Economic Research-Ekonomska Istraživanja

ISSN: 1331-677X (Print) 1848-9664 (Online) Journal homepage: https://www.tandfonline.com/loi/rero20

Pythagorean fuzzy Muirhead mean operators in multiple attribute decision making for evaluating of emerging technology commercialization

Xiyue Tang, Guiwu Wei & Hui Gao

ECONOMIC RESEARCH-EKONOMSKA ISTRAŽIVANJA
2019, VOL. 32, NO. 1, 1667–1696
https://doi.org/10.1080/1331677X.2019.1638808

Pythagorean fuzzy Muirhead mean operators in multiple attribute decision making for evaluating of emerging technology commercialization

Xiyue Tang, Guiwu Wei and Hui Gao

School of Business, Sichuan Normal University, Chengdu, P.R. China

ABSTRACT

In today's world, with the advancement of technology, several emerging technologies are coming. Faced with massive emerging technologies which are the component of the technology pool, how to identify the commercial potential of emerging technologies in theory and practice is an important problem. The scientific approach to the selection of these emerging technologies is one of the main objectives of the research. In this paper, we extend Muirhead mean (MM) operator and dual MM (DMM) operator to process the Pythagorean fuzzy numbers (PFNs) and then to solve the multiple attribute decision making (MADM) problems. Firstly, we develop some Pythagorean fuzzy Muirhead mean operators by extending MM and DMM operators to Pythagorean fuzzy information. Then, we prove some properties and discuss some special cases with respect to the parameter vector. Moreover, we present some new methods to deal with MADM problems with the PFNs based on the proposed MM and DMM operators. Finally, we verify the validity and reliability of our methods by using an application example for potential evaluation of emerging technology commercialization, and analyze the advantages of our methods by comparing with other existing methods.

ARTICLE HISTORY
Received 17 March 2018
Accepted 15 February 2019

KEYWORDS
Multiple attribute decision making (MADM); Muirhead mean (MM) operator; dual Muirhead mean (DMM) operator; Pythagorean fuzzy number; Pythagorean Fuzzy Muirhead mean (PFMM) operator; Pythagorean Fuzzy dual Muirhead mean (PFDMM) operator; potential evaluation; emerging technology commercialization

1. Introduction

Atanassov (1986, 2000) introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set (Zadeh, 1965). Each element in the IFS is expressed by an ordered pair, and each ordered pair is characterized by a membership degree and a non-membership degree. The sum of the membership degree and the non-membership degree of each ordered pair is less than or equal to 1. More recently, a Pythagorean fuzzy set (PFS) (Yager, 2013, 2014) has emerged as an effective tool for depicting uncertainty of the MADM problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of

CONTACT Guiwu Wei weiguiwu@163.com

squares is less than or equal to 1; the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot; for example, if a DM gives the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. Zhang and Xu (2014) provided the detailed mathematical expression for PFS and introduced the concept of Pythagorean fuzzy number (PFN). Meanwhile, they also developed a Pythagorean fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) for handling the MCDM problem within PFNs. Peng and Yang (2015) proposed the division and subtraction operations for PFNs, and also developed a Pythagorean fuzzy superiority and inferiority ranking method to solve multicriteria group decision making problem with PFNs. Afterwards, Beliakov and James (2014) focused on how the notion of 'averaging' should be treated in the case of PFNs and how to ensure that the averaging aggregation functions produce outputs consistent with the case of ordinary fuzzy numbers. Reformat and Yager (2014) applied the PFNs in handling the collaborative-based recommender system. Gou, Xu, and Ren (2016) investigated the Properties of Continuous PFN. Ren, Xu, and Gou (2016) proposed the Pythagorean fuzzy TODIM approach to multi-criteria decision making. Garg (2016a) proposed the new generalized Pythagorean fuzzy information aggregation by using Einstein Operations. Zeng, Chen, and Li (2016) developed a hybrid method for Pythagorean fuzzy multiple-criteria decision making. Garg (2016b) studied a novel accuracy function under interval-valued PFSs for solving multicriteria decision making problems. Wei (2017a) utilized arithmetic and geometric operations (Wang, Wei, & Lu, 2018a; Wei, Gao, & Wei, 2018a; Wei, Lu, Tang, & Wei, 2018b; Wu, Wang, Wei, & Wei, 2018) to develop some Pythagorean fuzzy interaction aggregation operators: Pythagorean fuzzy interaction weighted average (PFIWA) operator, Pythagorean fuzzy interaction weighted geometric (PFIWG) operator, Pythagorean fuzzy interaction ordered weighted average (PFIOWA) operator, Pythagorean fuzzy interaction ordered weighted geometric (PFIOWG) operator, Pythagorean fuzzy interaction hybrid average (PFIHA) operator and Pythagorean fuzzy interaction hybrid geometric (PFIHG) operator. Wei and Lu (2018a) extended Maclaurin symmetric mean (Maclaurin, 1729) to Pythagorean fuzzy environment to propose the Pythagorean fuzzy Maclaurin symmetric mean (PFMSM) operator and Pythagorean fuzzy weighted Maclaurin symmetric mean (PFWMSM) operator. Wei and Lu (2018b) utilized power aggregation operators (Yager, 2001) to develop some Pythagorean fuzzy power aggregation operators: Pythagorean fuzzy power average (PFPA) operator, Pythagorean fuzzy power geometric (PFPG) operator, Pythagorean fuzzy power weighted average (PFPWA) operator, Pythagorean fuzzy power weighted geometric (PFPWG) operator, Pythagorean fuzzy power ordered weighted average (PFPOWA) operator, Pythagorean fuzzy power ordered weighted geometric (PFPOWG) operator, Pythagorean fuzzy power hybrid average (PFPHA) operator and Pythagorean fuzzy power hybrid geometric (PFPHG) operator. Lu, Wei, Alsaadi, Hayat, and Alsaedi (2017) proposed some hesitant pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making. Wei and Lu (2017a) defined some dual hesitant Pythagorean fuzzy

Hamacher aggregation operators in multiple attribute decision making. Wei and Lu (2017b) developed some Pythagorean hesitant fuzzy Hamacher aggregation operators in multiple attribute decision making. Wu and Wei (2017) gave some Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Wei, Lu, Alsaadi, Hayat, and Alsaedi (2017a) proposed some Pythagorean 2-tuple linguistic aggregation operators: Pythagorean 2-tuple linguistic weighted average (P2TLWA) operator, Pythagorean 2-tuple linguistic weighted geometric (P2TLWG) operator, Pythagorean 2-tuple linguistic ordered weighted average (P2TLOWA) operator, Pythagorean 2-tuple linguistic ordered weighted geometric (P2TLOWG) operator, Pythagorean 2-tuple linguistic hybrid average (P2TLHA) operator and Pythagorean 2-tuple linguistic hybrid geometric (P2TLHG) operator. Lu and Wei (2017) proposed some Pythagorean uncertain linguistic aggregation operators for multiple attribute decision making.

In some real decision making, there exist the interrelationships among the attributes in MADM problems. Bonferroni mean (BM) operators (Bonferroni, 1950; Deng, Wei, Gao, & Wang, 2018; Liu, Chen, & Liu, 2017; Wang, Wei, & Wei, 2018b; Wei, 2017b; Zhu, Xu, & Xia, 2012) and the Heronian mean (HM) (Beliakov, Pradera, & Calvo, 2007; Chu & Liu, 2015; Liu, Liu, & Zhang, 2014; Liu, Zhu, Liu, & Hao, 2013; Yu, 2013; Yu, Zhou, Chen, & Wang, 2015) operators provided a tool to consider the interrelationships of aggregated arguments; however, they can only consider the interrelationships between two attributes and cannot process the interrelationships among three or more than three attributes. Muirhead mean (MM) (Muirhead, 1902) is a well-known aggregation operator which can consider interrelationships among any number of arguments assigned by a variable vector, and some existing operators, such as arithmetic and geometric operators (not considering the interrelationships), both BM operator and Maclaurin symmetric mean (Maclaurin, 1729) are the special cases of MM operator. Therefore, the MM can offer a flexible and robust mechanism to process the information fusion problem and make it more adequate to solve MADM problems. However, the original MM can only deal with the numeric arguments, in order to make the MM operator to process the linguistic information, Qin and Liu (2016) extended the MM operator to process the 2-tuple linguistic information, and proposed some 2-tuple linguistic MM operators and applied the proposed operators to solve the MADM problems.

Because PFNs can easily describe the fuzzy information, and the MM operator and dual MM (DMM) operator can capture interrelationships among any number of arguments assigned by a variable vector, it is necessary to extend the MM and DMM operator to deal with the PFNs. The purpose of this paper is to propose some Pythagorean fuzzy MM operators by extending MM and DMM operators to Pythagorean fuzzy information, then to study some properties of these operators, and applied them to solve the MADM problems in which the attributes take the form of PFNs.

In order to achieve this purpose, the rest of this paper is set out as follows. Section 2 reviews some basic concepts and theory of PFSs. In Section 3, we propose the some Pythagorean fuzzy MM operators, and study some properties of these operators. In Section 4, we develop two MADM methods for PFNs based on the PFWMM operator and PFWDMM operator. In Section 5, an illustrative example for potential evaluation of emerging technology commercialization is given to verify the validity of the proposed methods and to show their advantages. In Section 6, we give some conclusions of this study.

2. Preliminaries

In this section, we review some fundamental concept of Pythagorean fuzzy set and MM, which will be used in the next section.

2.1. Pythagorean fuzzy set

The basic concepts of PFSs (Yager, 2013, 2014) are briefly reviewed in this section. Afterwards, novel score and accuracy functions for PFNs are proposed. Furthermore, a new comparison method for PFNs is developed.

Definition 1 (Yager, 2013, 2014). Let X be a fix set. A PFS is an object having the form

$$P = \left\{ \langle x, (\mu_P(x), \nu_P(x)) \rangle | x \in X \right\} \tag{1}$$

where the function $\mu_P : X \rightarrow [0, 1]$ defines the degree of membership and the function $\nu_P : X \rightarrow [0, 1]$ defin1es the degree of non-membership of the element $x \in X$ to P, respectively, and, for every $x \in X$, it holds that

$$(\mu_p(x))^2 + (\nu_p(x))^2 \leq 1. \tag{2}$$

Definition 2 (Wei, 2017a). Let $\tilde{a} = (\mu, \nu)$ be a Pythagorean fuzzy number, a score function S of a Pythagorean fuzzy number can be represented as follows:

$$S(\tilde{a}) = \frac{1}{2}(1 + \mu^2 - \nu^2), S(\tilde{a}) \in [0, 1]. \tag{3}$$

Definition 3 (Ren et al., 2016). Let $\tilde{a} = (\mu, \nu)$ be a Pythagorean fuzzy number, an accuracy function H of a Pythagorean fuzzy number can be represented as follows:

$$H(\tilde{a}) = \mu^2 + \nu^2, H(\tilde{a}) \in [0, 1]. \tag{4}$$

to evaluate the degree of accuracy of the Pythagorean fuzzy number $\tilde{a} = (\mu, \nu)$, where $H(\tilde{a}) \in [0, 1]$. The larger the value of $H(\tilde{a})$ is, the more the degree of accuracy of the Pythagorean fuzzy number $\tilde{a}$.

Based on the score function S and the accuracy function H, in the following, we shall give an order relation between two Pythagorean fuzzy numbers, which is defined as follows:

Definition 4 (Wei, 2017a). Let $\tilde{a}_1 = (\mu_1, \nu_1)$ and $\tilde{a}_2 = (\mu_2, \nu_2)$ be two Pythagorean fuzzy numbers, $s(\tilde{a}_1) = \frac{1}{2}(1 + (\mu_1)^2 - (\nu_1)^2)$ and $s(\tilde{a}_2) = \frac{1}{2}(1 + (\mu_2)^2 - (\nu_2)^2)$ be the scores of $\tilde{a}$ and $\tilde{b}$, respectively, and let $H(\tilde{a}_1) = (\mu_1)^2 + (\nu_1)^2$ and $H(\tilde{a}_2) = (\mu_2)^2 + (\nu_2)^2$ be the accuracy degrees of $\tilde{a}$ and $\tilde{b}$, respectively, then if $S(\tilde{a})<S(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a}<\tilde{b}$; if $S(\tilde{a}) = S(\tilde{b})$, then

1. if $H(\tilde{a}) = H(\tilde{b})$, then $\tilde{a}$ and $\tilde{b}$ represent the same information, denoted by $\tilde{a} = \tilde{b}$; (2) if $H(\tilde{a})<H(\tilde{b})$, $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a}<\tilde{b}$.

Definition 5 (Reformat & Yager, 2014). Let $\tilde{a}_1 = (\mu_1, \nu_1)$, $\tilde{a}_2 = (\mu_2, \nu_2)$, and $\tilde{a} = (\mu, \nu)$ be three Pythagorean fuzzy numbers, and some basic operations on them are defined as follows:

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \left(\sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1)^2(\mu_2)^2}, \nu_1\nu_2\right)$;
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \left(\mu_1\mu_2, \sqrt{(\nu_1)^2 + (\nu_2)^2 - (\nu_1)^2(\nu_2)^2}\right)$;
3. $\lambda\tilde{a} = \left(\sqrt{1-(1-\mu^2)^\lambda}, \nu^\lambda\right), \lambda>0$;
4. $(\tilde{a})^\lambda = \left(\mu^\lambda, \sqrt{1-(1-\nu^2)^\lambda}\right), \lambda>0$;
5. $\tilde{a}^c = (\nu, \mu)$.

2.2. Muirhead mean (MM)

The MM was first introduced by Muirhead (1902), the advantage of the MM operator is that it can capture the overall interrelationships among the multiple input arguments and it is a generalization of some existing aggregation operators. It was defined as follows:

Definition 6 (Muirhead, 1902). Let $a_j (j = 1, 2, \ldots, n)$ be a set of crisp numbers and $[\lambda] = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R$, then the Muirhead mean (MM) operator is defined as

$$\mathrm{MM}^\lambda(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n!}\sum_{\vartheta \in S_n}\prod_{j=1}^{n} a_{\vartheta(j)}^{\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \tag{5}$$

Where $\vartheta(j) (j = 1, 2, \ldots, n)$ is any permutation of $(1, 2, \ldots, n)$ and S_n is the set of all permutation of $(1, 2, \ldots, n)$.

By assigning some special vectors to λ, we can obtain some special cases of the MM operator:

1. If $\lambda = (1, 0, 0, \ldots, 0)$ the MM is reduced to

$$\mathrm{MM}^{(1,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \frac{1}{n}\sum_{j=1}^{n} a_j \tag{6}$$

Which is the arithmetic averaging operator.

2. If $\lambda = (1, 1, 0, 0, \ldots, 0)$ the MM is reduced to

$$\mathrm{MM}^{(1,1,0,0,\cdots,0)}(a_1, a_2, \cdots, a_n) = \frac{1}{n(n+1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i a_j \tag{7}$$

Which is the BM operator (Bonferroni, 1950).

3. If

$$\mathrm{MM}^{(\overbrace{1,1,\cdots,1}^{k}, \overbrace{0,0,\cdots,0}^{n-k})}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \prod_{j=1}^{k} a_{i_j}}{C_n^k} \right)^{1/k}$$

the MM is reduced to

$$\mathrm{MM}^{(\overbrace{1,1,\cdots,1}^{k}, \overbrace{0,0,\cdots,0}^{n-k})}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \prod_{j=1}^{k} a_{i_j}}{C_n^k} \right)^{1/k} \tag{8}$$

which is the Maclaurin symmetric mean (MSM) operator (Maclaurin, 1729).

4. If $P = (1/n, 1/n, \ldots, 1/n)$ the MM is reduced to

$$\mathrm{MM}^{(1/n, 1/n, \ldots, 1/n)}(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} a_j^{1/n} \tag{9}$$

which is the arithmetic averaging operator.

3. Pythagorean fuzzy Muirhead mean operators

In this section, we shall develop some Pythagorean fuzzy Muirhead mean operators based on the operations of PFNs.

3.1. PFMM operator

The MM operator has usually been applied to a situation where the aggregation assessments exhibit interaction relationship. Next, we extend MM operator to PFS. From definition 5 and 6, we can obtain:

Definition 7. Let $p_j = (\mu_j, v_j)(j = 1, 2, \ldots, n)$ be a set of PFN and $[\lambda] = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R$ be a vector of parameters, then the Pythagorean Fuzzy Muirhead mean (PFMM) operator is defined as

$$\mathrm{PFMM}^{\lambda}(p_1, p_2, \ldots, p_n) = \left(\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \tag{10}$$

Where $\vartheta_j(j = 1, 2, ..., n)$ is any permutation of $(1, 2, ..., n)$, and S_n is the collection of all permutations of $(1, 2, ..., n)$.

Based on the operations of the PFN described, we can drive the Theorem 1.

Theorem 1. Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFMM operator is also a PFN, and

$$\mathrm{PFMM}^{\lambda}(p_1, p_2, ..., p_n) = \left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$= \left(\left(\sqrt{1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}},\right.$$

$$\left.\sqrt{1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^2\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right) \tag{11}$$

Proof :

$$p_{\vartheta(j)}^{\lambda_j} = \left(\mu_{\vartheta(j)}^{\lambda_j}, \sqrt{1-\left(1-\nu_{\vartheta(j)}^2\right)^{\lambda_j}}\right) \tag{12}$$

$$\prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j} = \left(\prod_{j=1}^{n}\mu_{\vartheta(j)}^{\lambda_j}, \sqrt{1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^2\right)^{\lambda_j}}\right) \tag{13}$$

Thereafter,

$$\sum_{\vartheta\in S_n}\prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j} = \left(\sqrt{1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(j)}^{2\lambda_j}\right)}, \prod_{\vartheta\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^2\right)^{\lambda_j}}\right) \tag{14}$$

$$\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j} = \left(\sqrt{1-\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(j)}^{2\lambda_j}\right)\right)\right)^{\frac{1}{n!}}}, \left(\prod_{\vartheta\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^2\right)^{\lambda_j}}\right)^{\frac{1}{n!}}\right) \tag{15}$$

Therefore,

$$\left(\frac{1}{n!}\sum\nolimits_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\vartheta(j)}^{\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}=\left(\left(\sqrt{1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}},\right.$$

$$\left.\sqrt{1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right) \tag{16}$$

And then, we can know:

$$0\leq\left(\sqrt{1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\leq 1 \tag{17}$$

$$0\leq\sqrt{1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\leq 1 \tag{18}$$

We can obtain $\mu_{\vartheta(j)}^{2}+v_{\vartheta(j)}^{2}\leq 1$ from the definition of PFS, so

$$\begin{aligned}&\left(\left(\sqrt{1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)^{2}\\&+\left(\sqrt{1-\left(1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)^{2}\\&\leq\left(1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\\&+1-\left(1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}=1\end{aligned} \tag{19}$$

We complete the proof.

Example 1. Let $x_1 = (0.6, 0.7)$,$x_2 = (0.4, 0.3)$,$x_3 = (0.8, 0.1)$ be three PFNs, and $[\lambda] = (0.2, 0.5, 0.3)$, then we have

$$\mathrm{PFMM}^{(0.2,0.5,0.3)}(x_1, x_2, x_3)$$

$$= \left(\left(\sqrt{1 - \begin{pmatrix} (1-0.6^{0.4}\times 0.4^{1}\times 0.8^{0.6})\times(1-0.4^{0.4}\times 0.6^{1}\times 0.8^{0.6})\times \\ (1-0.6^{0.4}\times 0.8^{1}\times 0.4^{0.6})\times(1-0.4^{0.4}\times 0.6^{1}\times 0.8^{0.6})\times \\ (1-0.8^{0.4}\times 0.4^{1}\times 0.6^{0.6})\times(1-0.8^{0.4}\times 0.6^{1}\times 0.4^{0.6}) \end{pmatrix}^{\frac{1}{3!}}}\right)\right)^{\frac{1}{0.2+0.5+0.3}},$$

$$\sqrt{1 - \left(1 - \begin{pmatrix} (1-0.51^{0.2}\times 0.91^{0.5}\times 0.99^{0.3})\times(1-0.51^{0.2}\times 0.99^{0.5}\times 0.91^{0.3})\times \\ (1-0.91^{0.2}\times 0.51^{0.5}\times 0.99^{0.3})\times(1-0.91^{0.2}\times 0.99^{0.5}\times 0.51^{0.3})\times \\ (1-0.99^{0.2}\times 0.51^{0.5}\times 0.91^{0.3})\times(1-0.99^{0.2}\times 0.91^{0.5}\times 0.51^{0.3}) \end{pmatrix}^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}$$

$$= (0.5818, 0.4673)$$

In the following, we give some properties of PFMM operator.

Property 1. (Idempotency) let $p_j = (\mu_{p_j}, \nu_{p_j}) = p = (\mu_p, \nu_p)(j = 1, 2, 3, ..., n)$, then

$$\mathrm{PFMM}^{\lambda}(p_1, p_2, ..., p_n) = p \tag{20}$$

Proof :

$$\begin{aligned} \mathrm{PFMM}^{\lambda}(p_1, p_2, ..., p_n) &= \left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n} p^{\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \\ &= \left(\frac{1}{n!}\cdot n!\cdot p^{\sum_{j=1}^{n}\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \\ &= p \end{aligned} \tag{21}$$

Property 2. (Monotonicity) let $p_j = (\mu_{p_j}, \nu_{p_j})$ and $q_j = (\mu_{q_j}, \nu_{q_j})$ $(j = 1, 2, 3, ..., n)$ be two sets of PFNs, If $(\mu_{p_j})^2 \le (\mu_{q_j})^2$ and $(\nu_{p_j})^2 \ge (\nu_{q_j})^2$ then

$$\mathrm{PFMM}^{\lambda}(p_1, p_2, ..., p_n) \le \mathrm{PFMM}^{\lambda}(q_1, q_2, ..., q_n) \tag{22}$$

Proof :

$$\prod_{j=1}^{n}\mu_{\vartheta(p_j)}^{2\lambda_j} \le \prod_{j=1}^{n}\mu_{\vartheta(q_j)}^{2\lambda_j} \tag{23}$$

$$\left(\sum_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(p_j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}\geq\left(\sum_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(q_j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}\tag{24}$$

Therefore,

$$\left(\sqrt{1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(p_j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\leq\left(\sqrt{1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\mu_{\vartheta(q_j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\tag{25}$$

Similarly, we also can obtain

$$\sqrt{1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(p_j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\geq\sqrt{1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(q_j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\tag{26}$$

then, the proof is completed. Then

If $\mu_{p_j}^2<\mu_{q_j}^2$ and $v_{p_j}^2>v_{q_j}^2$ then

$$\mathrm{PFMM}^{\lambda}(p_1,p_2,\ldots,p_n)<\mathrm{PFMM}^{\lambda}(q_1,q_2,\ldots,q_n);$$

If $\mu_{p_j}^2<\mu_{q_j}^2$ and $v_{p_j}^2=v_{q_j}^2$ then

$$\mathrm{PFMM}^{\lambda}(p_1,p_2,\ldots,p_n)<\mathrm{PFMM}^{\lambda}(q_1,q_2,\ldots,q_n);$$

If $\mu_{p_j}^2=\mu_{q_j}^2$ and $v_{p_j}^2>v_{q_j}^2$ then

$$\mathrm{PFMM}^{\lambda}(p_1,p_2,\ldots,p_n)<\mathrm{PFMM}^{\lambda}(q_1,q_2,\ldots,q_n);$$

If $\mu_{p_j}^2=\mu_{q_j}^2$ and $v_{p_j}^2=v_{q_j}^2$ then

$$\mathrm{PFMM}^{\lambda}(p_1,p_2,\ldots,p_n)=\mathrm{PFMM}^{\lambda}(q_1,q_2,\ldots,q_n);$$

Property 3. (Boundedness) Let $p_j=(\mu_j,v_j)(j=1,2,\ldots,n)$ be a set of PFNs. If $p^+=(\max_j(\mu_j),\min_j(v_j))$ and $p^-=(\min_j(\mu_j),\max_j(v_j))$, According the process of property of Monotonicity and Idempotency, it is easy to got that

$$p^{-} \leq \text{P2TLMM}^{\lambda}(p_1, p_2, ..., p_n) \leq p^{+} \tag{27}$$

3.2. PFWMM operator

In Section 3.1, it can be seen that the PFMM operator doesn't consider the importance of the aggregated arguments. However, in many real practical situations, especially in multiple attribute decision making, the weights of attributes plays an important role in the process of aggregation. To overcome the limitation of PFMM, we shall propose the Pythagorean fuzzy weighted MM (PFWMM) operator as follows.

Definition 8. Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a set of PFNs with weights vector being $W = (w_1, w_2, ...w_n)^T$, $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$ and $[\lambda] = (\lambda_1, \lambda_2, ..., \lambda_n) \in R$, then the Pythagorean fuzzy weighted Muirhead mean (PFWMM) operator is defined as

$$\text{PFWMM}_{w}^{\lambda}(p_1, p_2, ..., p_n) = \left(\frac{1}{n!}\sum_{\vartheta \in S_n}\prod_{j=1}^{n}\left(nw_{\vartheta(j)}p_{\vartheta(j)}\right)^{\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \tag{28}$$

Based on the operations of the PFN described, we can drive the Theorem 2.

Theorem 2. Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFWMM operator is also a PFN, and

$$\begin{aligned}\text{PFWMM}_{w}^{\lambda}(p_1, p_2, ...p_n) &= \left(\tfrac{1}{n!}\sum\nolimits_{\vartheta \in S_n}\prod_{j=1}^{n}\left(nw_{\vartheta(j)}p_{\vartheta(j)}\right)^{\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \\ &= \left(\left(\sqrt{1-\left(\prod\nolimits_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}},\right. \\ &\left.\sqrt{1-\left(1-\left(\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\vartheta(j)}^{2nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\end{aligned} \tag{29}$$

proof :

$$nw_{\vartheta(j)}p_{\vartheta(j)} = \left(\sqrt{1-\left(1-\mu_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}}}, \nu_{\vartheta(j)}^{nw_{\vartheta(j)}}\right) \tag{30}$$

$$\left(nw_{\vartheta(j)}p_{\vartheta(j)}\right)^{\lambda_j}=\left(\left(\sqrt{1-\left(1-\mu^2_{\vartheta(j)}\right)^{nw_{\vartheta(j)}}}\right)^{\lambda_j},\sqrt{1-\left(1-\nu^{2nw_{\vartheta(j)}}_{\vartheta(j)}\right)^{\lambda_j}}\right) \tag{31}$$

Thereafter,

$$\prod_{j=1}^{n}\left(nw_{\vartheta(j)}p_{\vartheta(j)}\right)^{\lambda_j}=\left(\prod_{j=1}^{n}\left(\sqrt{1-\left(1-\mu^2_{\vartheta(j)}\right)^{nw_{\vartheta(j)}}}\right)^{\lambda_j},\sqrt{1-\prod_{j=1}^{n}\left(1-\nu^{2nw_{\vartheta(j)}}_{\vartheta(j)}\right)^{\lambda_j}}\right) \tag{32}$$

$$\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(nw_{\vartheta(j)}p_{\vartheta(j)}\right)^{\lambda_j}=\left(\sqrt{1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu^2_{\vartheta(j)}\right)^{nw_{\vartheta(j)}}\right)^{\lambda_j}\right)},\right.$$

$$\left.\prod_{\vartheta\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\nu^{2nw_{\vartheta(j)}}_{\vartheta(j)}\right)^{\lambda_j}}\right) \tag{33}$$

Thus,

$$\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(nw_{\vartheta(j)}p_{\vartheta(j)}\right)^{\lambda_j}=\left(\sqrt{1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu^2_{\vartheta(j)}\right)^{nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}},\right.$$

$$\left.\left(\prod_{\vartheta\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\nu^{2nw_{\vartheta(j)}}_{\vartheta(j)}\right)^{\lambda_j}}\right)^{\frac{1}{n!}}\right) \tag{34}$$

Therefore,

$$\left(\tfrac{1}{n!}\textstyle\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(nw_{\vartheta(j)}p_{\vartheta(j)}\right)^{\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$=\left(\left(\sqrt{1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu^2_{\vartheta(j)}\right)^{nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}},\right.$$

$$\left.\left(\sqrt{1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu^{2nw_{\vartheta(j)}}_{\vartheta(j)}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right) \tag{35}$$

and we can get followed easily,

$$0 \leq \left(\sqrt{1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^{2}\right)^{nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \leq 1 \tag{36}$$

$$0 \leq \sqrt{1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}} \leq 1 \tag{37}$$

Therefore,

$$\begin{aligned}&\left(\left(\sqrt{1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^{2}\right)^{nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)^{2}\\&+\left(\sqrt{1-\left(1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)^{2}\end{aligned} \tag{38}$$

$$\begin{aligned}&\leq\left(\left(\sqrt{1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^{2}\right)^{nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)^{2}\\&+\left(\sqrt{1-\left(1-\left(\prod\nolimits_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^{2}\right)^{nw_{\vartheta(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)^{2}=1\end{aligned} \tag{39}$$

We complete the proof.

Example 2. Let $x_1 = (0.6, 0.7)$,$x_2 = (0.4, 0.3)$,$x_3 = (0.8, 0.1)$ be three PFNs, and $[\lambda] = (0.2, 0.5, 0.3)$, $W = (0.3, 0.4, 0.3)$ then we have

$$\mathrm{PFWMM}^{(0.2,0.5,0.3)}_{(0.3,0.4,03)}(x_1,x_2,x_3)$$

$$= \left(\left(\left(\sqrt[\frac{1}{3!}]{1 - \begin{pmatrix} (1-0.6013^{0.2}\times 0.1888^{0.5}\times 0.3308^{0.3}) \\ \times(1-0.6013^{0.2}\times 0.3308^{0.5}\times 0.1888^{0.3}) \\ \times(1-0.1888^{0.2}\times 0.6013^{0.5}\times 0.3308^{0.3}) \\ \times(1-0.1888^{0.2}\times 0.3308^{0.5}\times 0.6013^{0.3}) \\ \times(1-0.3308^{0.2}\times 0.1888^{0.5}\times 0.6013^{0.3}) \\ \times(1-0.3308^{0.2}\times 0.6013^{0.5}\times 0.1888^{0.3}) \end{pmatrix}} \right)^{\frac{1}{3!}} \right)^{\frac{1}{0.2+0.5+0.3}}, \right.$$

$$\left. \sqrt{1 - \left(1 - \begin{pmatrix} (1-0.0158^{0.2}\times 0.0556^{0.5}\times 0.5262^{0.3}) \\ \times(1-0.0158^{0.2}\times 0.5262^{0.5}\times 0.0556^{0.3}) \\ \times(1-0.0556^{0.2}\times 0.0158^{0.5}\times 0.5262^{0.3}) \\ \times(1-0.0556^{0.2}\times 0.5262^{0.5}\times 0.0158^{0.3}) \\ \times(1-0.5262^{0.2}\times 0.0556^{0.5}\times 0.0158^{0.3}) \\ \times(1-0.5262^{0.2}\times 0.0158^{0.5}\times 0.0556^{0.3}) \end{pmatrix}^{\frac{1}{3!}} \right)^{\frac{1}{0.2+0.5+0.3}}} \right)$$

$$= (0.5821, 0.4766)$$

The PFWMM operator has the property of boundedness and monotonicity, but it does not satisfy the property of idempotency. In the following, we omitted the process of proof, because it is similar with the PFMM monotonicity property.

Property 4. (Monotonicity) let $p_j = (\mu_{p_j}, v_{p_j})$ and $q_j = (\mu_{q_j}, v_{q_j})$ $(j = 1,2,3,\ldots,n)$ be two sets of PFNs, If $\mu_{p_j} \le \mu_{q_j}$ and $v_{p_j} \ge v_{q_j}$ then

$$\mathrm{PFWMM}^{\lambda}_{w}(p_1,p_2,\ldots,p_n) \le \mathrm{PFWMM}^{\lambda}_{w}(q_1,q_2,\ldots,q_n) \tag{40}$$

Property 5. (Boundedness) Let $p_j = (\mu_j, v_j)(j = 1,2,\ldots,n)$ be a set of PFNs with weights vector being $W = (w_1, w_2, \ldots w_n)^T$, $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$. If $p^+ = (\max_j(\mu_j), \min_j(v_j))$ and $p^- = (\min_j(\mu_j), \max_j(v_j))$, because of property 4, then

$$\begin{aligned} &\mathrm{P2TLWMM}^{\lambda}_{w}(p^-,p^-,\ldots,p^-) \\ &\le \mathrm{P2TLWMM}^{\lambda}_{w}(p_1,p_2,\ldots,p_n) \\ &\le \mathrm{P2TLWMM}^{\lambda}_{w}(p^+,p^+,\ldots,p^+) \end{aligned} \tag{41}$$

3.3. PFDMM operator

Qin and Liu (2016) proposed the dual Muirhead mean (DMM) based on MM operator.

Definition 9. Let $a_j(j = 1, 2, ..., n)$ be a set of non-negative real numbers, and $P = (p_1, p_2, ..., p_n) \in R^n$ be a vector of parameters. If

$$\mathrm{DMM}^{\mathrm{P}}(a_1, a_2, ..., a_n) = \frac{1}{\sum_{j=1}^{n} p_j}\left(\prod_{\sigma \in S_n} \sum_{j=1}^{n} p_j a_{\sigma(j)}\right)^{\frac{1}{n!}} \tag{42}$$

Then we called $\mathrm{DMM}^{\mathrm{P}}$ the dual Muirhead mean (DMM) operator, where $\sigma(j)(j = 1, 2, ..., n)$ is any a permutation of $\{1, 2, ..., n\}$ and S_n is the set of all permutation of $\{1, 2, ..., n\}$.

In the following, we proposed the Pythagorean fuzzy dual MM (PFDMM) operator for PFNs.

Definition 10. Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a collection of PFNs and there exists parameter vector $[\lambda] = (\lambda_1, \lambda_2, ..., \lambda_n) \in R^n$, then

$$\mathrm{PFDMM}^{[\lambda]}(p_1, p_2, ..., p_n) = \frac{1}{\sum_{j=1}^{n} \lambda_j}\left(\prod_{\sigma \in S_n} \sum_{j=1}^{n} \lambda_j p_{\sigma(j)}\right)^{\frac{1}{n!}} \tag{43}$$

Based on the operations of the PFN described, we can drive the Theorem 3.

Theorem 3. Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFDMM operator is also a PFN, and

$$\mathrm{PFDMM}^{\lambda}(p_1, p_2, ..., p_n) = \frac{1}{\sum_{j=1}^{n} \lambda_j}\left(\prod_{\sigma \in S_n} \sum_{j=1}^{n} \lambda_j p_{\sigma(j)}\right)^{\frac{1}{n!}}$$

$$= \left(\sqrt{1 - \left(1 - \left(\prod\nolimits_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \mu_{\sigma(j)}^2\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}},\right.$$

$$\left.\left(\sqrt{1 - \left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n} \nu_{\sigma(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right) \tag{44}$$

Proof :

$$\lambda_j p_{\sigma(j)} = \left(\sqrt{1 - \left(1 - \mu_{\sigma(j)}^2\right)^{\lambda_j}}, \nu_{\sigma(j)}^{\lambda_j}\right) \tag{45}$$

$$\sum_{j=1}^{n}\lambda_j p_{\sigma(j)} = \left(\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^2\right)^{\lambda_j}}, \prod_{j=1}^{n}\nu_{\sigma(j)}^{\lambda_j}\right) \tag{46}$$

Therefore,

$$\prod_{\sigma\in S_n}\sum_{j=1}^{n}\lambda_j p_{\sigma(j)} = \left(\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^2\right)^{\lambda_j}}, \sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\nu_{\sigma(j)}^{2\lambda_j}\right)}\right) \tag{47}$$

$$\left(\prod_{\sigma\in S_n}\sum_{j=1}^{n}\lambda_j p_{\sigma(j)}\right)^{\frac{1}{n!}} = \left(\left(\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^2\right)^{\lambda_j}}\right)^{\frac{1}{n!}}, \sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\nu_{\sigma(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right) \tag{48}$$

Then, we can get

$$\frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\prod_{\sigma\in S_n}\sum_{j=1}^{n}\lambda_j p_{\sigma(j)}\right)^{\frac{1}{n!}} = \left(\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^2\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}},\right.$$

$$\left.\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\nu_{\sigma(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right) \tag{49}$$

From the aggregation result above, we prove the result of PFDMM aggregation is also a PFN in the following, then

$$0 \le \sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^2\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}} \le 1 \tag{50}$$

$$0 \le \left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\nu_{\sigma(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \le 1 \tag{51}$$

And, we can prove

$$\left(\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)^{2} + \left(\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\nu_{\sigma(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)^{2} \le 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} + \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} = 1 \quad (52)$$

So, we proved that the aggregation result of PFDMM is also a PFN.

Example 3. Let $x_1 = (0.6, 0.7)$, $x_2 = (0.4, 0.3)$, $x_3 = (0.8, 0.1)$ be three PFNs, and $[\lambda] = (0.2, 0.5, 0.3)$, then we have

$$\mathrm{PFDMM}^{(0.2,0.5,0.3)}(x_1, x_2, x_3)$$

$$= \left(\left(\sqrt{1-\left(1-\begin{pmatrix}(1-0.64^{0.2}\times 0.84^{0.5}\times 0.36^{0.3})\times(1-0.64^{0.2}\times 0.36^{0.5}\times 0.84^{0.3})\times\\(1-0.84^{0.2}\times 0.64^{0.5}\times 0.36^{0.3})\times(1-0.84^{0.2}\times 0.36^{0.5}\times 0.64^{0.3})\times\\(1-0.36^{0.2}\times 0.84^{0.5}\times 0.64^{0.3})\times(1-0.36^{0.2}\times 0.64^{0.5}\times 0.84^{0.3})\end{pmatrix}^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}\right)\right),$$

$$\left(\sqrt{1-\begin{pmatrix}(1-0.49^{0.2}\times 0.09^{0.5}\times 0.01^{0.3})\times(1-0.49^{0.2}\times 0.01^{0.5}\times 0.09^{0.3})\times\\(1-0.09^{0.2}\times 0.49^{0.5}\times 0.01^{0.3})\times(1-0.09^{0.2}\times 0.01^{0.5}\times 0.49^{0.3})\times\\(1-0.01^{0.2}\times 0.49^{0.5}\times 0.09^{0.3})\times(1-0.01^{0.2}\times 0.09^{0.5}\times 0.49^{0.3})\end{pmatrix}^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.5+0.3}}$$

$$= (0.6447, 0.2891)$$

Property 6. (Idempotency) if all $p_j (j = 1, 2, ..., n)$ are equal, i.e.,$p_j = p = (\mu, \nu)$, then

$$\mathrm{PFDMM}^{\lambda}(p_1, p_2, ..., p_n) = p \quad (53)$$

Property 7. (Monotonicity) let $p_j = (\mu_{p_j}, \nu_{p_j})$ and $q_j = (\mu_{q_j}, \nu_{q_j})$ $(j = 1, 2, 3, ..., n)$ be two sets of PFNs, If $\mu_{p_j} \le \mu_{q_j}$ and $\nu_{p_j} \ge \nu_{q_j}$ then

$$\mathrm{PFDMM}^{\lambda}(p_1, p_2, ..., p_n) \le \mathrm{PFDMM}^{\lambda}(q_1, q_2, ..., q_n) \tag{54}$$

Property 8. (Boundedness). Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a set of PFNs. If $p^+ = (\max_j(\mu_j), \min_j(\nu_j))$ and $p^- = (\min_j(\mu_j), \max_j(\nu_j))$, because of property 7 and property 8, then $\mathrm{PFDMM}^{\lambda}(p^-, p^-, ..., p^-) = p^-, \le \mathrm{PFDMM}^{\lambda}(p^+, p^+, ..., p^+) = p^+$.

$$p^- \le \mathrm{PFDMM}^{\lambda}(p_1, p_2, ..., p_n) \le p^+ \tag{55}$$

3.4. PFWDMM operator

In Section 3.3, it can be seen that the PFDMM operator doesn't consider the importance of the aggregated arguments. However, in many real practical situations, especially in multiple attribute decision making, the weights of attributes plays an important role in the process of aggregation. To overcome the limitation of PFDMM operator, we shall propose the Pythagorean fuzzy weighted DMM (PFWDMM) operator as follows.

Definition 11. Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a collection of PFNs with weights vector being $W = (w_1, w_2, ...w_n)^T, w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1$ and there exists parameter vector $[\lambda] = (\lambda_1, \lambda_2, ..., \lambda_n) \in R^n$, then

$$\mathrm{PFWDMM}_w^{[\lambda]}(p_1, p_2, ..., p_n) = \frac{1}{\sum_{j=1}^{n} \lambda_j}\left(\prod_{\sigma \in S_n} \sum_{j=1}^{n} \lambda_j p_{\sigma(j)}^{n w_{\sigma(j)}}\right)^{\frac{1}{n!}} \tag{56}$$

Based on the operations of the PFN described, we can drive the Theorem 4.

Theorem 4. Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFWDMM operator is also a PFN, and

$$\mathrm{PFMM}^{\lambda}(p_1, p_2, \cdots, p_n) = \left(\frac{1}{n!}\sum_{\vartheta \in S_n} \prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j}\right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}$$

$$= \left(\left(\sqrt{1 - \left(\prod_{\vartheta \in S_n}\left(1 - \prod_{j=1}^{n} \mu_{\vartheta(j)}^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}},\right.$$

$$\left.\sqrt{1 - \left(1 - \left(\prod_{\vartheta \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \nu_{\vartheta(j)}^2\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}}\right) \tag{57}$$

Proof :

$$\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}} = \left(\sqrt{1-\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}}, \left(\sqrt{1-\left(1-\nu_{\sigma(j)}^{2}\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right) \tag{58}$$

$$\sum_{j=1}^{n} \lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}} = \left(\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}}, \prod_{j=1}^{n}\left(\sqrt{1-\left(1-\nu_{\sigma(j)}^{2}\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right) \tag{59}$$

Thereafter

$$\prod_{\sigma\in S_n}\sum_{j=1}^{n} \lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}} = \left(\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}}, \sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\nu_{\sigma(j)}^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)}\right) \tag{60}$$

$$\left(\prod\nolimits_{\sigma\in S_n}\sum_{j=1}^{n}\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)^{\frac{1}{n!}} = \left(\left(\prod\nolimits_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)^{\frac{1}{n!}}, \sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\nu_{\sigma(j)}^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right) \tag{61}$$

Therefore,

$$\frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\prod_{\sigma\in S_n}\sum_{j=1}^{n}\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)^{\frac{1}{n!}} = \left(\left(\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}, \left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\nu_{\sigma(j)}^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right) \tag{62}$$

Then, we can get

$$0 \le \sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)}^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \le 1 \tag{63}$$

$$0 \leq \left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-v_{\sigma(j)}^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \leq 1 \tag{64}$$

Because $\mu^2 + v^2 \leq 1$, therefore

$$\begin{aligned}&\left(\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)^2 + \\ &\left(\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-v_{\sigma(j)}^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)^2 \leq \\ &1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} + \\ &\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} = 1\end{aligned} \tag{65}$$

So, the aggregation result of PFWDMM is also PFN.

Example 4. Let $x_1 = (0.6, 0.7)$,$x_2 = (0.4, 0.3)$,$x_3 = (0.8, 0.1)$ be three PFNs, and $[\lambda] = (0.2, 0.5, 0.3)$, $W = (0.3, 0.4, 0.3)$ then we have

$$\begin{aligned}&\text{PFWDMM}_{(0.3,0.4,03)}^{(0.2,0.5,0.3)}(x_1, x_2, x_3) \\ &= \left(\left(\left(\sqrt{1-\begin{pmatrix}(1-0.8181^{0.2}\times 0.3330^{0.5}\times 0.6314^{0.3}) \\ \times(1-0.8181^{0.2}\times 0.6314^{0.5}\times 0.3330^{0.3}) \\ \times(1-0.3330^{0.2}\times 0.8181^{0.5}\times 0.6314^{0.3}) \\ \times(1-0.3330^{0.2}\times 0.6314^{0.5}\times 0.8181^{0.3}) \\ \times(1-0.6314^{0.2}\times 0.3330^{0.5}\times 0.8181^{0.3}) \\ \times(1-0.6314^{0.2}\times 0.8181^{0.5}\times 0.3330^{0.3})\end{pmatrix}^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.5+0.3}}\right.\right., \\ &\left.\sqrt{1-\left(1-\begin{pmatrix}(1-0.0090^{0.2}\times 0.1070^{0.5}\times 0.4545^{0.3}) \\ \times(1-0.0090^{0.2}\times 0.4545^{0.5}\times 0.1070^{0.3}) \\ \times(1-0.1070^{0.2}\times 0.0090^{0.5}\times 0.4545^{0.3}) \\ \times(1-0.1070^{0.2}\times 0.4545^{0.5}\times 0.0090^{0.3}) \\ \times(1-0.4545^{0.2}\times 0.1070^{0.5}\times 0.0090^{0.3}) \\ \times(1-0.4545^{0.2}\times 0.0090^{0.5}\times 0.1070^{0.3})\end{pmatrix}^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}\right) \\ &= (0.6567, 0.2890)\end{aligned}$$

PFWDMM is also satisfying property boundedness and monotonicity, but it is not satisfying the property of idempotency.

Property 10. (Monotonicity) let $p_j = (\mu_{p_j}, \nu_{p_j})$ and $q_j = (\mu_{q_j}, \nu_{q_j})$ $(j = 1, 2, 3, ..., n)$ be two sets of PFNs with weights vector being $W = (w_1, w_2, ...w_n)^T$, $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, If $\mu_{p_j} \leq \mu_{q_j}$ and $\nu_{p_j} \geq \nu_{q_j}$ then

$$\mathrm{PFWDMM}_w^{\lambda}(p_1, p_2, ..., p_n) \leq \mathrm{PFWDMM}_w^{\lambda}(q_1, q_2, ..., q_n) \tag{66}$$

Property 11. (Boundedness) Let $p_j = (\mu_j, \nu_j)(j = 1, 2, ..., n)$ be a set of PFNs with weights vector being $W = (w_1, w_2, ...w_n)^T$, $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. If $p^+ = (\max_j(\mu_j), \min_i(\nu_j))$ and $p^- = (\min_j(\mu_j), \max_j(\nu_j))$, because of property 10, then

$$\begin{aligned} &\mathrm{PFWDMM}_w^{\lambda}(p^-, p^-, ..., p^-) \\ &\leq \mathrm{PFWDMM}_w^{\lambda}(p_1, p_2, ..., p_n) \\ &\leq \mathrm{PFWDMM}_w^{\lambda}(p^+, p^+, ..., p^+) \end{aligned} \tag{67}$$

5. Models for MADM with PFNs

Based the PFWMM and PFWDMM operators, in this section, we shall propose the model for MADM with PFNs. Let $A = \{A_1, A_2, ..., A_m\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, ..., G_n\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weighting vector of the attribute $G_j (j = 1, 2, ..., n)$, where $\omega_j \in [0, 1], \sum_{j=1}^{n} \omega_j = 1$. Suppose that $P = (p_{ij})_{m \times n} = (\mu_{ij}, \nu_{ij})_{m \times n}$ is the Pythagorean fuzzy decision matrix, where μ_{ij} indicates the degree that the alternative A_i satisfies the attribute G_j given by the decision maker, ν_{ij} indicates the degree that the alternative A_i doesn't satisfy the attribute G_j given by the decision maker, $\mu_{ij} \subset [0, 1]$, $\nu_{ij} \subset [0, 1]$, $(\mu_{ij})^2 + (\nu_{ij})^2 \leq 1$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

In the following, we apply the PFWMM (PFWDMM) operator to the MADM problems with PFNs.

Step 1. We utilize the PFNs given in matrix $\tilde{R}$, and the PFWMM operator

$$\begin{aligned} p_i &= \mathrm{PFWMM}_w^{\lambda}(p_{i1}, p_{i2}, ..., p_{in}) = \left(\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} \left(n w_{\vartheta(j)} p_{\vartheta(ij)} \right)^{\lambda_j} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \\ &= \left(\left(\sqrt{1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(ij)}^2 \right)^{n w_{\vartheta(j)}} \right)^{\lambda_j} \right) \right)^{\frac{1}{n!}}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}, \right. \\ &\left. \sqrt{1 - \left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \nu_{\vartheta(ij)}^{2 n w_{\vartheta(j)}} \right)^{\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}} \right), i = 1, 2, ..., m. \end{aligned} \tag{68}$$

Or

$$p_i = \mathrm{PFWDMM}_W^{\lambda}(p_{i1}, p_{i2}, \ldots, p_{in})$$
$$= \frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\prod_{\sigma\in S_n}\sum_{j=1}^{n}\lambda_j p_{\sigma(ij)}^{nw_{\sigma(j)}}\right)^{\frac{1}{n!}}$$
$$= \left(\sqrt{1-\left(1-\left(\prod\nolimits_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(ij)}^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}},\right.$$
$$\left.\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-v_{\sigma(ij)}^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right), i = 1, 2, \ldots, m. \quad (69)$$

to derive the $p_i (i = 1, 2, \ldots, m)$ of the alternative A_i.

Step 2. Calculate the scores $S(p_i)(i = 1, 2, \ldots, m)$ of the overall PFNs $p_i (i = 1, 2, \ldots, m)$ to rank all the alternatives $A_i (i = 1, 2, \ldots, m)$ and then to select the best one(s). If there is no difference between two scores $S(p_i)$ and $S(p_j)$, then we need to calculate the accuracy degrees $H(p_i)$ and $H(p_j)$ of the overall PFNs p_i and p_j, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(p_i)$ and $H(p_j)$.

Step 3. Rank all the alternatives $A_i (i = 1, 2, \ldots, m)$ and select the best one(s) in accordance with $S(p_i)(i = 1, 2, \ldots, m)$.

Step 4. End.

6. Numerical example and comparative analysis

6.1. Numerical example

For the time being, emerging technologies have mushroomed up gradually with the rapid development of science and technology, and emerging technologies have brought obvious impacts on states, industries and societies. There are no countries which can oversee the development of emerging technologies, it can be said that the competition between countries and states is the one of emerging technologies, especially the quality, quantity and speed of commercialization and industrialization of emerging technologies. It is not only a difficult task for countries and enterprises to identify, further evaluate and select emerging technologies, finally commercialize and industrialize emerging technologies; but also a weak issue for researchers to study theoretically. Thus, in this section we shall present a numerical example to show potential evaluation of emerging technology commercialization with Pythagorean fuzzy information in order to illustrate the method proposed in this paper. There is a panel with five possible emerging technology enterprises $O_i (i = 1, 2, 3, 4, 5)$ to select. The experts selects four attributes to evaluate the five possible emerging technology enterprises: ①G_1 is the technical advancement; ②G_2 is the potential market and market risk; ③G_3 is the industrialization infrastructure, human resources and financial

Table 1. PFN decision matrix.

	C_1	C_2	C_3	C_4
O_1	(0.50,0.80)	(0.60,0.50)	(0.30,0.60)	(0.60,0.70)
O_2	(0.70,0.50)	(0.70,0.40)	(0.60,0.20)	(0.40,0.60)
O_3	(0.70,0.50)	(0.50,0.70)	(0.50,0.30)	(0.60,0.20)
O_4	(0.80,0.20)	(0.60,0.30)	(0.40,0.50)	(0.60,0.60)
O_5	(0.60,0.40)	(0.40,0.70)	(0.70,0.50)	(0.60,0.80)

Table 2. The aggregating Result of PFMM, PFWMM, PFDMM and PFWDMM operators.

	O_1	O_2	O_3	O_4	O_5
PFMM	(0.4827,0.6742)	(0.5862,0.4590)	(0.5695,0.4882)	(0.5833,0.4411)	(0.5639,0.6448)
PFWMM	(0.4556,0.708)	(0.5620,0.5247)	(0.5362,0.6065)	(0.5542,0.4671)	(0.5262,0.6842)
PFDMM	(0.522,0.6407)	(0.6229,0.3947)	(0.5874,0.382)	(0.637,0.3673)	(0.5936,0.5794)
PFWDMM	(0.5960,0.6007)	(0.6926,0.3695)	(0.6400,0.3673)	(0.6929,0.3416)	(0.6280,0.5410)

Table 3. The rank and score of emerging technology enterprises by using PFMM, PFWMM, PFDMM and PFWDMM operators.

	O_1	O_2	O_3	O_4	O_5	Order
PFMM	0.3892	0.5665	0.543	0.5728	0.4511	$O_4 > O_2 > O_3 > O_5 > O_1$
PFWMM	0.3531	0.5203	0.4599	0.5445	0.4043	$O_4 > O_2 > O_3 > O_5 > O_1$
PFDMM	0.4310	0.6161	0.5996	0.6354	0.5084	$O_4 > O_2 > O_3 > O_5 > O_1$
PFDWMM	0.4972	0.6716	0.6373	0.6817	0.5508	$O_4 > O_2 > O_3 > O_5 > O_1$

conditions; ④G_4 is the employment creation and the development of science and technology. The five possible emerging technology enterprises $O_i(i = 1, 2, 3, 4, 5)$ are to be evaluated using the PFNs according to four attributes (whose weighting vector $W = (0.2, 0.1, 0.3, 0.4)$,$\lambda = (0.2, 0.2, 0.3, 0.3)$, as shown in Table 1.

In the following, in order to show potential evaluation of emerging technology commercialization of five possible emerging technology enterprises, we utilize the PFMM, PFWMM, PFDMM and PFWDMM operators to solve MADM problem with PFNs, which concludes the following calculating steps:

Step 1. According to Table 1, aggregate all PFNs $p_{ij}(j = 1, 2, ..., n)$ by using the PFMM, PFWMM, PFDMM and PFWDMM operators to derive the overall PFNs $p_i(i = 1, 2, 3, 4)$ of the emerging technology enterprises O_i. The aggregating results are shown in Table 2.

Step 2. According to the aggregating results shown in Table 2 and the score functions of the emerging technology enterprises are shown in Table 3.

According the result of emerging technology enterprises order, we can know that the best choice is emerging technology enterprise O_4, we get same result by different aggregation, that proved the effectiveness of the result.

6.2. Influence of the parameter on the final result

The aggregation method of extend PFS with MM has two advantages, one is that it can reduce the bad effects of the unduly high and low assessments on the final result,

Table 4. Ranking results by utilizing different parameter vector *R* in the PFWMM operator.

		Scores				
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	O_1	O_2	O_3	O_4	O_5	Order
(1,0,0,0)	0.4126	0.5869	0.6269	0.6000	0.5147	$O_3>O_4>O_2>O_5>O_1$
(1,1,0,0)	0.3793	0.5401	0.5521	0.5708	0.4696	$O_4>O_3>O_2>O_5>O_1$
(2,0,0,0)	0.4389	0.5962	0.6440	0.6166	0.5412	$O_3>O_4>O_2>O_5>O_1$
(3,0,0,0)	0.4640	0.6052	0.6574	0.6309	0.5611	$O_3>O_4>O_2>O_5>O_1$
(1,1,1,0)	0.3638	0.5273	0.4993	0.5548	0.4374	$O_4>O_2>O_3>O_5>O_1$
(1,1,1,1)	0.3524	0.5197	0.4576	0.5438	0.4026	$O_4>O_2>O_3>O_5>O_1$

Table 5. Ranking results by utilizing different parameter vector *R* in the PFWDMM operator.

		Scores				
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	O_1	O_2	O_3	O_4	O_5	Order
(1,0,0,0)	0.3750	0.5261	0.5875	0.5353	0.4572	$O_3>O_4>O_2>O_5>O_1$
(1,1,0,0)	0.4317	0.6145	0.6137	0.6081	0.5234	$O_3>O_2>O_4>O_5>O_1$
(2,0,0,0)	0.3535	0.4827	0.5734	0.4913	0.4233	$O_3>O_4>O_2>O_5>O_1$
(3,0,0,0)	0.3368	0.4486	0.5609	0.4603	0.3939	$O_3>O_4>O_2>O_5>O_1$
(1,1,1,0)	0.4647	0.6502	0.6285	0.6582	0.5401	$O_4>O_2>O_3>O_5>O_1$
(1,1,1,1)	0.4991	0.6730	0.6379	0.6832	0.5515	$O_4>O_2>O_3>O_5>O_1$

the other is that it can capture the interrelationship between PFNs. These aggregation operators have a parameter vector, which make extended operator more flexible, so the different vector leads to different aggregation results, different scores and ranking results. In order to illustrate the influence of the parameter vector R on the ranking result, we discuss the influence with several parameter vectors, the result you can find in Tables 4 and 5.

We can see that the different parameters lead to a different result and different ranking order. More attributes we consider, the bigger the scores, the bigger the attribute value and the more lower the scores. Therefore, the parameter vector can be considered as decision maker's risk preference.

6.3. Comparative analysis

The prominent characteristic of the PFMM, PFWMM, PFDMM and PFWDMM operators is that they can consider the interrelationship among the PFNs. We investigate some comparative analyses to demonstrate the advantages of the proposed operators. Table 6 presents further details.

Table 6 shows that the aggregation operators introduced in (Yager, 2014; Garg, 2016a; Zeng et al., 2016; Wei & Lu, 2017a; Wu & Wei, 2017; Wei, 2017c) cannot consider the interrelationship between the PFNs. Although PFCIA, PFCIG, PFPWA and PFPWG can capture the interrelationship between the PFNs, they only change the weight vector of the aggregation operators. In addition, the correlations of the aggregated arguments are measured subjectively by the decision makers. PFMSM, PFWMSM, PFIWA, PFIWG, PFIOWA, PFIOWG, PFIHA, PFIHG, GPFWBM, GPFWBGM, GPFWHM, PFGWHM, PFMM, PFDMM, PFWMM, PFWDMM operators focus on the aggregated PFNs. In addition, the GPFWBM, GPFWBGM, PFMM,

Table 6. The Comparison of the different aggregation operators under PFNs.

Aggregation operators	Whether the operator can capture the interrelationship between any two or three PFNs	Whether the operator can capture the interrelationship among all PFNs	Whether a parameter vector exists to manipulate the ranking results
PFWA and PFWG (Yager, 2014)	No	No	No
SPFWA and SPFWG (Ma & Xu, 2016)	No	No	No
PFOWAWAD (Zeng et al., 2016)	No	No	No
PFEWA, PFEOWA (Garg, 2016a)	No	No	No
PFHWA and PFHWG (Wu & Wei, 2017)	No	No	No
PFHOWA, PFHOWG (Wu & Wei, 2017)	No	No	No
PFHHA, PFHHG (Wu & Wei, 2017)	No	No	No
DHPFHWA, DHPFHWG (Wei & Lu, 2017a)	No	No	No
DHPFHOWA,DHPFHOWG (Wei & Lu, 2017a)	No	No	No
DHPFHHA, DHPFHHG (Wei & Lu, 2017a)	No	No	No
HPFHWA, HPFHWG (Lu et al., 2017)	No	No	No
HPFHOWA, HPFHOWG (Lu et al., 2017)	No	No	No
HPFHHA, HPFHHG (Lu et al., 2017)	No	No	No
PFCIA and PFCIG (Peng & Yang, 2016)	Yes	No	No
PFPWA, PFPWG (Wei & Lu, 2018b)	Yes	No	No
PFIWA, PFIWG (Wei, 2017a)	Yes	No	No
PFIOWA, PFIOWG (Wei, 2017a)	Yes	No	No
PFIHA, PFIHG (Wei, 2017a)	Yes	No	No
PFMSM,PFWMSM (Wei & Lu, 2018a)	Yes	Yes	No
GPFWBM,GPFWBGM (Zhang, Wang, Zhu, Xia, & Yu, 2017)	Yes	Yes	Yes
PFMM, PFDMM	Yes	Yes	Yes
PFWMM, PFWDMM	Yes	Yes	Yes

Table 7. Ordering of the emerging technology enterprises.

	Ordering
PFWA operator (Yager, 2014)	$A_3 > {}_4 > {}_2 > {}_5 > {}_1$
PFWG operator (Yager, 2014)	$A_3 > {}_4 > {}_2 > {}_5 > {}_1$
SPFWA operator (Ma & Xu, 2016)	$A_3 > {}_4 > {}_2 > {}_5 > {}_1$
SPFWG operator (Ma & Xu, 2016)	$A_3 > {}_4 > {}_2 > {}_5 > {}_1$

PFDMM, PFWMM, PFWDMM operators have a parameter vector, thereby enabling the aggregation process to be substantially flexible.

At the same time, we compare our proposed method with other existing methods including the Pythagorean fuzzy weighted averaging (PFWA) operator (Yager, 2014), Pythagorean fuzzy weighted geometric (PFWG) operator (Yager, 2014), symmetric Pythagorean fuzzy weighted averaging (SPFWA) operator (Ma & Xu, 2016) and symmetric Pythagorean fuzzy weighted geometric (SPFWG) operator (Ma & Xu, 2016).

From the Table 7, we can get the same optimal emerging technology enterprises and four methods' ranking results are slightly different from the proposed approaches. However, the existing aggregation operators, such as PFWA operator, PFWG operator, SPFWA operator and SPFWG operator, do not consider the information about the relationship between arguments being aggregated, and thus cannot eliminate the influence of unfair arguments on decision results. Our proposed operators, such as PFMM operator, PFDMM operator, PFWMM operator and PFWDMM operator consider the information about the relationship among arguments being aggregated.

7. Conclusion

Aggregation operators have become a hot issue and an important tool in the decision making fields in recent years. However, they still have some limitations in practical applications. For example, some aggregation operators suppose the attributes are independent of each other. However, the MM operator and dual MM operator have a prominent characteristic that it can consider the interaction relationships among any number of attributes by a parameter vector λ. Motivated by the studies about MM operator and dual MM operator, in this paper, we proposed some new MM and DMM aggregation operators to deal with MADM problems under a Pythagorean fuzzy environment, included the Pythagorean fuzzy MM (PFMM) operator, Pythagorean fuzzy weighted MM (PFWMM) operator, Pythagorean fuzzy dual MM (PFWMM) operator and the Pythagorean fuzzy weighted dual MM (PFWDMM) operator. Then, the desirable properties were proved. Moreover, these proposed operators are utilized to solve the MADM problems with PFNs. Finally, we used an illustrative example for potential evaluation of emerging technology commercialization to show the feasibility and validity of the proposed operators by comparing with the other existing methods. In the future, we shall extend the proposed operators to dynamic and complex decision making (Chen, 2015; De & Sana, 2014; Gao, 2018; Gao, Wei, & Huang, 2018; Huang & Wei, 2018; Tang & Wei, 2018; Wang, Wei, & Lu, 2018c; Wei, 2017c, 2018a, 2018b; Wei, Gao, Wang, & Huang, 2018c; Yue & Jia, 2013), risk analysis (Wei, Liu, Lai, & Hu, 2017b; Wei, Yu, Liu, & Cao, 2018d) and many other fields under uncertain environment (Chen, 2017; Mardani et al., 2015, 2018; Peng & Selvachandran, 2017; Rostamzadeh, Esmaeili, Nia, Saparauskas, & Ghorabaee, 2017; Wang, Wei, & Gao, 2018d; Wei, 2018c; Wei & Wei, 2018; Zeng, Mu, & Balezentis, 2018).

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The work was supported by the National Natural Science Foundation of China under Grant No. 71571128 and the Humanities and Social Sciences Foundation of Ministry of Education of the People's Republic of China (No. 17XJA630003) and the Construction Plan of Scientific Research Innovation Team for Colleges and Universities in Sichuan Province (15TD0004).

References

Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87–96. doi:10.1016/S0165-0114(86)80034-3

Atanassov, K. (2000). Two theorems for intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *110*(2), 267–269. doi:10.1016/S0165-0114(99)00112-8

Beliakov, G., & James, S. (2014). *Averaging aggregation functions for preferences expressed as Pythagorean membership grades and fuzzy orthopairs*. FUZZ-IEEE, pp. 298–305.

Beliakov, G., Pradera, A., & Calvo, T. (2007). *Aggregation functions: A guide for practitioners*. Berlin. Springer.

Bonferroni, C. (1950). Sulle medie multiple di potenze. *Bolletino Matematica Italiana*, *5*, 267–270.
Chen, T. Y. (2017). Multiple criteria decision analysis using prioritised interval type-2 fuzzy aggregation operators and its application to site selection. *Technological and Economic Development of Economy*, *23*(1), 1–21. doi:10.3846/20294913.2016.1209249
Chen, T.-Y. (2015). The inclusion-based TOPSIS method with interval-valued intuitionistic fuzzy sets for multiple criteria group decision making. *Applied Soft Computing*, *26*, 57–73. doi:10.1016/j.asoc.2014.09.015
Chu, Y. C., & Liu, P. D. (2015). Some two-dimensional uncertain linguistic Heronian mean operators and their application in multiple-attribute decision making. *Neural Computing and Applications*, *26*(6), 1461–1480. doi:10.1007/s00521-014-1813-8
De, S. K., & Sana, S. S. (2014). A multi-periods production-inventory model with capacity constraints for multi-manufacturers-A global optimality in intuitionistic fuzzy environment. *Applied Mathematics and Computation*, *242*, 825–841. doi:10.1016/j.amc.2014.06.075
Deng, X. M., Wei, G. W., Gao, H., & Wang, J. (2018). Models for safety assessment of construction project with some 2-tuple linguistic Pythagorean fuzzy Bonferroni mean operators. *IEEE Access*, *6*, 52105–52137. doi:10.1109/ACCESS.2018.2869414
Gao, H. (2018). Pythagorean fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *35*(2), 2229–2245. doi:10.3233/JIFS-172262
Gao, H., Wei, G. W., & Huang, Y. H. (2018). Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. *IEEE Access*, *6*(1), 11508–11522. doi:10.1109/ACCESS.2017.2784963
Garg, H. (2016a). A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. *International Journal of Intelligent Systems*, *31*(9), 886–920. doi:10.1002/int.21809
Garg, H. (2016b). A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem. *Journal of Intelligent & Fuzzy Systems*, *31*(1), 529–540. doi:10.3233/IFS-162165
Gou, X., Xu, Z., & Ren, P. (2016). The properties of continuous Pythagorean fuzzy information. *International Journal of Intelligent Systems*, *31*(5), 401–424. doi:10.1002/int.21788
Huang, Y. H., & Wei, G. W. (2018). TODIM method for Pythagorean 2-tuple linguistic multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *35*(1), 901–915. doi:10.3233/JIFS-171636
Liu, P. D., Liu, Z. M., & Zhang, X. (2014). Some intuitionistic uncertain linguistic Heronian mean operators and their application to group decision making. *Applied Mathematics and Computation*, *230*, 570–586. doi:10.1016/j.amc.2013.12.133
Liu, P.-D., Chen, S.-M., & Liu, J. (2017). Multiple attribute group decision making based on intuitionistic fuzzy interaction partitioned Bonferroni mean operators. *Information Sciences.*, *411*, 98–121. doi:10.1016/j.ins.2017.05.016
Liu, X. D., Zhu, J. J., Liu, G. D., & Hao, J. J. (2013). A multiple attribute decision making method based on uncertain linguistic Heronian mean. *Mathematical Problems in Engineering*, *2013*, 1–11. doi:10.1155/2013/597671
Lu, M., & Wei, G. W. (2017). Pythagorean uncertain linguistic aggregation operators for multiple attribute decision making. *International Journal of Knowledge-Based and Intelligent Engineering Systems*, *21*(3), 165–179. doi:10.3233/KES-170361
Lu, M., Wei, G. W., Alsaadi, F. E., Hayat, T., & Alsaedi, A. (2017). Hesitant Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *33*(2), 1105–1117. doi:10.3233/JIFS-16554
Ma, Z., & Xu, Z. (2016). Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multicriteria decision-making problems. *International Journal of Intelligent Systems*, *31*(12), 1198–1219. no. doi:10.1002/int.21823

Maclaurin, C. (1729). A second letter to Martin Folkes, Esq.; concerning the roots of equations, with demonstration of other rules of algebra. *Philosophical Transactions of the Royal Society of London Series A*, *36*, 59–96.

Mardani, A., Jusoh, A., Nor, K. M., Khalifah, Z., Zakwan, N., & Valipour, A. (2015). Multiple criteria decision-making techniques and their applications-a review of the literature from 2000 to 2014. *Economic Research-Ekonomska Istraživanja*, *28*(1), 516–571. doi:10.1080/1331677X.2015.1075139

Mardani, A., Nilashi, M., Zavadskas, E. K., Awang, S. R., Zare, H., & Jamal, N. M. (2018). Decision-making methods based on fuzzy aggregation operators: Three decades review from 1986 to 2017. *International Journal of Information Technology & Decision Making*, *17*, 391–466. doi:10.1142/S021962201830001X

Muirhead, R. F. (1902). Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters. *Proceedings of the Edinburgh Mathematical Society*, *21*(3), 144–162. doi:10.1017/S001309150003460X

Peng, X. D., & Yang, Y. (2016). Pythagorean fuzzy choquet integral based MABAC method for multiple attribute group decision making. *International Journal of Intelligent Systems*, *31*(10), 989–1020. doi:10.1002/int.21814

Peng, X., & Selvachandran, G. (2017). Pythagorean fuzzy set: State of the art and future directions. *Artificial Intelligence Review*, *1*, 1–55.

Peng, X., & Yang, Y. (2015). Some results for Pythagorean fuzzy sets. *International Journal of Intelligent Systems*, *30*(11), 1133–1160. doi:10.1002/int.21738

Qin, J. Q., & Liu, X. W. (2016). 2-tuple linguistic Muirhead mean operators for multiple attribute group decision making and its application to supplier selection. *Kybernetes*, *45*(1), 2–29. doi:10.1108/K-11-2014-0271

Reformat, M., & Yager, R. R. (2014). Suggesting recommendations using Pythagorean fuzzy sets illustrated using Netflix movie data. *IPMU*, *1*, 546–556.

Ren, P., Xu, Z., & Gou, X. (2016). Pythagorean fuzzy TODIM approach to multi-criteria decision making. *Applied Soft Computing*, *42*, 246–259. doi:10.1016/j.asoc.2015.12.020

Rostamzadeh, R., Esmaeili, A., Nia, A. S., Saparauskas, J., & Ghorabaee, M. K. (2017). A fuzzy Aras method for supply chain management performance measurement in SMEs under uncertainty. *Transformations in Business & Economics*, *16*(2A), 319–348.

Tang, X. Y., & Wei, G. W. (2018). Models for green supplier selection in green supply chain management with Pythagorean 2-tuple linguistic information. *IEEE Access*, *6*, 18042–18060. doi:10.1109/ACCESS.2018.2817551

Wang, J., Wei, G. W., & Wei, Y. (2018b). Models for green supplier selection with some 2-tuple linguistic neutrosophic number Bonferroni mean operators. *Symmetry*, *10*(5), 131. doi:10.3390/sym10050131

Wang, J., Wei, G., & Gao, H. (2018d). Approaches to multiple attribute decision making with interval-valued 2-tuple linguistic pythagorean fuzzy information. *Mathematics*, *6*(10), 201. doi:10.3390/math6100201

Wang, J., Wei, G., & Lu, M. (2018a). TODIM method for multiple attribute group decision making under 2-tuple linguistic neutrosophic environment. *Symmetry*, *10*(10), 486. doi:10.3390/sym10100486

Wang, J., Wei, G., & Lu, M. (2018c). An extended VIKOR method for multiple criteria group decision making with triangular fuzzy neutrosophic numbers. *Symmetry*, *10*(10), 497. doi:10.3390/sym10100497

Wei, G. W. (2017a). Pythagorean fuzzy interaction aggregation operators and their application to multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *33*(4), 2119–2132. doi:10.3233/JIFS-162030

Wei, G. W. (2017b). Picture uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making. *Kybernetes*, *46*(10), 1777–1800. doi:10.1108/K-01-2017-0025

Wei, G. W. (2017c). Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making. *Informatica*, *28*(3), 547–564. doi:10.15388/Informatica.2017.144

Wei, G. W. (2018a). Some similarity measures for picture fuzzy sets and their applications. *Iranian Journal of Fuzzy Systems*, *15*(1), 77–89.

Wei, G. W. (2018b). Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *Fundamenta Informaticae*, *157*(3), 271–320. doi:10.3233/FI-2018-1628

Wei, G. W. (2018c). TODIM method for picture fuzzy multiple attribute decision making. *Informatica*, *29*(3), 555–566. doi:10.15388/Informatica.2018.181

Wei, G. W., & Lu, M. (2017a). Dual hesitant Pythagorean fuzzy Hamacher aggregation operators in multiple attribute decision making. *Archives of Control Sciences*, *27*(3), 365–395. doi:10.1515/acsc-2017-0024

Wei, G. W., & Lu, M. (2017b). Pythagorean hesitant fuzzy Hamacher aggregation operators in multiple attribute decision making. *Journal of Intelligent Systems*. DOI: doi:10.1515/jisys-2017-0106

Wei, G. W., & Lu, M. (2018a). Pythagorean fuzzy Maclaurin symmetric mean operators in multiple attribute decision making. *International Journal of Intelligent Systems*, *33*(5), 1043–1070. doi:10.1002/int.21911

Wei, G. W., & Lu, M. (2018b). Pythagorean fuzzy power aggregation operators in multiple attribute decision making. *International Journal of Intelligent Systems*, *33*(1), 169–186. doi:10.1002/int.21946

Wei, G. W., Gao, H., & Wei, Y. (2018a). Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making. *International Journal of Intelligent Systems*, *33*(7), 1426–1458. doi:10.1002/int.21985

Wei, G. W., Gao, H., Wang, J., & Huang, Y. H. (2018c). Research on risk evaluation of enterprise human capital investment with interval-valued bipolar 2-tuple linguistic information. *IEEE Access*, *6*, 35697–35712. doi:10.1109/ACCESS.2018.2836943

Wei, G. W., Lu, M., Alsaadi, F. E., Hayat, T., & Alsaedi, A. (2017a). Pythagorean 2-tuple linguistic aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *33*(2), 1129–1142. doi:10.3233/JIFS-16715

Wei, G. W., Lu, M., Tang, X. Y., & Wei, Y. (2018b). Pythagorean hesitant fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *International Journal of Intelligent Systems*, *33*(6), 1197–1233. doi:10.1002/int.21978

Wei, G., & Wei, Y. (2018). Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *35*(2), 2001–2013. doi:10.3233/JIFS-171741

Wei, Y., Liu, J., Lai, X., & Hu, Y. (2017b). Which determinant is the most informative in forecasting crude oil market volatility: Fundamental, speculation, or uncertainty? *Energy Economics*, *68*, 141–150. doi:10.1016/j.eneco.2017.09.016

Wei, Y., Yu, Q., Liu, J., & Cao, Y. (2018d). Hot money and China's stock market volatility: Further evidence using the GARCH-MIDAS model. *Physica A: Statistical Mechanics and Its Applications*, *492*, 923–930. doi:10.1016/j.physa.2017.11.022

Wu, S. J., & Wei, G. W. (2017). Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. *International Journal of Knowledge-Based and Intelligent Engineering Systems*, *21*(3), 189–201. doi:10.3233/KES-170363

Wu, S., Wang, J., Wei, G., & Wei, Y. (2018). Research on construction engineering project risk assessment with some 2-tuple linguistic neutrosophic Hamy mean operators. *Sustainability*, *10*(5), 1536. doi:10.3390/su10051536

Yager, R. R. (2001). The power average operator. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, *31*(6), 724–731. doi:10.1109/3468.983429

Yager, R. R. (2013). *Pythagorean fuzzy subsets*. Proceeding of The Joint IFSA Wprld Congress and NAFIPS Annual Meeting, Edmonton, Canada, pp. 57–61.

Yager, R. R. (2014). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, *22*(4), 958–965. doi:10.1109/TFUZZ.2013.2278989

Yu, D. J. (2013). Intuitionistic fuzzy geometric Heronian mean aggregation operators. *Applied Soft Computing*, *13*(2), 1235–1246. doi:10.1016/j.asoc.2012.09.021

Yu, S. M., Zhou, H., Chen, X. H., & Wang, J. Q. (2015). A multi-criteria decision-making method based on Heronian mean operators under a linguistic hesitant fuzzy environment. *Asia-Pacific Journal of Operational Research*, *32*(05), 1550035. doi:10.1142/S0217595915500359

Yue, Z., & Jia, Y. (2013). A method to aggregate crisp values into interval-valued intuitionistic fuzzy information for group decision making. *Applied Soft Computing*, *13*(5), 2304–2317. doi:10.1016/j.asoc.2012.12.032

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–356. doi:10.1016/S0019-9958(65)90241-X

Zeng, S. Z., Mu, Z. M., & Balezentis, T. (2018). A novel aggregation method for Pythagorean fuzzy multiple attribute group decision making. *International Journal of Intelligent Systems*, *33*(3), 573–585. doi:10.1002/int.21953

Zeng, S., Chen, J., & Li, X. (2016). A hybrid method for Pythagorean fuzzy multiple-criteria decision making. *International Journal of Information Technology & Decision Making*, *15*(02), 403–422. doi:10.1142/S0219622016500012

Zhang, R. T., Wang, J., Zhu, X. M., Xia, M. M., & Yu, M. (2017). Some generalized Pythagorean fuzzy Bonferroni mean aggregation operators with their application to multiattribute group decision-making. *Complexity*, *2017*, 1–16. Volume Article ID 5937376, pages. doi:10.1155/2017/5937376

Zhang, X. L., & Xu, Z. S. (2014). Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *International Journal of Intelligent Systems*, *29*(12), 1061–1078. doi:10.1002/int.21676

Zhu, B., Xu, Z. S., & Xia, M. M. (2012). Hesitant fuzzy geometric Bonferroni means. *Information Sciences*, *205*, 72–85. doi:10.1016/j.ins.2012.01.048