_i \in h_s ands_i \geq s_j, \forall i;$
2. Upper bound: $h_S^+ = \max(s_i) = s_j, s_i \in h_s ands_i \leq s_j, \forall i.;$
3. Complement operation:$H_S^c = S - H_S = \{s_i | s_i \in Sands_i \notin H_S\}.$
4. Union operation: $H_S^1 \cup H_S^2 = \{s_i | s_i \in H_S^1 ors_i \in H_S^2\}.$
5. Intersection operation:$H_S^1 \cap H_S^2 = \{s_i | s_i \in H_S^1 ands_i \in H_S^2\}.$

Some dice similarity measure for hesitant fuzzy linguistic information

The Dice similarity measure can't induce this undefined situation when one vector is zero, which overcomes the disadvantage of the cosine similarity measure (Dice, 1945). Therefore, the concept of the Dice similarity measure is introduced in the section.

Definition 4 (Dice, 1945). Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two vectors of length n where all the coordinates are positive real numbers. Then the Dice similarity measure is defined as follows:

$$D(X, Y) = \frac{2X \cdot Y}{\|X\|_2^2 + \|Y\|_2^2} = \frac{2\sum_{j=1}^{n} x_j y_j}{\sum_{j=1}^{n} (x_j)^2 + \sum_{j=1}^{n} (y_j)^2} \tag{2}$$

where $X \cdot Y = \sum_{j=1}^{n} x_j y_j$ is called the inner product of the vector X and Y and $\|X\|_2 = \sqrt{\sum_{j=1}^{n} (x_j)^2}$and $\|Y\|_2 = \sqrt{\sum_{j=1}^{n} (y_j)^2}$ are the Euclidean norms of X and Y(also called the L_2 norms.

The Dice similarity measure takes value in the interval$[0, 1]$. However, it is undefined if $x_j = y_j = 0 (j = 1, 2, \ldots, n)$. In this case, let the Dice measure value be zero when $x_j = y_j = 0 (j = 1, 2, \ldots, n)$.

Dice similarity measure for hesitant fuzzy linguistic information

In this section, we shall propose some Dice similarity measure and some weighted Dice similarity measure between HFLTSs based on the concept of the Dice similarity measure (Dice, 1945).

Definition 5. Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. For two HFLTSs $H_S^1 = \{\langle x_j, h_S^1(x_j)\rangle | x_j \in X\}$ and $H_S^2 = \{\langle x_j, h_S^2(x_j)\rangle | x_j \in X\}$ with $h_S^2(x_j) = \{s_{\delta_l^k}(x_j) | s_{\delta_l^k}(x_j) \in S, l = 1, 2, \ldots, L_i\}$,$k = 1, 2$, a Dice similarity measure between HFLTSs H_S^1 and H_S^2 is proposed as follows:

$$D^1\left(H_S^1, H_S^2\right) = \frac{1}{n} \sum_{j=1}^{n} \frac{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \tag{3}$$

The Dice similarity measure between HFLTSs H_S^1 and H_S^2 also satisfies the following properties:

1. $0 \le D^1(H_S^1, H_S^2) \le 1$;
2. $D^1(H_S^1, H_S^2) = D^1(H_S^2, H_S^1)$;
3. $D^1(H_S^1, H_S^2) = 1, if H_S^1 = H_S^2$, i.e. $s_{\delta_l^1}(x_j) = s_{\delta_l^2}(x_j)$, $j = 1, 2, \ldots, n$.

Proof.

1. Let us consider the jth item of the summation in Eq.(3).

$$D^1\left(H_S^1(x_j), H_S^2(x_j)\right) = \frac{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}$$

It is obvious that$D^1(H_S^1(x_j), H_S^2(x_j)) \ge 0$,and

$$\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2 \ge 2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)$$

according to the inequality $a^2 + b^2 \ge 2ab$. Thus, $0 \le D^1(H_S^1(x_j), H_S^2(x_j)) \le 1$.

From Eq.(3), the summation of n terms is $0 \le D^1(H_S^1, H_S^2) \le 1$.

1. It is obvious that the proposition is true.
2. When $H_S^1 = H_S^2$, there are $s_{\delta_l^1}(x_j) = s_{\delta_l^2}(x_j)$, for $j = 1, 2, ..., n$. So, there is

$$D^1\left(H_S^1, H_S^2\right) = \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}$$
$$= \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2} .$$
$$= \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2}{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2} = 1$$

Therefore, we have finished the proofs.

If we consider the weights x_j, a weighted Dice similarity measure between HFLTSs H_S^1 and H_S^2 is proposed as follows:

$$WD^1\left(H_S^1, H_S^2\right) = \sum_{j=1}^{n} \omega_j \frac{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \tag{4}$$

where$\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of $x_j (j = 1, 2, ..., n)$,with $w_j \in [0, 1]$,$i = 1, 2, ..., n$,$\sum_{j=1}^{n} w_j = 1$. In particular, if $\omega = (1/n, 1/n, ..., 1/n)^T$, then the weighted Dice similarity measure reduces to Dice similarity measure. That is to say, if we take $\omega_j = \frac{1}{n}, j = 1, 2 \cdots, n$, then there is $WD^1(H_S^1, H_S^2) = D^1(H_S^1, H_S^2)$.

Obviously, the weighted Dice similarity measure of between two HFLTSs H_S^1 and H_S^2 also satisfies the following properties:

1. $0 \le WD^1(H_S^1, H_S^2) \le 1$;
2. $WD^1(H_S^1, H_S^2) = WD^1(H_S^2, H_S^1)$;
3. $WD^1(H_S^1, H_S^2) = 1, if H_S^1 = H_S^2$, i.e. $s_{\delta_l^1}(x_j) = s_{\delta_l^2}(x_j)$, $j = 1, 2, ..., n$.

Similar to the previous proof method, we can prove the above three properties.

Another form of the dice similarity measure for hesitant fuzzy linguistic information

In this section, we shall develop another form of Dice similarity measure for hesitant fuzzy linguistic information, which is defined as follows:

Definition 6. Let$S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. For two HFLTSs $H_S^1 = \{\langle x_j, h_S^1(x_j)\rangle | x_j \in X\}$ and $H_S^2 = \{\langle x_j, h_S^2(x_j)\rangle | x_j \in X\}$ with $h_S^2(x_j) = \{s_{\delta_l^k}(x_j) | s_{\delta_l^k}(x_j) \in S, l = 1, 2, \ldots, L_i\}$,$k = 1, 2$, a Dice similarity measure between HFLTSsH_S^1 and H_S^2 is proposed as follows:

$$D^2\left(H_S^1, H_S^2\right) = \frac{\sum_{j=1}^{n} 2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\sum_{j=1}^{n}\left(\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2\right)} \tag{5}$$

The Dice similarity measure between HFLTSsH_S^1 and H_S^2 also satisfies the following properties:

1. $0 \le D^2(H_S^1, H_S^2) \le 1$;
2. $D^2(H_S^1, H_S^2) = D^2(H_S^2, H_S^1)$;
3. $D^2(H_S^1, H_S^2) = 1, if H_S^1 = H_S^2$, i.e. $s_{\delta_l^1}(x_j) = s_{\delta_l^2}(x_j)$, $j = 1, 2, \ldots, n$.

Similar to the previous proof method, we can prove the above three properties.

If we consider the weights x_j, a weighted Dice similarity measure between HFLTSs H_S^1 and H_S^2 is proposed as follows:

$$WD^2\left(H_S^1, H_S^2\right) = \frac{\sum_{j=1}^{n} 2\omega_j^2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\sum_{j=1}^{n} \omega_j^2\left(\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2\right)} \tag{6}$$

where$\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j (j = 1, 2, \ldots, n)$,with $w_j \in [0, 1]$,$i = 1, 2, \ldots, n$,$\sum_{j=1}^{n} w_j = 1$. In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the weighted Dice similarity measure reduces to Dice similarity measure. That is to say, if we take $\omega_j = \frac{1}{n}, j = 1, 2 \cdots, n$, then there is $WD^2(\phi, \varphi) = D^2(\phi, \varphi)$.

Obviously, the weighted Dice similarity measure of between two HFLTSs H_S^1 and H_S^2 also satisfies the following properties:

1. $0 \le WD^2(H_S^1, H_S^2) \le 1$;
2. $WD^2(H_S^1, H_S^2) = WD^2(H_S^2, H_S^1)$;
3. $WD^2(H_S^1, H_S^2) = 1, if H_S^1 = H_S^2$, i.e. $s_{\delta_l^1}(x_j) = s_{\delta_l^2}(x_j)$, $j = 1, 2, \ldots, n$.

The generalized dice similarity measure for hesitant fuzzy linguistic information

In this section, we develop the generalized Dice similarity measure between two HFLTSsH_S^1 and H_S^2. As the generalization of the Dice similarity measure, the generalized Dice similarity measures between two HFLTSs H_S^1 and H_S^2 are defined below.

Definition 7. Let$\phi = \{(\gamma_1, \alpha_1), (\gamma_2, \alpha_2), \ldots, (\gamma_n, \alpha_n)\}$ and $\varphi = \{(\eta_1, \beta_1), (\eta_2, \beta_2), \ldots, (\eta_n, \beta_n)\}$ be two groups of HFLTSs, a generalized Dice similarity measure between two HFLTSs H_S^1 and H_S^2 is proposed as follows:

$$GD^1\left(H_S^1, H_S^2\right) = \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-\lambda)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \tag{7}$$

$$GD^2\left(H_S^1, H_S^2\right) = \frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-\lambda)\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \tag{8}$$

where λ is a positive parameter for $0 \leq \lambda \leq 1$.

Then, the generalized Dice similarity measure includes some special cases by altering the parameter value λ.

If $\lambda = 0.5$, the two generalized Dice similarity measures (7) and (8) reduced to Dice similarity measures (3) and (5):

$$\begin{aligned} &GD^1\left(H_S^1, H_S^2\right) \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-\lambda)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{0.5\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-0.5)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \end{aligned} \tag{9}$$

$$\begin{aligned}
&GD^2\left(H_S^1, H_S^2\right)\\
&= \frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-\lambda)\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&= \frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{0.5\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-0.5)\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&= \frac{2\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}
\end{aligned} \tag{10}$$

If $\lambda = 0, 1$, the two generalized Dice similarity measures reduced to the following asymmetric similarity measures, respectively:

$$\begin{aligned}
&GD^1\left(H_S^1, H_S^2\right)\\
&= \frac{1}{n}\sum_{j=1}^{n}\frac{\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-\lambda)\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&= \frac{1}{n}\sum_{j=1}^{n}\frac{\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{0\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-0)\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&= \frac{1}{n}\sum_{j=1}^{n}\frac{\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}, for \lambda = 0.
\end{aligned} \tag{11}$$

$$\begin{aligned}
&GD^1\left(H_S^1, H_S^2\right)\\
&=\frac{1}{n}\sum_{j=1}^{n}\frac{\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-\lambda)\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&=\frac{1}{n}\sum_{j=1}^{n}\frac{\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{1\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-1)\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&=\frac{1}{n}\sum_{j=1}^{n}\frac{\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2}, for \lambda=1.
\end{aligned}\tag{12}$$

$$\begin{aligned}
&GD^2\left(H_S^1, H_S^2\right)\\
&=\frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-\lambda)\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&=\frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{0\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-0)\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&=\frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}, for \lambda=0
\end{aligned}\tag{13}$$

$$
\begin{aligned}
&GD^2\left(H_S^1, H_S^2\right)\\
&= \frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-\lambda)\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&= \frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{1\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-1)\sum_{j=1}^{n}\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}\\
&= \frac{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\sum_{j=1}^{n}\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2}, for \lambda = 1.
\end{aligned} \tag{14}
$$

From above analysis, it can be seen that the above four asymmetric similarity measures are the extension of the relative projection measure of the HFLTSs.

In many situations, the weight of the elements $x_j \in X$ should be taken into account. For example, in multiple attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights. Thus, we further propose the following two weighted generalized Dice similarity measures for HFLTSs, respectively, as follows:

$$
WGD^1\left(H_S^1, H_S^2\right) = \sum_{j=1}^{n}\omega_j\frac{\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-\lambda)\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \tag{15}
$$

$$
WGD^2\left(H_S^1, H_S^2\right) = \frac{\sum_{j=1}^{n}\omega_j^2\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\cdot\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\sum_{j=1}^{n}\omega_j^2\left(\sum_{l=1}^{L_1}\frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2+(1-\lambda)\sum_{j=1}^{n}\omega_j^2\left(\sum_{l=1}^{L_2}\frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \tag{16}
$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of $x_j (j = 1, 2, ..., n)$,with $\omega_j \in [0, 1]$,$j = 1, 2, ..., n$,$\sum_{j=1}^{n}\omega_j = 1$. In particular, if $\omega = (1/n, 1/n, ..., 1/n)^T$, then the weighted generalized Dice similarity measures reduce to generalized Dice similarity measures. That is to say, if we take $\omega_j = \frac{1}{n}, j = 1, 2\cdots, n$, then there is $WGD^k_{2TLV}(\phi, \varphi) = GD^k_{2TLV}(\phi, \varphi)(k = 1, 2)$.

Then, the weighted generalized Dice similarity measure includes some special cases by altering the parameter value λ.

If $\lambda = 0.5$, the two weighted generalized Dice similarity measures (15) and (16) reduced to weighted Dice similarity measures (4) and (6):

$$
\begin{aligned}
&WGD^1\left(H_S^1, H_S^2\right) \\
&= \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-\lambda)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\
&= \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{0.5 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-0.5)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\
&= \sum_{j=1}^{n} \omega_j \frac{2\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&WGD^2\left(H_S^1, H_S^2\right) \\
&= \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-\lambda)\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\
&= \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{0.5 \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-0.5)\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\
&= \frac{2\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}
\end{aligned}
\tag{18}
$$

If $\lambda = 0, 1$, the two weighted generalized Dice similarity measures reduced to the following asymmetric weighted similarity measures, respectively:

$$
\begin{aligned}
& WGD^1\left(H_S^1, H_S^2\right) \\
&= \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-\lambda)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\
&= \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{0\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-0)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\
&= \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2}, for \lambda = 0.
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
& WGD^1\left(H_S^1, H_S^2\right) \\
&= \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\lambda\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-\lambda)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\
&= \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{1\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2 + (1-1)\left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)^2} \\
&= \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1}\right)}{\left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1}\right)^2}, for \lambda = 1.
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&WGD^2\left(H_S^1, H_S^2\right)\\
&= \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)}{\lambda \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \right)^2 + (1-\lambda) \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)^2}\\
&= \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)}{0 \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \right)^2 + (1-0) \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)^2}\\
&= \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)}{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)^2}, for \lambda = 0.
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&WGD^2\left(H_S^1, H_S^2\right)\\
&= \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)}{\lambda \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \right)^2 + (1-\lambda) \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)^2}\\
&= \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)}{1 \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \right)^2 + (1-1) \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)^2}\\
&= \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \cdot \sum_{l=1}^{L_2} \frac{|\delta_l^2(x_j)|}{2\tau+1} \right)}{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{L_1} \frac{|\delta_l^1(x_j)|}{2\tau+1} \right)^2}, for \lambda = 1.
\end{aligned}
\tag{22}
$$

From above analysis, it can be seen that the above four asymmetric weighted similarity measures are the extension of the relative weighted projection measure of the HFLTSs.

The generalized dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information

In this section, we shall extend the generalized Dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information. Let $A =$

$\{A_1, A_2, ..., A_m\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, ..., G_n\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weighting vector of the attributes $G_j(j = 1, 2, ..., n)$, where $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$. Suppose that $\tilde{H} = (H_S^{ij})_{m\times n}$ is the hesitant fuzzy linguistic decision matrix, where $H_S^{ij} = \cup_{S_{\delta_l^{ij}} \in H_S^{ij}} \{s_{\delta_l^{ij}} | l = 1, 2, ..., H_S^{ij}\}(i = 1, 2, ..., m; j = 1, 2, ..., n)$ is the hesitant fuzzy linguistic values, which take the form of HFLTSs, given by the decision maker for the alternative $A_i \in A$ with respect to the attribute $G_j \in G$.

Then, in the following, we shall develop an algorithm to utilize the generalized Dice similarity measures to solve the multiple attribute decision making with hesitant fuzzy linguistic information.

Step 1. Defining the hesitant fuzzy linguistic positive ideal solution (HFLPIS)A^+ as

$$A^+ = \left(H_S^{1+}, H_S^{2+}, \mathrm{L}, H_S^{n+}\right) \tag{23}$$

where

$$H_S^{j+} = \begin{cases} \max\limits_{i=1,\ldots,m} H_S^{ij} = \max\limits_{\substack{i=1,\ldots,m \\ l=1,\ldots,H_S^{ij}}} \left\{s_{\delta_l^{ij}}\right\}, \text{for benefit attribute } G_j \\ \min\limits_{i=1,\ldots,m} H_S^{ij} = \min\limits_{\substack{i=1,\ldots,m \\ l=1,\ldots,H_S^{ij}}} \left\{s_{\delta_l^{ij}}\right\}, \text{for cost attribute } G_j \end{cases}, \quad j = 1, 2, \mathrm{L}, n. \tag{24}$$

Note that the hesitant fuzzy linguistic positive ideal solution A^+ is linguistic term sets. Hence, they certainly can be taken as special HFLTSs with only one linguistic term in each HFLTS.

Step 2. Calculating the weighted generalized Dice similarity measures between $A_i(i = 1, 2, ..., m)$ and A^+ as follows:

$$WGD^1\left(A_i, A^+\right) = \sum_{j=1}^{n} \omega_j \frac{\left(\sum_{l=1}^{H_S^{ij}} \frac{|\delta_l^{ij}|}{2\tau+1} \cdot \frac{\max\limits_{\substack{i=1,\ldots,m \\ l=1,\ldots,H_S^{ij}}} \left\{\delta_l^{ij}\right\}}{2\tau+1}\right)}{\lambda\left(\sum_{l=1}^{H_S^{ij}} \frac{|\delta_l^{ij}|}{2\tau+1}\right)^2 + (1-\lambda)\left(\frac{\max\limits_{\substack{i=1,\ldots,m \\ l=1,\ldots,H_S^{ij}}} \left\{\delta_l^{ij}\right\}}{2\tau+1}\right)^2} \tag{25}$$

or

$$WGD^2(A_i, A^+) = \frac{\sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{H_S^{ij}} \frac{|\delta_l^{ij}|}{2\tau+1} \cdot \frac{\max\limits_{\substack{i=1,\dots,m \\ l=1,\dots,H_S^{ij}}} \left\{ \delta_l^{ij} \right\}}{2\tau+1} \right)}{\lambda \sum_{j=1}^{n} \omega_j^2 \left(\sum_{l=1}^{H_S^{ij}} \frac{|\delta_l^{ij}|}{2\tau+1} \right)^2 + (1-\lambda) \sum_{j=1}^{n} \omega_j^2 \left(\frac{\max\limits_{\substack{i=1,\dots,m \\ l=1,\dots,H_S^{ij}}} \left\{ \delta_l^{ij} \right\}}{2\tau+1} \right)^2} \tag{26}$$

Step 3. Rank all the alternatives $A_i (i = 1, 2, \dots, m)$ and select the best one(s) in accordance with the weighted generalized Dice similarity measures $WGD^1(A_i, A^+)(WGD^2(A_i, A^+))$ $(i = 1, 2, \dots, m)$. If any alternative has the highest $WGD^1(A_i, A^+)(WGD^2(A_i, A^+))$ value, then, it is the most important alternative.
Step 4. End.

Numerical example and comparative analyses

In this section, we consider a movie recommender system (adapted from Liao et al., 2014) to demonstrate the efficiency of the proposed generalized Dice similarity measures. Suppose that a company intends to give ratings on five movies$A_i (i = 1, 2, \dots, 5)$ with respect to four attributes: story (G_1), acting (G_2), visuals (G_3) and direction (G_4). The weighing vector of these four attributes is:$\omega = (0.4, 0.2, 0.2, 0.2)$. The ratings provide information about the quality of the movies as well as the taste of the users who give the ratings. Since these criteria are all qualitative, it is convenient and only feasible for the decision makers to express their feelings by using linguistic terms. As pointed out by Miller (1956), most decision makers cannot handle more than nine factors when making their decision. Hence, the company constructs a seven-point linguistic scale to assess the movies, which is

$$S = \{s_{-3} = extremelypoor, s_{-2} = verypoor, s_{-1} = poor, s_0 = medium, \\ s_1 = good, s_2 = verygood, s_3 = extremelygood\}$$

The five possible movies $A_i (i = 1, 2, \dots, 5)$ are to be evaluated using the linguistic sets S by the the decision makers under the above four attributes, and construct the hesitant fuzzy linguistic decision matrix as follows $\tilde{H} = (H_S^{ij})_{5\times4}$ in Table 1.

In the following, we shall utilize the proposed approach in this paper getting the most desirable movies

Step 1. Defining the hesitant fuzzy linguistic positive ideal solution (HFLPIS) A^+ as

$$A^+ = (\{s_3\}, \{s_3\}, \{s_3\}, \{s_2\})^T$$

Step 2. According to Equations (25) and (26) and different values of the parameter λ, the weighted generalized Dice measure values between $A_i (i = 1, 2, 3, 4, 5)$ and A^+ can be obtained, which are shown in Tables 2 and 3, respectively.

Table 1. The hesitant fuzzy linguistic decision matrix.

	G_1	G_2
A_1	$\{S_{-2}, S_{-1}, S_0\}$	$\{S_0, S_1\}$
A_2	$\{S_0, S_1, S_2\}$	$\{S_1, S_2\}$
A_3	$\{S_2, S_3\}$	$\{S_1, S_2, S_3\}$
A_4	$\{S_0, S_1, S_2\}$	$\{S_{-1}, S_0, S_1\}$
A_5	$\{S_{-1}, S_0\}$	$\{S_0\ S_1, S_2\}$
	G_3	**G_4**
A_1	$\{S_0\ S_1, S_2\}$	$\{S_1, S_2\}$
A_2	$\{S_0, S_1\}$	$\{S_0, S_1, S_2\}$
A_3	$\{S_1, S_2\}$	$\{S_2\}$
A_4	$\{S_1, S_2, S_3\}$	$\{S_1, S_2\}$
A_5	$\{S_0, S_1, S_2\}$	$\{S_0\ S_1\}$

Table 2. The generalized Dice similarity measures of Equation (25) and ranking orders.

λ	(A_1, A^+)	(A_2, A^+)	(A_3, A^+)	(A_4, A^+)	(A_5, A^+)	Ranking orders
0	0.383	0.367	0.700	0.461	0.250	$A_3 > A_4 > A_1 > A_2 > A_5$
0.2	0.449	0.439	0.746	0.531	0.306	$A_3 > A_4 > A_1 > A_2 > A_5$
0.5	0.617	0.625	0.843	0.701	0.464	$A_3 > A_4 > A_2 > A_1 > A_5$
0.7	0.850	0.878	0.947	0.920	0.707	$A_3 > A_4 > A_2 > A_1 > A_5$
1.0	3.267	3.200	2.180	2.667	4.400	$A_5 > A_1 > A_2 > A_4 > A_3$

Table 3. The generalized Dice similarity measures of Equation (26) and ranking orders.

λ	(A_1, A^+)	(A_2, A^+)	(A_3, A^+)	(A_4, A^+)	(A_5, A^+)	Ranking orders
0	0.336	0.345	0.716	0.397	0.224	$A_3 > A_4 > A_2 > A_1 > A_5$
0.2	0.407	0.418	0.782	0.474	0.276	$A_3 > A_4 > A_2 > A_1 > A_5$
0.5	0.595	0.611	0.910	0.670	0.709	$A_3 > A_5 > A_4 > A_2 > A_1$
0.7	0.861	0.883	1.020	0.924	0.661	$A_3 > A_4 > A_2 > A_1 > A_5$
1.0	2.600	2.667	1.248	2.151	4.000	$A_5 > A_2 > A_1 > A_4 > A_3$

From the Tables 2 and 3, different ranking orders are shown by taking different values of λ and different Dice similarity measures. Then the best movies should belong to A_3 or A_5 according to the principle of the maximum degree of Dice similarity measures between HFLTSs.

Furthermore, for the special cases of the two generalized Dice measures we obtain the following results:

- When $\lambda = 0$, the two weighted generalized Dice measures are reduced to the weighted projection measures of $A_i (i = 1, 2, 3, 4, 5)$ on A^+. Thus, the best movies should belong to A_3 according to the principle of the maximum degree of Dice similarity measures between HFLTSs. For this case, we can derive the same best alternative as the method proposed in Ref. Liao et al. (2014). Thus our method is effective.
- When $\lambda = 0.5$, the two weighted generalized Dice measures are reduced to the weighted Dice similarity measures of $A_i (i = 1, 2, 3, 4, 5)$ and A. Thus, the best movies should belong to A_3 according to the principle of the maximum degree of Dice similarity measures between HFLTSs. For this case, we can derive the same best alternative as the method proposed in Ref. Liao et al. (2014). Thus our method is effective.

- When $\lambda = 1$, the two weighted generalized Dice measures are reduced to the weighted projection measures of A^+ on $A_i(i = 1, 2, 3, 4, 5)$. Thus, the best movies should belong to A_5 according to the principle of the maximum degree of Dice similarity measures between HFLTSs.

Therefore, according to different Dice similarity measures and different values of the parameter λ, ranking orders may be also different. Thus the proposed multiple attribute decision making methods can be assigned some value of λ and some measure to satisfy the requirements of decision makers' preference and flexible decision making.

Obviously, the multiple attribute decision making methods based on the Dice measures and the projection measures are the special cases of the proposed multiple attribute decision making methods based on generalized Dice measures. Therefore, in the multiple attribute decision making process, the multiple attribute decision making models developed in this paper are more general and more flexible than existing multiple attribute decision making models under a hesitant fuzzy linguistic environment.

Conclusion and future work

In this paper, we present some novel Dice similarity measures of HFLTSs and the generalized Dice similarity measures of HFLTSs and indicate that the Dice similarity measures and asymmetric measures (projection measures) are the special cases of the generalized Dice similarity measures in some parameter values. Then, we propose the generalized Dice similarity measures-based multiple attribute decision making models with HFLTSs. Finally, an illustrative example for concerning the evaluation of the quality of movies is given to demonstrate the efficiency of the similarity measures. In the future, the application of the proposed Dice similarity measure of HFLTSs needs to be explored in dynamic and complex decision making, risk analysis and many other fields under an uncertain environment (Gao, 2018; Gao, Lu, Wei, & Wei, 2018; Gao, Wei, & Huang, 2018; Huang & Wei, 2018; Liao, Li, & Lu, 2007; Tang, Wen, & Wei, 2017; Tang & Wei, 2018; Wei & Wei, 2018).

Acknowledgment

The work was supported by the National Natural Science Foundation of China under Grant No. 71571128 and the Humanities and Social Sciences Foundation of Ministry of Education of the People's Republic of China (No. 17XJA630003) and the Construction Plan of Scientific Research Innovation Team for Colleges and Universities in Sichuan Province (15TD0004).

ORCID

Guiwu Wei http://orcid.org/0000-0001-9074-2005

References

Beg, I., & Rashid, T. (2015). Hesitant 2-tuple linguistic information in multiple attributes group decision making. *Journal of Intelligent & Fuzzy Systems*, *30*(1), 109–116. doi:10.3233/IFS-151737

Dice, L. R. (1945). Measures of the amount of ecologic association between species. *Ecology*, *26*(3), 297–302. doi:10.2307/1932409

Dong, Y. C., & Herrera-Viedma, E. (2015). Consistency-Driven Automatic Methodology to Set Interval Numerical Scales of 2-Tuple Linguistic Term Sets and Its Use in the Linguistic GDM With Preference Relation. *IEEE Trans. Cybernetics*, *45*(4), 780–792.

Dutta, B., & Guha, D. (2015). Partitioned Bonferroni mean based on linguistic 2-tuple for dealing with multi-attribute group decision making. *Appl. Soft Comput*, *37*, 166–179. doi:10.1016/j.asoc.2015.08.017

Dutta, B., Guha, D., & Mesiar, R. (2015). A Model Based on Linguistic 2-Tuples for Dealing With Heterogeneous Relationship Among Attributes in Multi-expert Decision Making. *IEEE Transactions on Fuzzy Systems*, *23*(5), 1817–1831. doi:10.1109/TFUZZ.2014.2379291

Estrella, F. J., Espinilla, M., Herrera, F., & Martínez, L. (2014). FLINTSTONES: A fuzzy linguistic decision tools enhancement suite based on the 2-tuple linguistic model and extensions. *Information Sciences*, *280*, 152–170. doi:10.1016/j.ins.2014.04.049

Fan, Z. P., Feng, B., Sun, Y. H., & Ou, W. (2009). Evaluating knowledge management capability of organizations: A fuzzy linguistic method. *Expert Systems with Applications*, *36*(2), 3346–3354. doi:10.1016/j.eswa.2008.01.052

Fan, Z. P., & Liu, Y. (2010). A method for group decision-making based on multi-granularity uncertain linguistic information. *Expert Systems with Applications*, *37*(5), 4000–4008. doi:10.1016/j.eswa.2009.11.016

Gao, H. (2018). Pythagorean Fuzzy Hamacher Prioritized Aggregation Operators in Multiple Attribute Decision Making. *Journal of Intelligent & Fuzzy Systems*, *35*(2)2018), 2229–2245. doi:10.3233/JIFS-172262

Gao, H., Lu, M., Wei, G. W., & Wei, Y. (2018). Some novel Pythagorean fuzzy interaction aggregation operators in multiple attribute decision making. *Fundamenta Informaticae*, *159*(4)2018), 385–428. doi:10.3233/FI-2018-1669

Gao, H., Wei, G. W., & Huang, Y. H. (2018). Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. *IEEE Access*, *6*(1)2018), 11508–11522. doi:10.1109/ACCESS.2017.2784963

Herrera, F., & Herrera-Viedma, E. (2000a). Choice functions and mechanisms for linguistic preference relations. *European Journal of Operational Research*, *120*(1), 144–161. doi:10.1016/S0377-2217(98)00383-X

Herrera, F., & Herrera-Viedma, E. (2000b). Linguistic decision analysis: Steps for solving decision problems under linguistic information. *Fuzzy Sets and Systems*, *115*(1), 67–82. doi:10.1016/S0165-0114(99)00024-X

Herrera, F., Herrera-Viedma, E., & Martínez, L. (2008). A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions on Fuzzy Systems*, *16*(2), 354–370. doi:10.1109/TFUZZ.2007.896353

Herrera, F., & Martínez, L. (2001a). A model based on linguistic 2-tuple for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making. *IEEE Transactions on Systems, Man and Cybernetics, Part B (Cybernetics)*, *31*(2), 227–234. doi:10.1109/3477.915345

Herrera, F., & Martínez, L. (2001b). The 2-tuple linguistic computational model: Advantages of its linguistic description, accuracy and consistency. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *09*(supp01), 33–49. doi:10.1142/S0218488501000971

Herrera, F., Martínez, L., & Sánchez, P. J. (2005). Managing non-homogeneous information in group decision making. *European Journal of Operational Research*, *166*(1), 115–132. doi:10.1016/j.ejor.2003.11.031

Herrera-Viedma, E., Martinez, L., Mata, F., & Chiclana, F. (2005). A consensus support system model for group decision-making problems with multigranular linguistic preference relations. *IEEE Transactions on Fuzzy Systems*, *13*(5), 644–658. doi:10.1109/TFUZZ.2005.856561

Hu, Z., Rao, C. J., Zheng, Y., & Huang, D. (2015). Optimization Decision of Supplier Selection in Green Procurement under the Mode of Low Carbon Economy. *International Journal of Computational Intelligence Systems*, *8*(3), 407–421. doi:10.1080/18756891.2015.1017375

Huang, Y. H., & Wei, G. W. (2018). TODIM Method for Pythagorean 2-tuple Linguistic Multiple Attribute Decision Making. *Journal of Intelligent & Fuzzy Systems*, *35*(1)2018), 901–915. doi:10.3233/JIFS-171636

Hung, K. C. (2012). Applications of medical information: Using an enhanced likelihood measured approach based on intuitionistic fuzzy sets. *IIE Transactions on Healthcare Systems Engineering*, *2*(3), 224–231. doi:10.1080/19488300.2012.713443

Hung, W. L., & Yang, M. S. (2004). Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. *Pattern Recognition Letters*, *25*(14), 1603–1611. doi:10.1016/j.patrec.2004.06.006

Hung, W. L., & Yang, M. S. (2007). Similarity measures of intuitionistic fuzzy sets based on Lp metric. *International Journal of Approximate Reasoning*, *46*(1), 120–136. doi:10.1016/j.ijar.2006.10.002

Jaccard, P. (1901). Distribution de la flore alpine dans le Bassin des Drouces et dans quelques regions voisines. *Bull. Soc. Sci. Nat*, *37*(140), 241–272.

Li, Y. H., Olson, D. L., & Zheng, Q. (2007). Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis. *Pattern Recognition Letters*, *28*(2), 278–285. doi:10.1016/j.patrec.2006.07.009

Liao, X. W., Li, Y., & Lu, B. (2007). A model for selecting an ERP system based on linguistic information processing. *Information Systems*, *32*(7), 1005–1017. doi:10.1016/j.is.2006.10.005

Liao, H. C., & Xu, Z. S. (2015). Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making. *Expert Syst. Appl*, *42*(12), 5328–5336. doi:10.1016/j.eswa.2015.02.017

Liao, H. C., Xu, Z. S., & Zeng, X. J. (2014). Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making. *Information Sciences.*, *271*, 125–142. doi:10.1016/j.ins.2014.02.125

Liu, H. W. (2005). New similarity measures between intuitionistic fuzzy sets and between elements. *Mathematical and Computer Modelling*, *42*(1-2), 61–70. doi:10.1016/j.mcm.2005.04.002

Liu, H. C., Lin, Q. L., & Wu, J. (2014). Dependent Interval 2-Tuple Linguistic Aggregation Operators and Their Application to Multiple Attribute Group Decision Making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *22*(05), 717–736. doi:10.1142/S0218488514500366

Martínez, L., & Herrera, F. (2012). An overview on the 2-tuple linguistic model for Computing with Words in Decision Making: Extensions, applications and challenges. *Information Sciences*, *207*(1), 1–18. 2012. doi:10.1016/j.ins.2012.04.025

Martínez-López, L., Rodríguez,R, M., & Herrera, F. (2015). *The 2-tuple Linguistic Model - Computing with Words in Decision Making*1–168). Springer:., pp.

Miller, G. A. (1956). The magical number seven plus or minus two: Some limitations on our capacity for processing information. *Psychol. Rev*, *63*(2), 81–97. doi:10.1037/h0043158

Qin, J. D., & Liu, X. W. (2016). 2-tuple linguistic Muirhead mean operators for multiple attribute group decision making and its application to supplier selection. *Kybernetes*, *45*(1), 2–29. doi:10.1108/K-11-2014-0271

Rajarajeswari, P., & Uma, N. (2013). Intuitionistic fuzzy multi similarity measure based on cotangent function. *International Journal of Engineering Research & Technology*, *2*(11), 1323–1329.

Rao, C., Zheng, J., Wang, C., & Xiao, X. (2016). A hybrid multi-attribute group decision making method based on grey linguistic 2-tuple. *Iranian Journal of Fuzzy Systems*, *13*(2), 37–59.

Rodríguez, R. M., & Martínez, L. (2013). An Analysis of Symbolic Linguistic Computing Models in Decision Making. *International Journal of General Systems*, *42*(1), 121–136. doi: 10.1080/03081079.2012.710442
Rodríguez, R. M., Martínez, L., & Herrera, F. (2012). Hesitant fuzzy linguistic terms sets for decision making. *IEEE Transactions on Fuzzy Systems*, *20*, 109–119. doi:10.1109/TFUZZ.2011.2170076
Salton, G., & McGill, M. J. (1987). *Introduction to Modern Information Retrieval*New York, NY, USA. McGraw-Hill:
Shi, L. L., & Ye, J. (2013). Study on fault diagnosis of turbine using an improved cosine similarity measure for vague sets. *Journal of Applied Sciences*, *13*(10), 1781–1786. doi:10.3923/jas.2013.1781.1786
Singh, P. (2014). Correlation coefficients for picture fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, *27*, 2857–2868. doi:10.3233/JIFS-18719
Su, Z., Xu, Z. S., Liu, H. F., & Liu, S. S. (2015). Distance and similarity measures for dual hesitant fuzzy sets and their applications in pattern recognition. *Journal of Intelligent & Fuzzy Systems*, *29*(2), 731–745. doi:10.3233/IFS-141474
Szmidt, E. (2014). *Distances and similarities in intuitionistic fuzzy sets. Studies in Fuzziness and Soft Computing*(*307*, Springer:, Vol.,.
Szmidt, E., & Kacprzyk, J. (2000). Distances between intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *114*(3), 505–518. doi:10.1016/S0165-0114(98)00244-9
Tai, W. S., & Chen, C. T. (2009). A new evaluation model for intellectual capital based on computing with linguistic variable. *Expert Systems with Applications*, *36*(2), 3483–3488. doi: 10.1016/j.eswa.2008.02.017
Tang, X. Y., & Wei, G. W. (2018). Models for green supplier selection in green supply chain management with Pythagorean 2-tuple linguistic information. *IEEE Access*, *6*(2018), 18042–18060. doi:10.1109/ACCESS.2018.2817551
Tang, Y., Wen, L. L., & Wei, G. W. (2017). Approaches to multiple attribute group decision making based on the generalized Dice similarity measures with intuitionistic fuzzy information. *International Journal of Knowledge-Based and Intelligent Engineering Systems*, *21*(2), 85–95. doi:10.3233/KES-170354
Tian, M. Y. (2013). A new fuzzy similarity based on cotangent function for medical diagnosis. *Advanced Modeling and Optimization*, *15*(2), 151–156.
Torra, V. (2010). Hesitant fuzzy sets. *International Journal of Intelligent Systems*, *25*(2010), 529–539. doi:10.1002/int.20418
Wang, W. P. (2009). Evaluating new product development performance by fuzzy linguistic computing. *Expert Systems with Applications*, *36*(6), 9759–9766. doi:10.1016/j.eswa.2009.02.034
Wang, J. Q., Wang, D. D., Zhang, H. Y., & Chen, X. H. (2015). Multi-criteria group decision making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators. *Soft Computing*, *19*(2), 389–405. doi:10.1007/s00500-014-1259-z
Wei, G. W. (2017a). Interval-valued dual hesitant fuzzy uncertain linguistic aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *33*(3)2017):, 1881–1893. doi:10.3233/JIFS-161811
Wei, G. W. (2017b). Picture 2-tuple linguistic Bonferroni mean operators and their application to multiple attribute decision making. *International Journal of Fuzzy Systems*, *19*(4)2017):, 997–1010. doi:10.1007/s40815-016-0266-x
Wei, G. W. (2017c). Picture uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making. *Kybernetes*, *46*(10)2017), 1777–1800. doi:10.1108/K-01-2017-0025
Wei, G. W. (2017d). Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making. *Informatica*, *28*(3)2017):, 547–564. doi:10.15388/Informatica.2017.144

Wei, G. W. (2018a). Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *Fundamenta Informaticae*, *157*(3)2018), 271–320. doi:10.3233/FI-2018-1628
Wei, G. W. (2018b). Some similarity measures for picture fuzzy sets and their applications. *Iranian Journal of Fuzzy Systems*, *15*(1), 77–89.
Wei, G. W., & Gao, H. (2018). The generalized Dice similarity measures for picture fuzzy sets and their applications. *Informatica*, *29*(1), 107–118. doi:10.15388/Informatica.2018.160
Wei, G. W., Gao, H., Wang, J., & Huang, Y. H. (2018). Research on Risk Evaluation of Enterprise Human Capital Investment with Interval-valued bipolar 2-tuple linguistic Information. *IEEE Access*, *6*(2018), 35697–35712. doi:10.1109/ACCESS.2018.2836943
Wei, G. W., Gao, H., & Wei, Y. (2018). Some q-Rung Orthopair Fuzzy Heronian Mean Operators in Multiple Attribute Decision Making. *International Journal of Intelligent Systems*, *33*(7)2018), 1426–1458. doi:10.1002/int.21985
Wei, G. W., & Lu, M. (2018a). Pythagorean Fuzzy Maclaurin Symmetric Mean Operators in multiple attribute decision making. *International Journal of Intelligent Systems*, *33*(5)2018), 1043–1070. doi:10.1002/int.21911
Wei, G. W., & Lu, M. (2018b). Pythagorean fuzzy power aggregation operators in multiple attribute decision making. *International Journal of Intelligent Systems*, *33*(1)2018), 169–186. doi:10.1002/int.21946
Wei, G. W., Lu, M., Tang, X. Y., & Wei, Y. (2018). Pythagorean Hesitant Fuzzy Hamacher Aggregation Operators and Their Application to Multiple Attribute Decision Making. *International Journal of Intelligent Systems*, *33*(6), 1197–1233. doi:10.1002/int.21978
Wei, G. W., & Wei, Y. (2018a). Similarity measures of Pythagorean fuzzy sets based on cosine function and their applications. *International Journal of Intelligent Systems*, *33*(3)2018), 634–652. doi:10.1002/int.21965
Wei, G., & Wei, Y. (2018b). Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *35*(2)2018), 2001–2013. doi:10.3233/JIFS-171741
Wu, Q., Wu, P., Zhou, Y. Y., Zhou, L. G., Chen, H. Y., & Ma, X. Y. (2015). Some 2-tuple linguistic generalized power aggregation operators and their applications to multiple attribute group decision making. *Journal of Intelligent & Fuzzy Systems*, *29*(1), 423–436. doi:10.3233/IFS-151609
Xia, M. M., & Xu, Z. S. (2011). Hesitant fuzzy information aggregation in decision making. *International Journal of Approximate Reasoning*, *52*(3), 395–407. doi:10.1016/j.ijar.2010.09.002
Xu, Z. S., & Xia, M. M. (2010). Some new similarity measures for intuitionistic fuzzy values and their application in group decision making. *Journal of Systems Science and Systems Engineering*, *19*, 430–452. doi:10.1007/s11518-010-5151-9
Xu, Z. S. (2004). A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Information Sciences*, *166*(1-4), 19–30. doi:10.1016/j.ins.2003.10.006
Xu, Z. S. (2004). Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. *Inform. Sci*, *168*(1-4) (2004)., 171–184. doi:10.1016/j.ins.2004.02.003
Xu, Z. S. (2006). A note on linguistic hybrid arithmetic averaging operator in multiple attribute group decision making with linguistic information. *Group Decision and Negotiation*, *15*(6), 593–604. doi:10.1007/s10726-005-9008-4
Xu, Y. J., Ma, F., Tao, F. F., & Wang, H. M. (2014). Some methods to deal with unacceptable incomplete 2-tuple fuzzy linguistic preference relations in group decision making. *Knowl.-Based Syst*, *56*, 179–190. doi:10.1016/j.knosys.2013.11.008
Yang, Y. (2017). Model for evaluating the visual design quality with 2-tuple linguistic information. *Journal of Intelligent & Fuzzy Systems*, *33*(3), 1741–1748. doi:10.3233/JIFS-161637
Yao, S. B., & Khalid, A. (2018). Completing 2-tuple linguistic preference relations based on upper bound condition. *Soft Computing*, *22*(18), 6215–6227. doi:10.1007/s00500-017-2762-9

Ye, J. (2011). Cosine similarity measures for intuitionistic fuzzy sets and their applications. *Mathematical and Computer Modelling*, *53*(1-2), 91–97. doi:10.1016/j.mcm.2010.07.022

Ye, J. (2012a). Multicriteria decision-making method using the Dice similarity measure between expected intervals of trapezoidal fuzzy numbers. *Journal of Decision Systems*, *21*(4), 307–317. doi:10.1080/12460125.2012.734265

Ye, J. (2012b). Multicriteria decision-making method using the Dice similarity measure based on the reduct intuitionistic fuzzy sets of interval-valued intuitionistic fuzzy sets. *Applied Mathematical Modelling*, *36*(9), 4466–4472. doi:10.1016/j.apm.2011.11.075

Ye, J. (2012c). Multicriteria group decision-making method using vector similarity measures for trapezoidal intuitionistic fuzzy numbers. *Group Decision and Negotiation*, *21*(4), 519–530. doi:10.1007/s10726-010-9224-4

Ye, J. (2014). Vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making. *International Journal of Fuzzy Systems*, *16*(2), 204–211.

Ye, J. (2015). Similarity measures of intuitionistic fuzzy sets based on cosine function for the decision making of mechanical design schemes. *Journal of Intelligent & Fuzzy Systems*, *30*(1), 151–158. doi:10.3233/IFS-151741

Ye, J. (2016). The generalized Dice measures for multiple attribute decision making under simplified neutrosophic environments. *Journal of Intelligent & Fuzzy Systems*, *31*(1), 663–671. doi:10.3233/IFS-162179

Ye, J. (2017). Generalized Dice measures for multiple attribute decision making under intuitionistic and interval-valued intuitionistic fuzzy environments. *Neural Computing & Applications*, (8)2017), 1–10.

Zhang, Z. F., & Chu, X. N. (2009). Fuzzy group decision-making for multi-format and multi-granularity linguistic judgments in quality function deployment. *Expert Systems with Applications*, *36*(5), 9150–9158. doi:10.1016/j.eswa.2008.12.027

Zhang, X., & Liu, P. D. (2010). Method for aggregating triangular fuzzy intuitionistic fuzzy information and its application to decision making. *Technological and Economic Development of Economy*, *16*(2), 280–290. doi:10.3846/tede.2010.18

Zhang, L., Wang, Y. Z., & Zhao, X. Y. (2018). A new emergency decision support methodology based on multi-source knowledge in 2-tuple linguistic model. *Knowl.-Based Syst*, *144*, 77–87. doi:10.1016/j.knosys.2017.12.026

Zhang, W. C., Xu, Y. J., & Wang, H. M. (2016). A consensus reaching model for 2-tuple linguistic multiple attribute group decision making with incomplete weight information. *Int. J. Systems Science*, *47*(2), 389–405. doi:10.1080/00207721.2015.1074761