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THETA LIFTS OF GENERIC REPRESENTATIONS: THE CASE OF ODD ORTHOGONAL GROUPS

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Abstract. We determine the occurrence and explicitly describe the theta lifts on all levels of all the irreducible generic representations of the odd orthogonal group defined over a local nonarchimedean field of characteristic zero.

1. Introduction

In this paper, we investigate the representation theory of the odd-orthogonal group $\mathrm{O}(V)$ defined over a nonarchimedean local field $\mathbb{F}$ of characteristic 0. Our main results describe the theta correspondence for generic representations of $\mathrm{O}(V)$.

To explain the contents of this paper in detail, let us briefly recall the basic setting of theta correspondence. Let V_m be a quadratic space of odd dimension m over $\mathbb{F}$ (i.e., a space endowed with a non-degenerate symmetric $\mathbb{F}$-bilinear form). The odd orthogonal group is the corresponding group of isometries, denoted $\mathrm{O}(V_m)$. In order to introduce theta correspondence we need to consider another group, the so-called metaplectic group. To define it, we let W_n be a symplectic space of (even) dimension n over $\mathbb{F}$. The corresponding group in this case is the symplectic group $\mathrm{Sp}(W_n)$. We define the metaplectic group $\mathrm{Mp}(W_n)$ as the (unique) non-trivial central extension

$$1 \to \{\pm 1\} \to \mathrm{Mp}(W_n) \to \mathrm{Sp}(W_n) \to 1$$

(see section 2.1). The groups $\mathrm{O}(V_m)$ and $\mathrm{Mp}(W_n)$ form a dual pair inside a larger metaplectic group $\mathrm{Mp}(W_{mn})$. Fixing a non-trivial additive character ψ of $\mathbb{F}$, we obtain the so-called Weil representation of the metaplectic group

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$\mathrm{Mp}(W_{mn})$. Restricting this representation to $O(V_m) \times \mathrm{Mp}(W_n)$ we obtain the Weil representation $\omega_{m,n}$ of this dual pair.

For any $\pi \in \mathrm{Irr}(\mathrm{O}(V_m))$ we may look at the maximal π-isotypic quotient of $\omega_{m,n}$. We denote it $\Theta(\pi, W_n)$ and call it the full theta lift of π to V. This representation, when non-zero, has a unique irreducible quotient, denoted $\theta(\pi, W_n)$—the small theta lift of π. This basic fact, called the Howe duality conjecture, was first formulated by Howe in [14], proven by Waldspurger in [34] (for odd residue characteristic) and by Gan and Takeda ([8]) in general.

The Howe duality establishes a map $\pi \mapsto \theta(\pi)$ which is called the theta correspondence. It is an exceptionally useful tool in the representation theory of p-adic groups. However, its importance also stems from number-theoretic considerations, since the global variant of theta correspondence can be used for constructing automorphic representations. For this reason, theta correspondence has been an area of active research for the last forty years. The study of theta correspondence was initiated by Roger Howe in [14, 15] and continued by Kudla in [16, 17], Rallis in [27], Kudla-Rallis in [18], Moeglin-Vigneras-Waldspurger in [20], Waldspurger in [34] and many others. In recent years this topic has seen a major revival of interest, with many new developments and many old problems being resolved. However, the two main problems concerning theta lifts still remain open: determining when $\Theta(\pi, W_n)$ is non-zero and identifying $\theta(\pi, W_n)$ explicitly. The main contribution of this paper is the complete resolution of these problems when π is a generic representation of the odd orthogonal group.

To explain our results, we first recall the definition of generic representations (see section 2.7 for additional details). Let $B = TU$ be the standard Borel subgroup of $\mathrm{O}(V_m)$. Every non-trivial additive character ψ of $\mathbb{F}$ induces a non-degenerate character χ of U. We say that a representation (π, V) of $O(V_m)$ is χ-generic if there is a non-trivial linear functional $l_\pi \colon V \to \mathbb{C}$ such that

$$l_\pi(\pi(u)v) = \chi(u) l_\pi(v).$$

for all $v \in V$ and $u \in U$. If χ is fixed, we often say simply that π is generic. An important result concerning generic representations is the so-called standard module conjecture, proven by Muić in [21]. It asserts that any generic representation of $\mathrm{Irr}(\mathrm{O}(V_m))$ is isomorphic to its standard module. This allows us to prove the following (cf. Theorem 4.1).

Theorem 1.1. *The first non-zero lift of a generic representation* $\pi \in \mathrm{Irr}(\mathrm{O}(V_m))$ *occurs at* $n = m - 1$.

To interpret this theorem, we recall a basic fact about theta correspondence in towers (cf. Proposition 3.3): if $\Theta(\pi, W_n) \neq 0$, then $\Theta(\pi, W_{n+2}) \neq 0$. Thus, to answer the question of occurrence, it is enough to know the first non-zero occurrence of π, which is given by the above theorem. Note also that any $\pi \in \mathrm{Irr}(\mathrm{O}(V_m))$ comes paired with $\pi \otimes \det$. It is natural to consider the lifts of these two representations simultaneously. The above theorem states that one of the two representations $\{\pi, \pi \otimes \det\}$ first appears on level $n = m - 1$. The conservation relation (see §3.2) then implies that the other must first appear when $n = m+1$. To differentiate the two, we use the results of Gan and Savin from [7].

The second problem, i.e. that of explicitly describing the theta lifts $\theta(\pi, W_n)$ is resolved by the following theorem (see Theorem 5.1). To simplify our calculations we modify the notation: for $\pi \in \mathrm{Irr}(\mathrm{O}(V_m)$ we denote $\theta(\pi, W_n)$ by $\theta_l(\pi)$, where $l = m - n - \epsilon$. We have the following result.

THEOREM 1.2. *Let π be an irreducible generic representation of* $\mathrm{Irr}(O(V_m))$ *isomorphic to its standard module,*

$$\chi_W \delta_r \nu^{s_r} \times \cdots \times \chi_W \delta_1 \nu^{s_1} \rtimes \pi_0.$$

Let l be an even integer such that $\theta_l(\pi) \neq 0$. Then

$$\chi_V \delta_r \nu^{s_r} \times \cdots \times \chi_V \delta_1 \nu^{s_1} \rtimes \theta_l(\pi_0) \twoheadrightarrow \theta_l(\pi).$$

Furthermore, if $\theta_l(\pi_0) = L(\chi_V \delta'_k \nu^{s'_k} \times \cdots \times \chi_V \delta'_1 \nu^{s'_1} \rtimes \tau)$, then $\theta_l(\pi)$ is uniquely determined by

$$\theta_l(\pi) = L(\chi_V \delta_r \nu^{s_r}, \ldots, \chi_V \delta_1 \nu^{s_1}, \chi_V \delta'_k \nu^{s'_k}, \ldots, \chi_V \delta'_1 \nu^{s'_1}; \tau).$$

This paper is the continuation of that of M. Hanzer and the author [4]. Together, these two papers provide a complete description of theta lifts of generic representations for the dual pair $(\mathrm{Mp}(W_n), \mathrm{O}(V_m))$. Analogous results were obtained in [3] for the dual pair $(\mathrm{Sp}(W_n), \mathrm{O}(V_m))$ when m is even. The results and techniques we use in this paper are similar to those of [3] and [4]. We rely heavily on Jacquet module computations similar to those utilized by Muić in [22, 24, 25]. We also use the results of Muić from [21] and Hanzer from [12] on generic representations, the results of Atobe and Gan from [2] on the lifts of tempered representations and those of Gan and Savin from [7] on the theta correspondence for the dual pair $(\mathrm{Mp}(W_n), \mathrm{O}(V_m))$.

Let us briefly describe the contents of this paper. In Section 2 we go over the basic notation and the results regarding the representation theory of the (quasi-split) classical p-adic groups. In Section 3 we review the main results concerning theta correspondence in general. We also derive a number of useful corollaries (3.6, 3.7, 3.8) of Kudla's filtration (Theorem 3.4) which we use in subsequent sections. Section 4 contains the proof of Theorem 4.1 which determines the first occurrence index. The proof relies on Kudla's filtration and the standard module conjecture to reduce the question of occurrence to the case of tempered representations, which is known by the work of Atobe and Gan ([2]). In the fifth section we state our main result and prove it in some special cases. Section 6 contains a number of auxiliary technical results based on the work of Zelevinsky ([35]). These results are used in Section 7, which contains the rest of the proof of Theorem 5.1, providing a complete description of the lifts.

2. Preliminaries

2.1. *Groups.* Let $\mathbb{F}$ be a nonarchimedean local field of characteristic 0 and let $|\cdot|$ be the absolute value on $\mathbb{F}$ (normalized as usual).

All the groups considered in this paper will be defined over $\mathbb{F}$. For $\epsilon = \pm 1$ fixed, we let

$$\begin{cases} W_n = \text{a } (-\epsilon)\text{-Hermitian space of dimension } n, \\ V_m = \text{an } \epsilon\text{-Hermitian space of dimension } m. \end{cases}$$

When $\epsilon = 1$, this means that W_n is symplectic, whereas V_m is a quadratic space. In this case we denote by $Sp(W_n)$ the symplectic group (i.e., the group of isometries of W_n) and we define the corresponding metaplectic group $Mp(W_n)$ as the unique non-trivial central extension

$$1 \to \{\pm 1\} \to \mathrm{Mp}(W_n) \to \mathrm{Sp}(W_n) \to 1$$

with Rao's cocycle ([28]) used to define the extension. Although $\mathrm{Mp}(W_n)$ is not a linear group, it inherits a number of structural properties from the symplectic group—most importantly, we can define the standard parabolic subgroups (§2.3). For detailed accounts of the structural theory of the metaplectic group we refer to [17, 28, 9, 13]. Set

$$H(V_m) = \begin{cases} \mathrm{O}(V_m) \text{ (the orthogonal group)} & \text{if } \epsilon = 1, \\ \mathrm{Mp}(V_m) \text{ (the metaplectic group)} & \text{if } \epsilon = -1, \end{cases}$$

and define $G(W_n)$ similarly by switching the roles of the groups. These groups will also be denoted H_m and G_n. Furthermore, let $\mathrm{GL}(X)$ denote the general linear group of a vector space X over $\mathbb{F}$. Note that all the groups considered here are totally disconnected locally compact topological groups.

2.2. *Witt towers.* Every Hermitian space V_m has a Witt decomposition

$$V_m = V_{m_0} + V_{r,r} \quad (m = m_0 + 2r),$$

where V_{m_0} is anisotropic and $V_{r,r}$ is split (i.e. a sum of r hyperbolic planes). The space V_{m_0} is unique up to isomorphism, and so is the number $r \geqslant 0$, which is called the Witt index of V_m. The collection of spaces

$$\mathcal{V} = \{V_{m_0} + V_{r,r} : r \geqslant 0\}$$

is called a Witt tower. Since

$$\det(V_{m_0+2r}) = (-1)^r \det(V_{m_0}) \in \mathbb{F}^\times/(\mathbb{F}^\times)^2,$$

the quadratic character

$$\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}} \det(V))_{\mathbb{F}}$$

is the same for all the spaces V in a single Witt tower (see [17, §V.1]; here $(\cdot,\cdot)_{\mathbb{F}}$ denotes the Hilbert symbol).

REMARK 2.1. In this paper we consider Witt towers of odd dimension; this implies $m_0 = \dim(V_{m_0}) \in \{1,3\}$. However, if $\dim(V_{m_0}) = 3$, the orthogonal groups in the corresponding tower are not quasi-split, and thus have no generic representations. Consequently, we assume $m_0 = 1$ in the rest of the paper.

The symplectic spaces W_n can be organized in a Witt tower in the same way. This case is somewhat simpler: since the only anisotropic symplectic space is the trivial one, there is only one tower of symplectic spaces. The corresponding character χ_W is trivial.

2.3. *Parabolic subgroups.* Let V_m be a Hermitian space with a non-degenerate form $(\cdot,\cdot)$ and let $V_{m_0} \oplus V_{r,r}$ be its Witt decomposition. We can choose a basis $\{u_1,\dots,u_r,u_1',\dots,u_r'\}$ for $V_{r,r}$ such that $(u_1,u_j') = \delta_{ij}$. Such a basis determines a choice of a standard minimal parabolic (i.e. Borel, if $H(V_m)$ is quasi-split) subgroup. Furthermore, for any $t \leqslant r$ we can decompose

$$V_m = U_t \oplus V_{m-2t} \oplus U_t'$$

where $U_t = \operatorname{span}\{u_1,\dots,u_t\}$ and $U_t' = \operatorname{span}\{u_1',\dots,u_t'\}$ are isotropic subspaces of V_m. The subgroup P_t of $\mathrm{O}(V_m)$ which stabilizes U_t is a maximal parabolic subgroup; it has a Levi decomposition $P_t = M_tN_t$, with Levi component $M = \mathrm{GL}(U_t) \times \mathrm{O}(V_{m-2t})$ (we identify $\mathrm{GL}(U_t)$ with $\mathrm{GL}_t(\mathbb{F})$).

By letting t vary, we obtain a set $\{P_t : t \in \{1,\dots,r\}\}$ of standard maximal parabolic subgroups. By further partitioning t, we get the rest of the standard parabolic subgroups—generally, the Levi factor of a standard parabolic subgroup is of the form

$$\mathrm{GL}_{t_1}(\mathbb{F}) \times \cdots \times \mathrm{GL}_{t_k}(\mathbb{F}) \times \mathrm{O}(V_{m-2t}) \quad (t = t_1 + \cdots + t_k).$$

The standard parabolic subgroups of $\mathrm{Sp}(W_n)$ are constructed in the same way. For any parabolic subgroup Q of $\mathrm{Sp}(W_n)$ we denote by $\widetilde{Q}$ its preimage in $\mathrm{Mp}(W_n)$. This way, we obtain the set of standard parabolic subgroups of $\mathrm{Mp}(W_n)$. From the Levi decomposition $Q = MN$ we get $\widetilde{Q} = \widetilde{M}N'$ where N' is the image in $\mathrm{Mp}(W_n)$ of the unique monomorphism from N to $\mathrm{Mp}(W_n)$. In this case $\widetilde{M}$ is not a product of GL factors and a metaplectic group of smaller rank, but there is an epimorphism

$$\widetilde{\mathrm{GL}}_{t_1}(\mathbb{F}) \times \cdots \times \widetilde{\mathrm{GL}}_{t_k}(\mathbb{F}) \times \mathrm{Mp}(W_{n'}) \twoheadrightarrow \widetilde{M}.$$

Here $\widetilde{\mathrm{GL}}_t(\mathbb{F})$ denotes the twofold cover of $GL_t(F)$, i.e. $GL_t(F) \times \{\pm 1\}$ with multiplication given by $(g_1,\epsilon_1)(g_2,\epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2(\det g_1, \det g_2)_{\mathbb{F}})$. This allows us to view any representation π of $\widetilde{M}$ as a representation $\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ of $\widetilde{\mathrm{GL}}_{t_1}(\mathbb{F}) \times \cdots \times \widetilde{\mathrm{GL}}_{t_k}(\mathbb{F}) \times \mathrm{Mp}(W_{n'})$, where $\rho_1,\dots,\rho_k$ and σ are all either trivial or non-trivial when restricted to $\mu_2 = \{\pm 1\}$. Further information can be found in [13] and [31, Chapter 4].

We denote the maximal standard parabolic subgroups of $H(V_m)$ and $G(W_n)$ by P_t and Q_t, respectively.

2.4. *Representations.* Let G be a reductive p-adic group, such as $\mathrm{O}(V_m)$. By a representation of G we mean a pair (π, V) where V is a complex vector space and π is a homomorphism $G \to \mathrm{GL}(V)$. With V_∞ we denote the subspace of V comprised of all the smooth vectors, i.e. those having an open stabilizer in G. If $V = V_\infty$, we say that the representation (π, V) is smooth. Unless otherwise stated, we will assume that all the representations are smooth; the category of all smooth complex representations of G will be denoted $\mathcal{A}(G)$. The set of equivalence classes of smooth irreducible representations of G will be denoted $\mathrm{Irr}(G)$. All these concepts also apply in case G is the metaplectic group. In this paper, we only consider genuine representations of the metaplectic group, i.e. those which do not factor through the underlying symplectic group.

For each parabolic subgroup $P = MN$ of G we have the (normalized) induction and localization (Jacquet) functors, $\mathrm{Ind}_P^G \colon \mathcal{A}(M) \to \mathcal{A}(G)$ and $R_P \colon \mathcal{A}(G) \to \mathcal{A}(M)$. These are connected by the standard Frobenius reciprocity

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\pi')) \cong \mathrm{Hom}_M(R_P(\pi), \pi')$$

and by the second (Bernstein) form of Frobenius reciprocity,

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G(\pi'), \pi) \cong \mathrm{Hom}_M(\pi', R_{\overline{P}}(\pi))$$

(here $\overline{P} = M\overline{N}$ is the parabolic subgroup opposite to P).

If $P = MN$ is a parabolic subgroup of $\mathrm{O}(V_m)$ with Levi factor $M = \mathrm{GL}_{t_1}(\mathbb{F}) \times \cdots \times \mathrm{GL}_{t_k}(\mathbb{F}) \times \mathrm{O}(V_{m-2t})$, we write

$$\tau_1 \times \cdots \times \tau_k \rtimes \pi_0$$

for $\mathrm{Ind}_P^{\mathrm{O}(V)}(\tau_1 \otimes \cdots \otimes \tau_k \otimes \pi_0)$, where τ_i is a representation of $\mathrm{GL}_{t_i}(\mathbb{F})$ and π_0 is a representation of $\mathrm{O}(V_{m-2t})$ (with $t = t_1 + \cdots + t_k$). We use the same notation to denote the genuine representation of $\mathrm{Mp}(W_n)$ induced from $\tau_1, \dots, \tau_k$ and a genuine representation π_0 of $\mathrm{Mp}(W_{n-2t})$. This notation implies that the GL-representations τ_i are lifted to representations of the respective double covers which are all genuine; see Section 4.1 of [13] or Section 4.1 of [31].

To obtain a complete list of irreducible representations of $H(V_m)$, we use the Langlands classification: let $\delta_i \in \mathrm{GL}_{t_i}(\mathbb{F}), i = 1, \dots, r$ be irreducible discrete series representations, and let τ be an irreducible tempered representation of $H(V_{m-2t})$ (for $t = t_1 + \cdots + t_r$). Any representation of the form

$$\nu^{s_r}\delta_r \times \cdots \times \nu^{s_1}\delta_1 \rtimes \tau,$$

where $s_r \geqslant \cdots \geqslant s_1 > 0$ (and where ν denotes the character $|\det|$ of the corresponding general linear group) is called a standard representation (or a standard module). It possesses a unique irreducible quotient, the so-called Langlands quotient, denoted $L(\nu^{s_r}\delta_r \times \cdots \times \nu^{s_1}\delta_1 \rtimes \tau)$. Occasionally, we will also write $L(\nu^{s_r}\delta_r, \ldots, \nu^{s_1}\delta_1; \tau)$, implying that the representations $\{\nu^{s_r}\delta_r, \ldots, \nu^{s_1}\delta_1\}$ are to be sorted decreasingly with respect to s_i's before taking the quotient. Conversely, every irreducible representation can be represented as the Langlands quotient of a unique standard representation. In this way, we obtain a complete description of $\mathrm{Irr}(H(V_m))$. We note that the proof of the Langlands classification in cases which interest us (i.e. for $\mathrm{O}(V)$ and $\mathrm{Mp}(W)$) was provided by Ban and Jantzen ([5, 6]).

We will use this (quotient) form of the Langlands classification interchangeably with the subrepresentation form, by means of the following lemma ([2, Lemma 2.2]).

LEMMA 2.2. *Let P be a standard parabolic subgroup of $H(V_m)$ with Levi component equal to $GL_{t_1}(\mathbb{F}) \times \cdots \times GL_{t_r}(\mathbb{F}) \rtimes H(V_{m_0})$. Then, for $\tau_i \in \mathrm{Irr}(GL_{t_i}(\mathbb{F}))$, $\pi_0 \in \mathrm{Irr}(H(V_{m_0}))$ and $\pi \in \mathrm{Irr}(H(V_m))$ the following statements are equivalent:*

(i) $\pi \hookrightarrow \tau_1 \times \cdots \times \tau_r \rtimes \pi_0$;
(ii) $\tau_1^\vee \times \cdots \times \tau_r{}^\vee \rtimes \pi_0 \twoheadrightarrow \pi$.

Here (and in the rest of this paper) $\tau^\vee$ denotes the contragredient representation. When dealing with tempered representations, we often need the following result of Goldberg ([10], Theorems 6.4 and 6.5)

LEMMA 2.3. *Let $\delta_i \in \mathrm{Irr}(GL_{t_i}(\mathbb{F}))$ for $i = 1, \ldots, k$ and $\pi_{00} \in \mathrm{Irr}H(V)$ be discrete series representations. Then the induced representation $\delta_1 \times \cdots \times \delta_k \rtimes \pi_{00}$ is a direct sum of mutually non-equivalent tempered representations. It is of length 2^L where L is the number of non-equivalent δ_i such that $\delta_i \rtimes \pi_{00}$ reduces.*

This result is originally stated for the split classical groups, but it extends to representations of the metaplectic group ([11, Theorem 3.5]) and the full orthogonal group $\mathrm{O}(V)$. The latter is a consequence of the following fact: since $\mathrm{O}(V) = \mathrm{SO}(V) \times \{\pm I\}$, the restriction of any irreducible representation of $\mathrm{O}(V)$ to $\mathrm{SO}(V)$ is also irreducible.

REMARK 2.4. This connection between $\mathrm{O}(V)$ and $\mathrm{SO}(V)$ means that most of our results can be stated for either one of those groups. We note that any irreducible representation of $\mathrm{SO}(V)$ extends to two irreducible representations of $O(V)$, π and $\pi \otimes \det$, which differ by their central character $\nu \in \{\pm I\}$.

2.5. *Computing Jacquet modules.* We need to compute the Jacquet modules of various representations on a number of occasions. For any $\pi \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}))$ we let $m^*(\pi)$ denote the sum of the semi-simplifications of $R_P(\pi)$ when P varies over the set of maximal standard parabolic subgroups of

$\mathrm{GL}_n(\mathbb{F})$. The basic fact due to Zelevinsky (see Section 1.7 of [35] for additional details) is that

$$m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \times m^*(\pi_2).$$

Furthermore, this will mostly be required in the case when $\pi = \delta([\rho, \rho\nu^k])$ is an essentially square integrable representation corresponding to a segment $[\rho, \rho\nu^k]$ of cuspidal representations (see section 6). In that case, we have

$$\text{(JM1)} \qquad m^*(\delta[\rho, \nu^k\rho]) = \sum_{i=-1}^{k} \delta([\nu^{i+1}\rho, \nu^k\rho]) \otimes \delta([\rho, \nu^i\rho]).$$

This theory was extended by Tadić to the case of classical groups in [32]. For any $\pi \in \mathrm{Irr}(H(V_m))$ we let $\mu^*(\pi)$ be the sum of the semi-simplifications of $R_P(\pi)$ when P varies over the set of maximal standard parabolic subgroups of $H(V_m)$. The relevant formula is now

$$\mu^*(\delta \rtimes \pi) = M^*(\delta) \rtimes \mu^*(\pi).$$

The definition of M^* can be found in [32, Theorem 5.4], but we shall need it here only in the special case when $\delta = \delta([\rho, \rho\nu^k])$; in this case, we have ([33, §14])

$$M^*([\rho, \nu^k\rho]) = \sum_{i=-1}^{k}\sum_{j=i}^{k} \delta([\nu^{-i}\rho^\vee, \rho^\vee]) \times \delta([\nu^{j+1}\rho, \nu^k\rho]) \otimes \delta([\nu^{i+i}\rho, \nu^j\rho]).$$

The above results extend in the same form to the case of the metaplectic group ([13]).

2.6. *Local Langlands Correspondence.* Another way of classifying the irreducible representations of $H(V_m)$ is by means of the Local Langlands Correspondence (LLC). We use it mainly to harvest the results on lifts of tempered representations established recently by Atobe and Gan in [2]. Without going into detail, we give a brief description of the basic features of LLC; a concise overview of the theory along with the key references can be found in appendices A and B of [2].

The LLC parametrizes $\mathrm{Irr}(H(V_m))$ by representations of the Weil-Deligne group, $\mathrm{WD}_\mathbb{F} = \mathrm{W}_\mathbb{F} \times \mathrm{SL}_2(\mathbb{C})$ (here $W_\mathbb{F}$ denotes the Weil group of $\mathbb{F}$). More precisely, we define $\Phi(H(V_m))$ as a set of equivalence classes:

$$\begin{cases} \Phi(\mathrm{Mp}(V_m)) = \{\phi\colon \mathrm{WD}_\mathbb{F} \to \mathrm{Sp}(m, \mathbb{C})\}/\cong, & (m \text{ is even}) \\ \Phi(\mathrm{O}(V_m)) = \{\phi\colon \mathrm{WD}_\mathbb{F} \to \mathrm{Sp}(m-1, \mathbb{C})\}/\cong & (m \text{ is odd}). \end{cases}$$

The irreducible representations of $H(V_m)$ are then parametrized by the so called L-parameters, i.e., pairs of the form (ϕ, η), where $\phi \in \Phi(H(V_m))$, and η is a character of the (finite) component group of the centralizer of $\mathrm{Im}(\phi)$. The set of representations which correspond to the same ϕ is called an L-packet attached to ϕ.

Any $\phi \in \Phi(H(V_m))$ can be decomposed as

$$\phi = \bigoplus_{n\geqslant 1} \phi_n \otimes S_n,$$

where ϕ_n is a representation of $W_{\mathbb{F}}$, whereas S_n denotes the unique algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n. Tempered representations are parametrized by pairs (ϕ, η) in which $\phi(W_{\mathbb{F}})$ is bounded; the discrete series representations correspond to parameters which are bounded and multiplicity free.

Note that, unlike ϕ, the choice of η is non-canonical: it depends on the choice of a Whittaker datum of $H(V_m)$ (see [2, Remark B.2]). This choice will be fixed, and will correspond to the characters used in the definition of generic representations and the Weil representation (see Remark 2.5 and section 3.1).

2.7. *Generic representations.* Let $B = TU$ be the standard Borel subgroup of $\mathrm{O}(V_m)$ as fixed in section 2.3. Recall that we are assuming that $\mathrm{O}(V_m)$ is quasi-split (see Remark 2.1), so that the Borel subgroup is indeed defined over $\mathbb{F}$. Every non-trivial additive character ψ of $\mathbb{F}$ induces a non-degenerate character χ of U (see e.g. [26, §1]). We say that a representation (π, V) of $O(V_m)$ is χ-generic if there is a non-trivial linear functional $l_\pi \colon V \to \mathbb{C}$ such that

$$l_\pi(\pi(u)v) = \chi(u) l_\pi(v).$$

for all $v \in V$ and $u \in U$.

Remark 2.5. The character χ of U will be fixed throughout our calculations; this allows us to shorten the notation: instead of χ-generic, we will often simply refer to π being generic. Moreover, the choice of Whittaker datum needed to fix the LLC in section 2.6 coincides with the one we make here. Matching these choices has an important consequence: if (ϕ, η) is an L-parameter of a χ-generic representation, then η is necessarily equal to the trivial character, as shown by H. Atobe in Desideratum 1 of [1].

The following theorem contains the most important properties of generic representations which we often use. The first two (established by F. Rodier in [29]) are known as the heredity and the uniqueness of the Whittaker model, respectively. The third one is the standard module conjecture, established by G. Muić in [21].

Theorem 2.6. *(i) If $\tau_i \in \mathrm{Irr}(GL_{t_i}(\mathbb{F})), i = 1, \dots, r$ are irreducible generic representations, and π_0 is an irreducible representation of $O(V_m)$, then $\tau_1 \times \cdots \times \tau_k \rtimes \pi_0$ is χ-generic if and only if π_0 is χ-generic.*

(ii) If $\pi_0, \tau_i \in \mathrm{Irr}(GL_{t_i}(\mathbb{F})), i = 1, \dots, r$ are irreducible generic representations of $O(V_m)$, then $\tau_1 \times \cdots \times \tau_k \rtimes \pi_0$ contains a unique irreducible generic subquotient, which has multiplicity one.

(iii) The standard module of any irreducible generic representation of $O(V_m)$ is itself irreducible.

Note that the (iii) can be viewed as a consequence of the so-called generalized injectivity conjecture, established by M. Hanzer in [12]. We often combine it with the following result ([23, Introduction]).

Proposition 2.7. *A standard representation of the form $\nu^{s_r}\delta_r \times \cdots \times \nu^{s_1}\delta_1 \rtimes \tau$ reduces if and only if one of the following holds*

(i) $\nu^{s_i}\delta_i \times \nu^{s_j}\delta_j$ reduces for some pair $i \neq j$;
(ii) $\nu^{s_i}\delta_i \times \nu^{-s_j}\delta_j^\vee$ reduces for some pair $i \neq j$;
(iii) $\nu^{s_i} \rtimes \tau$ reduces for some i.

3. Theta correspondence

In this section, we review the basic facts concerning the local theta correspondence established in [16, 14, 34]. We also fix the notation, roughly following [17].

3.1. *Howe duality.* Let $\omega_{m,n}$ be the Weil representation of $H(V_m) \times G(W_n)$. The Weil representation depends on the choice of a non-trivial additive character $\psi\colon \mathbb{F} \to \mathbb{C}$. This character will be fixed throughout (see the end of section 2.6 for the choice we make), so we omit it from the notation. Similarly, if the dimensions m and n are known, we will often simply write ω instead of $\omega_{m,n}$.

For any $\pi \in \mathrm{Irr}(H(V_m))$, a basic structural fact about the Weil representation ([20, Chapter II, III.4]) guarantees that the maximal π-isotypic quotient of $\omega_{m,n}$ is of the form

$$\pi \otimes \Theta(\pi, W_n)$$

for a certain smooth representation $\Theta(\pi, W_n)$ of $G(W_n)$, called the full theta lift of π. When the target Witt tower is fixed, we will often denote it by $\Theta(\pi, n)$ or, more often, by $\Theta_l(\pi)$, where $l = m - n - \epsilon$.

The key result which establishes the theta correspondence is the following:

Theorem 3.1 (Howe duality). *If $\Theta(\pi, W_n)$ is non-zero, it possesses a unique irreducible quotient, denoted $\theta(\pi, W_n)$.*

Originally conjectured by Howe in [14, p. 279], it was first proven by Waldspurger in [34] when the residual characteristic of $\mathbb{F}$ is different from 2, and by Gan and Takeda in [8] in general. The representation $\theta(\pi, W_n)$ is called the (small) theta lift of π; like the full lift, it will also be denoted $\theta(\pi, n)$ and $\theta_l(\pi)$.

For future reference, we state the following simple but useful fact from [24, Lemma 1.1].

Lemma 3.2. *For $\pi \in \mathrm{Irr}(H(V_m))$ we have*

$$\Theta^\vee(\pi, n) = \mathrm{Hom}_{H(V_m)}(\omega_{m,n}, \pi)_\infty.$$

3.2. *First occurrence in towers.* The study of theta correspondence in towers is motivated by the following facts (Propositions 4.1 and 4.3 of [17]).

PROPOSITION 3.3. *Let π be an irreducible representation of $H(V_m)$.*
(i) If $\Theta(\pi, W_n) \neq 0$, then $\Theta(\pi, W_{n+2r}) \neq 0$ for all $r \geqslant 0$.
(ii) For n large enough, we have $\Theta(\pi, W_n) \neq 0$.

The above proposition implies that we can define, for any Witt tower $\mathcal{W} = (W_n)$,

$$n_{\mathcal{W}}(\pi) = \min\{n \geqslant 0 : \Theta(\pi, W_n) \neq 0\}.$$

This number (also denoted $n(\pi)$ when the choice of $\mathcal{W}$ is implicit) is called the first occurrence index of π. Note that we are using the term "index" here to signify the dimension, although it would be more appropriate to use it for the Witt index of the corresponding space.

An important result which helps us compute the first occurrence indices is the so-called conservation relation. The Witt towers of quadratic spaces can be appropriately organized into pairs, with the towers comprising a pair denoted $\mathcal{W}^+$ and $\mathcal{W}^-$ (a complete list of pairs of dual towers can be found in [17, Chapter V]). Thus, instead of observing just one target tower, we can simultaneously look at two of them. This way, for each $\pi \in \mathrm{Irr}(H(V_m))$ we get two corresponding first occurrence indices, $n^+(\pi)$ and $n^-(\pi)$.

If $\epsilon = 1$ then V_m is a quadratic space, and there is only one tower of symplectic spaces (W_n). We proceed as follows: since $H(V_m)$ is now equal to $\mathrm{O}(V_m)$, any $\pi \in \mathrm{Irr}(O(V_m))$ is naturally paired with its twist, $\pi \otimes \det$. This allows us to define

$$n^{\pm}(\pi) = \min\{n(\pi') : \pi' \in \{\pi, \pi \otimes \det\} \text{ such that } \pi'(-I) = \pm \mathrm{id}\}.$$

We are now able to set

$$n^{\mathrm{down}}(\pi) = \min\{n^+(\pi), n^-(\pi)\}, \quad n^{\mathrm{up}}(\pi) = \max\{n^+(\pi), n^-(\pi)\}$$

regardless of whether $\epsilon = 1$ or $\epsilon = -1$. The conservation relation (first conjectured by Kudla and Rallis in [19], completely proven by Sun and Zhu in [30]) states that

$$n^{\mathrm{up}}(\pi) + n^{\mathrm{down}}(\pi) = 2n - 2\epsilon + 2.$$

The tower in which $n(\pi) = n^{\mathrm{down}}(\pi)$ (resp. n^{up}) is often called the going-down (resp. going-up) tower. As we have already noted, there is only one possible target tower when lifting from the orthogonal group. However, we still use the tower terminology to differentiate between the two possible series of lifts, i.e. those of π or those of $\pi \otimes \det$.

3.3. *Kudla's filtration.* One of our main tools is Kudla's filtration of $R_P(\omega)$, the Jacquet module of the Weil representation ([16, Theorem 2.8]). We state it here (formulated as in [2, Theorem 5.1]) along with a few useful corollaries.

THEOREM 3.4. *The Jacquet module $R_{P_k}(\omega_{m,n})$ possesses an $GL_k(\mathbb{F}) \times H(V_{m-2k}) \times G(W_n)$-equivariant filtration*

$$R_{P_k}(\omega_{m,n}) = R^0 \supset R^1 \supset \cdots \supset R^k \supset R^{k+1} = 0$$

in which the successive quotients $J^a = R^a/R^{a+1}$ are given by

$$J^a = \mathrm{Ind}_{P_{k-a,a}\times H_{m-2k}\times Q_a}^{GL_k\times H_{m-2k}\times G_n} \left(\chi_W|\det{}_{GL_{k-a}}|^{\lambda_{k-a}} \otimes \Sigma_a \otimes \omega_{m-2k,n-2a}\right),$$

where

- $\lambda_{k-a} = (n-m+k-a+\epsilon)/2$;
- $P_{k-a,a}$ *is the standard parabolic subgroup of GL_k with Levi component $GL_{k-a} \times GL_a$;*
- $\Sigma_a = C_c^\infty(GL_a(\mathbb{F}))$*, the space of locally constant compactly supported functions on $GL_a(\mathbb{F})$. The action of $GL_a(\mathbb{F}) \times GL_a(\mathbb{F})$ on Σ_a is given by*

$$[(g,h)\cdot f](x) = \chi_W(\det(g))\chi_V(\det(h))f(g^{-1}\cdot x\cdot h).$$

If $n-2a$ is less than the dimension of the first (anisotropic) space in $\mathcal{W}$, we put $R^a = J^a = 0$.

We will often use the following proposition (see [24, Corollary 3.2], [2, Proposition 5.2]) derived from the previous theorem:

PROPOSITION 3.5. *Assume $l = m-n-\epsilon > 0$ and $k > 0$. Let $\pi_0 \in \mathrm{Irr}(H_{m-2k})$ and let δ be an irreducible essentially square integrable representation of $GL_k(\mathbb{F})$. Then the space* $\mathrm{Hom}_{GL_k(\mathbb{F})\times H_{m-2k}}(J^a, \chi_W\delta^\vee \otimes \pi_0)_\infty$*, viewed as a representation of G_n, is isomorphic to*

$$\begin{cases} \chi_V^{-1}\delta^\vee \rtimes \mathrm{Hom}_{H_{m-2k}}(\omega_{m-2k,n-2k},\pi_0)_\infty, & \text{if } a=k,\\ \chi_V^{-1}St_{k-1}\nu^{\frac{k-l+1}{2}} \rtimes \mathrm{Hom}_{H_{m-2k}}(\omega_{m-2k,n-2k+2},\pi_0)_\infty, & \text{if } \begin{array}{c} a=k-1\\ \&\\ \delta = St_k\nu^{\frac{l-k}{2}}\end{array},\\ 0, & \text{otherwise.}\end{cases}$$

Recall that, in the above proposition, we have $\mathrm{Hom}_G(\omega,\pi)_\infty = \Theta^\vee(\pi)$. Furthermore, St_k denotes the so-called Steinberg representation of $\mathrm{GL}_k(\mathbb{F})$, the square integrable representation attached to the segment $[|\cdot|^{\frac{1-k}{2}}, |\cdot|^{\frac{k-1}{2}}]$ (see the beginning of section 6).

We now list a few useful corollaries of Proposition 3.5. The first one is Corollary 5.3 of [2]. See also [24, Corollary 3.2].

COROLLARY 3.6. *Let $\pi \in \mathrm{Irr}(H_m), \pi_0 \in \mathrm{Irr}(H_{m-2k})$ and let δ be an irreducible essentially square integrable representation of $GL_k(\mathbb{F})$. Assume that $\delta \not\cong St_k\nu^{\frac{l-k}{2}}$, where $l = m-n-\epsilon$. Then*

$$\chi_W\delta \rtimes \pi_0 \twoheadrightarrow \pi$$

implies

$$\chi_V\delta \rtimes \Theta_l(\pi_0) \twoheadrightarrow \Theta_l(\pi).$$

The second corollary we state is a slight modification of the first: this time, we are unable to obtain information about the full lift $\Theta_l(\pi)$, but we allow the special case $\delta \cong \mathrm{St}_k\nu^{\frac{l-k}{2}}$.

COROLLARY 3.7. *Let δ be an irreducible essentially square integrable representation of $GL_k(\mathbb{F})$ and let $\pi \in \mathrm{Irr}(H_M), \pi_0 \in \mathrm{Irr}(H_{m-2k})$ be such that*

$$\chi_W\delta \rtimes \pi_0 \twoheadrightarrow \pi,$$

as in the preceding corollary. Then one of the following is true:

(i) $\chi_V\delta \rtimes \Theta_l(\pi_0) \twoheadrightarrow \theta_l(\pi)$*; or*

(ii) $\chi_V\delta([|\cdot|^a, |\cdot|^{b-1}]) \rtimes \Theta_{l-2}(\pi_0) \twoheadrightarrow \theta_l(\pi)$*.*

Option (ii) is possible only if δ is attached to the segment $[|\cdot|^a, |\cdot|^b]$ with $b = \frac{l-1}{2}$, i.e., $\delta \cong St_k\nu^{\frac{l-k}{2}}$ for some positive integer k.

PROOF. According to Lemma 2.2 we have $\pi \hookrightarrow \chi_W\delta^\vee \rtimes \pi_0$, and so

$$\begin{aligned}\Theta_l^\vee(\pi) &\cong \mathrm{Hom}_{H_m}(\omega_{m,n}, \pi)_\infty \\ &\hookrightarrow \mathrm{Hom}_{H_m}(\omega_{m,n}, \chi_W\delta^\vee \rtimes \pi_0)_\infty \\ &\cong \mathrm{Hom}_{\mathrm{GL}_k \times H_{m-2k}}(R_{P_k}(\omega_{m,n}), \chi_W\delta^\vee \otimes \pi_0)_\infty.\end{aligned}$$

We now use Kudla's filtration to analyze $R_{P_k}(\omega_{m,n})$. For each index $a = 0, \dots, k$ we have an exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}(J^a, \chi_W\delta^\vee \otimes \pi_0)_\infty \to \mathrm{Hom}(R^a, \chi_W\delta^\vee \otimes \pi_0)_\infty \\ &\to \mathrm{Hom}(R^{a+1}, \chi_W\delta^\vee \otimes \pi_0)_\infty.\end{aligned}$$

Since we know, by Proposition 3.5, that the space $\mathrm{Hom}(J^a, \chi_W\delta^\vee \otimes \pi_0)_\infty$ is trivial for $a = 0, \dots, k-2$, this leads to an inclusion

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}_k \times H_{m-2k}}(R_{P_k}(\omega_{m,n}), \chi_W\delta^\vee \otimes \pi_0)_\infty \\ &\qquad \hookrightarrow \mathrm{Hom}_{\mathrm{GL}_k \times H_{m-2k}}(R^{k-1}, \chi_W\delta^\vee \otimes \pi_0)_\infty.\end{aligned}$$

In particular, we have $\Theta_l^\vee(\pi) \hookrightarrow \mathrm{Hom}_{\mathrm{GL}_k \times H_{m-2k}}(R^{k-1}, \chi_W\delta^\vee \otimes \pi_0)_\infty$. As $\theta_l^\vee(\pi)$ is a subrepresentation of $\Theta_l^\vee(\pi)$, we conclude that there is an injective equivariant map

$$f\colon \theta_l^\vee(\pi) \hookrightarrow \mathrm{Hom}_{\mathrm{GL}_k \times H_{m-2k}}(R^{k-1}, \chi_W\delta^\vee \otimes \pi_0)_\infty.$$

On the other hand, we have the exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}(J^{k-1}, \chi_W\delta^\vee \otimes \pi_0)_\infty \xrightarrow{g} \mathrm{Hom}(R^{k-1}, \chi_W\delta^\vee \otimes \pi_0)_\infty \\ &\xrightarrow{h} \mathrm{Hom}(J^k, \chi_W\delta^\vee \otimes \pi_0)_\infty.\end{aligned}$$

We now consider two options:

(i) If $\mathrm{Im}(f) \cap \mathrm{Ker}(h) = 0$, then we have an injective map

$$h \circ f \colon \theta_l^\vee(\pi) \hookrightarrow \mathrm{Hom}(J^k, \chi_W \delta^\vee \otimes \pi_0)_\infty.$$

Proposition 3.5 describes $\mathrm{Hom}(J^k, \chi_W \delta^\vee \otimes \pi_0)_\infty$; by taking the contragredient we get

$$\chi_V \delta \rtimes \Theta_l(\pi_0) \twoheadrightarrow \theta_l(\pi).$$

(ii) If $\mathrm{Im}(f) \cap \mathrm{Ker}(h) \neq 0$, then the irreducibility of $\theta_l(\pi)$ implies $\theta_l^\vee(\pi) \hookrightarrow \mathrm{Ker}(h)$. By the exactness of the above sequence we have $\mathrm{Ker}(h) = \mathrm{Im}(g)$, and since g is injective, we also have $\mathrm{Im}(g) \cong \mathrm{Hom}(J^{k-1}, \chi_W \delta^\vee \otimes \pi_0)_\infty$. Thus, we can write

$$\theta_l^\vee(\pi) \hookrightarrow \mathrm{Hom}(J^{k-1}, \chi_W \delta^\vee \otimes \pi_0)_\infty$$

from which, by looking at the contragradient (and using Proposition 3.5), we arrive at

$$\chi_V \delta([|\cdot|^a, |\cdot|^{b-1}]) \rtimes \Theta_{l-2}(\pi_0) \twoheadrightarrow \theta_l(\pi).$$

Note that this second option is only possible if $\mathrm{Hom}(J^{k-1}, \chi_W \delta^\vee \otimes \pi_0)_\infty$ is non-trivial; in particular, by Proposition 3.5, $\delta = \delta([|\cdot|^a, |\cdot|^b])$ with $b = \frac{l-1}{2}$ is a necessary condition.

□

Finally, we state a generalization of the above corollary. The same proof, with an additional application of the exactness of the induction functor, yields the following result.

Corollary 3.8. *Let δ be an irreducible essentially square integrable representation of $GL_k(\mathbb{F})$ and let $\pi \in \mathrm{Irr}(G_n), \pi_0 \in \mathrm{Irr}(H_{m-2k})$ be such that*

$$\chi_W \delta \rtimes \pi_0 \twoheadrightarrow \pi.$$

Furthermore, let A be an irreducible representation of a general linear group. Assume that an irreducible representation σ satisfies

$$\chi_V A \rtimes \Theta_l(\pi) \twoheadrightarrow \sigma,$$

where $l = n - m - \epsilon$. Then one of the following is true:

(i) $\chi_V A \times \chi_V \delta \rtimes \Theta_l(\pi_0) \twoheadrightarrow \sigma$; or

(ii) $\chi_V A \times \chi_V \delta([|\cdot|^a, |\cdot|^{b-1}]) \rtimes \Theta_{l-2}(\pi_0) \twoheadrightarrow \sigma$.

Option (ii) is possible only if δ is attached to the segment $[|\cdot|^a, |\cdot|^b]$ with $b = \frac{l-1}{2}$.

Remark 3.9. At some point it will be useful to use the same notation for the outcomes of both options (i) and (ii). With this in mind, we set

$$(\delta) = \begin{cases} \delta, & \text{if we used option (i)} \\ \delta([|\cdot|^a, |\cdot|^{b-1}]), & \text{if we used option (ii).} \end{cases}$$

3.4. *Discrete series and tempered representations.* In this section we go over some of the important results concerning the theta lifts of discrete series and tempered representations. First, we recall the main results of Muić from [24] (Theorems 6.1 and 6.2), which give a complete description of theta lifts for discrete series representations, along with an insight into the structure of the full theta lift.

THEOREM 3.10. *Let $\sigma \in \mathrm{Irr}(H_m)$ be a discrete series representation. Set*

$$n_{temp}(\sigma) = \begin{cases} n(\sigma), & n(\sigma) > m - \epsilon \\ m - \epsilon, & n(\sigma) \leqslant m - \epsilon. \end{cases}$$

Then

(i) $\Theta(\sigma, n)$ *is an irreducible tempered representation for* $n(\sigma) \leqslant n \leqslant n_{temp}(\sigma)$.

(ii) If $n > n_{temp}(\sigma)$*, then* $\theta(\sigma, n)$ *is the unique irreducible (Langlands) quotient of*

$$\chi_V |\cdot|^{\frac{n-m+\epsilon-1}{2}} \times \ldots \times \chi_V |\cdot|^{\frac{n_{temp}(\sigma)-m+\epsilon+1}{2}} \rtimes \theta(\sigma, n_{temp}(\sigma)).$$

The remaining subqoutients of $\Theta(\sigma, n)$ *are either tempered, or equal to the Langlands quotient of*

$$\chi_V |\cdot|^{\frac{n-m+\epsilon-1}{2}} \times \ldots \times \chi_V |\cdot|^{\frac{n_1-m+\epsilon+1}{2}} \rtimes \sigma(n_1),$$

where $\sigma(n_1)$ *is a tempered irreducible subquotient of* $\Theta(\sigma, n_1)$ *for some* $n > n_1 \geqslant n_{temp}(\sigma)$.

PROOF. The above theorem is a summary of Theorems 6.1 and 6.2 of [24], which, although stated for a different dual pair (symplectic–even orthogonal), transfer easily to our setting. Furthermore, in [24] it is assumed that the residual characteristic of $\mathbb{F}$ is different from 2. We point out that this assumption is no longer needed, due to the fact that Howe duality is now proven in all cases ([8]). Finally, note that the first part of (ii) and the temperedness of $\Theta(\sigma, n)$ for $n < n_{\mathrm{temp}}$ also follow from the results of [2] on tempered representations (Theorems 4.3, 4.5 and Proposition 5.4). □

As mentioned earlier, the recent results of Atobe and Gan ([2]) on theta lifts of tempered representations subsume most of the aforeknown results on the lifts of discrete series. For the sake of brevity, we do not state the relevant theorems here; we shall however use them on more than one occasion in the following sections. For now, we state a useful auxiliary result concerning tempered representations, see [2, Proposition 5.5, Lemma 6.4].

PROPOSITION 3.11. *Let $\pi \in \mathrm{Irr}(G(W_n))$ be such that $\Theta(\pi, V_m) \neq 0$.*

(1) If one of the following is satisfied

(i) π *is tempered and* $m \leqslant n + 1 + \epsilon$*;*

(ii) *π is in discrete series and $\Theta(\pi, V_m)$ is the first lift to the going-up tower,*

then all the irreducible subquotients of $\Theta(\pi, V_m)$ are tempered.

(2) *If all the irreducible subquotients of $\Theta(\pi, V_m)$ are tempered, then they all belong to the same L-packet.*

4. First occurrence

In this section we describe the first occurrence index of a generic representation $\pi \in \mathrm{Irr}(\mathrm{O}(V_m))$. We now fix $\epsilon = 1$ so that $H(V_m) = \mathrm{O}(V_m)$. Note that this also implies $\chi_W = \mathbb{1}$. Recall that $n^{\mathrm{down}}(\pi)$ denotes the lower of the two possible first occurrence indices, $n^+(\pi)$ and $n^-(\pi)$. We set

$$l(\pi) = m - 1 - n^{\mathrm{down}}(\pi).$$

This notation is motivated by the one used by Atobe and Gan, but does not have quite the same meaning as in their paper [2]. By the standard module conjecture, π is isomorphic to its standard module:

$$\pi \cong \delta_r \nu^{s_r} \times \cdots \times \delta_1 \nu^{s_1} \rtimes \pi_0,$$

where $\delta_i \in \mathrm{Irr}_{\mathrm{disc}} \mathrm{GL}_{m_i}$ $(i = 1, \ldots, r)$, $s_r \geqslant \cdots \geqslant s_1 > 0$, and $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}} \mathrm{O}(V_{m_0})$ for $m_0 = m - \sum_{i=1}^{r} 2m_i$. Note that π_0 is also generic by the hereditary property.

The first occurrence index is determined by the following theorem:

Theorem 4.1. *Let $\pi \in \mathrm{Irr}(\mathrm{O}(V_m))$ be a generic representation. Then $l(\pi) = 0$.*

Since the first occurrence of tempered representations is described by [2, Theorem 4.1], it will be enough to show that $l(\pi) = l(\pi_0)$. Indeed, if (ϕ, η) is the L-parameter of π_0, we know that η must be trivial, since π_0 is generic (see Remark 2.5). This means that the alternating property of Theorem 4.1 in [2] is never satisfied, so that $l(\pi_0) = 0$.

Note that this gives us $n^{\mathrm{down}}(\pi) = m - 1$ and $n^{\mathrm{up}}(\pi) = m + 1$.

Remark 4.2. Before proving the theorem, we remind the reader of the notation: recall that $\Theta_l(\pi) = \Theta(\pi, m - \epsilon - l)$. Combined with our definition of $l(\pi)$ and the conservation relation, this means that $\Theta_l(\pi)$ denotes the first non-zero lift of π precisely when

$$l = \begin{cases} l(\pi), & \text{in the going-down tower;} \\ -l(\pi) - 2, & \text{in the going-up tower.} \end{cases}$$

Proof of Theorem 4.1. We first consider the going-up tower with respect to π_0. We compute $\Theta_l(\pi)$ with $l = -l(\pi_0)$. Since $s_i > 0$, we know that $\delta_i\nu^{s_i} \neq \mathrm{St}_k\nu^{\frac{l-k}{2}}$ for all $1,\dots,r$. This allows us to use Corollary 3.6. Repeatedly applying it to

$$(*) \qquad \delta_r\nu^{s_r} \times \cdots \times \delta_1\nu^{s_1} \rtimes \pi_0 \twoheadrightarrow \pi$$

we get

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \rtimes \Theta_l(\pi_0) \twoheadrightarrow \Theta_l(\pi).$$

This being the going-up tower, we have $\Theta_l(\pi_0) = 0$ (see Remark 4.2). Since the above map is surjective, this implies $\Theta_l(\pi) = 0$. We deduce that

- the going-up tower for π is the same as for π_0;
- we have $-l(\pi) \leqslant l$, i.e. $l(\pi) \geqslant l(\pi_0)$.

Now set $l = l(\pi_0)+2$; this time we consider the going-down tower with respect to π_0. We repeat the above argument to show that

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \rtimes \Theta_l(\pi_0) \twoheadrightarrow \Theta_l(\pi).$$

In this case it can happen that for some i we have $\delta_i\nu^{s_i} = \mathrm{St}_k\nu^{\frac{l-k}{2}}$. To justify the use of Corollary 3.6, we need a simple application of the MVW involution: since $\delta_r\nu^{s_r} \times \cdots \times \delta_1\nu^{s_1} \rtimes \pi_0$ is irreducible, we have

$$\delta_r\nu^{s_r} \times \cdots \times \delta_i\nu^{s_i} \times \cdots \times \delta_1\nu^{s_1} \rtimes \pi_0 \cong \delta_r\nu^{s_r} \times \cdots \times \delta_i\nu^{-s_i} \times \cdots \times \delta_1\nu^{s_1} \rtimes \pi_0.$$

We can thus replace $\delta_i\nu^{s_i}$ with $\delta_i\nu^{-s_i}$ in $(*)$ and thus bypass the restriction of Corollary 3.6. Since $l > l(\pi_0)$, we have $\Theta_l(\pi_0) = 0$, so the above map implies $\Theta_l(\pi) = 0$ (see Remark 4.2). This means that $l(\pi) < l$, i.e. $l(\pi) \leqslant l(\pi_0)$.

Combining the two inequalities we get the desired result, $l(\pi) = l(\pi_0)$. It is worth mentioning the following fact obtained in the proof: the going-up (going-down) tower for π coincides with the going-up (going-down) tower for π_0. ∎

5. The lifts

We are now ready to state the main result of this paper. The following theorem fully describes the theta lifts of a generic irreducible representation of $\mathrm{Irr}(\mathrm{O}(V_m))$.

Theorem 5.1. *Let π be an irreducible generic representation of $\mathrm{Irr}(O(V_m))$ isomorphic to its standard module,*

$$\delta_r\nu^{s_r} \times \cdots \times \delta_1\nu^{s_1} \rtimes \pi_0.$$

Let l be an even integer such that $\theta_l(\pi) \neq 0$. Then

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \rtimes \theta_l(\pi_0) \twoheadrightarrow \theta_l(\pi).$$

Furthermore, if $\theta_l(\pi_0) = L(\chi_V\delta'_k\nu^{s'_k} \times \cdots \times \chi_V\delta'_1\nu^{s'_1} \rtimes \tau)$, then $\theta_l(\pi)$ is uniquely determined by

$$\theta_l(\pi) = L(\chi_V\delta_r\nu^{s_r}, \dots, \chi_V\delta_1\nu^{s_1}, \chi_V\delta'_k\nu^{s'_k}, \dots, \chi_V\delta'_1\nu^{s'_1}; \tau).$$

In order to sketch our general approach, we now prove this theorem in case when $\theta_l(\pi_0)$ is tempered. The rest of the proof is more involved and will be given in several steps in §7.

PROOF. Theorem 4.1 shows that we need only consider $\theta_l(\pi)$ for $l \leqslant 0$. With this in mind, Theorems 4.3 and 4.5 of [2] imply that the only cases in which $\theta_l(\pi_0)$ is tempered are the first lifts of π_0: $l = 0$ in the going-down tower and $l = -2$ in the going-up tower. We treat them separately.

Case 1: $l = 0$, going-down tower
Since the left-hand side of

$$\delta_r\nu^{s_r} \times \cdots \times \delta_1\nu^{s_1} \rtimes \pi_0 \twoheadrightarrow \pi$$

has a unique irreducible quotient, we can repeatedly apply Corollary 3.6 to arrive at

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \rtimes \Theta_0(\pi_0) \twoheadrightarrow \Theta_0(\pi). \tag{1}$$

The use of Corollary 3.6 is justified: since $l = 0$ and $s_i > 0$, none of $\delta_i\nu^{s_i}$ are defined by a segment ending in $|\cdot|^{-\frac{1}{2}}$.

Notice that $\Theta_0(\pi_0)$ is irreducible and tempered: writing π_0 as a quotient of its tempered support, we let $\delta_1', \dots, \delta_k'$ and π_{00} be the discrete series representations such that

$$\delta_1' \times \cdots \times \delta_k' \rtimes \pi_{00} \twoheadrightarrow \pi_0$$

In this situation we can also use Corollary 3.6: the segment defining δ_i' cannot end in $|\cdot|^{-\frac{1}{2}}$ (which is the exceptional case for $l = 0$). We get

$$\chi_V\delta_1' \times \cdots \times \chi_V\delta_k' \rtimes \Theta_0(\pi_{00}) \twoheadrightarrow \Theta_0(\pi_0).$$

We can now use Theorem 3.10: $\Theta_0(\pi_{00})$ is irreducible and tempered. This shows, by Lemma 2.3, that the left-hand side in the above map is completely reducible, and that all of its irreducible subquotients are tempered. Thus, the same must hold for $\Theta_0(\pi_0)$. Since $\Theta_0(\pi_0)$ has a unique irreducible quotient, complete reducibility implies that $\Theta_0(\pi_0)$ is itself irreducible (and tempered).

This shows that the left-hand side of (1) is a standard module. Furthermore, since $\Theta_0(\pi) \twoheadrightarrow \theta_0(\pi)$, we can write

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \rtimes \Theta_0(\pi_0) \twoheadrightarrow \theta_0(\pi)$$

instead of (1) and in this way arrive at the standard module for $\theta_0(\pi)$.

Case 2: $l = -2$, going-up tower
This case is treated just like the previous one. Using Corollary 3.6 we get

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \rtimes \Theta_{-2}(\pi_0) \twoheadrightarrow \Theta_{-2}(\pi)$$

and it only remains to show that $\Theta_{-2}(\pi_0)$ is irreducible and tempered. Again we look at the tempered support of π_0; then the first lift of π_{00}(the classical part of the support) is also $\Theta_{-2}(\pi_{00})$. By Theorem 3.10, this means that

$\Theta_{-2}(\pi_{00})$ is irreducible and tempered, so we can deduce the same properties for $\Theta_{-2}(\pi_0)$ just like in Case 1. ∎

6. Interlude: irreducibility in $GL_n(\mathbb{F})$

Before advancing to the main part of the proof of Theorem 5.1, we prove some auxiliary results concerning certain induced representations of $\mathrm{GL}_n(\mathbb{F})$ which appear in our calculations. The reader is advised to skim through this section at first reading, since only the statements (and not their proofs) are crucial for the next section.

We recall the work of Zelevinsky ([35]): to each segment $[\rho, \nu^k\rho]$, $k \in \mathbb{Z}_{\geqslant 0}$, of irreducible cuspidal representations we can attach the induced representation

$$\nu^k\rho \times \nu^{k-1}\rho \times \cdots \times \nu\rho \times \rho.$$

A representation of this form has a unique Langlands quotient, but also a unique subrepresentation, denoted $\delta = \delta([\rho, \nu^k\rho])$. Such a representation is essentially square integrable; conversely, any essentially square integrable representation of the general linear group can be obtained in this way from a (uniquely determined) segment. In what follows, we will assume that ρ is always equal to the trivial character $\mathbb{1}$ of $\mathrm{GL}_1(\mathbb{F})$, although the same proofs work for any cuspidal ρ. Therefore we modify the traditional notation and omit ρ: the unique quotient of $\nu^b\rho \times \cdots \times \nu^a\rho$ attached to $[\nu^a\rho, \nu^b\rho]$ will be denoted simply by $\zeta(a,b)$, and we will write $\delta([a,b])$ instead of $\delta([\nu^a\rho, \nu^b\rho])$. At various points of this section, we freely use the terminology and results of [35] on linked segments.

We begin with the following lemma. We say that two (or more) numbers are congruent modulo $\mathbb{Z}$ if their difference is an integer.

Lemma 6.1. *Let $c \leqslant a \leqslant d < b \in \mathbb{R}$ be congruent mod $\mathbb{Z}$. Then*

$$\zeta(a,b) \times \delta([c,d]) \quad \textit{and} \quad \delta([c,d]) \times \zeta(a,b)$$

are irreducible and isomorphic.

Notice that $c \leqslant a \leqslant d < b$ implies that the segment $[a,b]$ intersects $[c,d]$ from "above". We first prove this in a special case when $a = d$ and $b = d+1$.

Lemma 6.2. *The representation $\zeta(d, d+1) \times \delta([c,d])$ is irreducible.*

Proof. We give a proof by induction on $d-c$. Set $k = d-c+3$, so that $\zeta(d,d+1) \times \delta([c,d])$ is a representation of $\mathrm{GL}_k(\mathbb{F})$. The base case, i.e. the fact that $\zeta(d,d+1) \times |\cdot|^d$ is irreducible, is known from [35].

Now let $c < d$; assume that the statement is true for all c' such that $c < c' < d$.

We compute the Jacquet modules (their semi-simplifications, to be precise) of $\zeta(d,d+1) \times \delta([c,d])$ with respect to standard parabolic subgroups

$P_{k-1,1}, P_{k-2,2}$ i $P_{k-2,1,1}$. We aim to apply a criterion of Tadić on the irreducibility of induced representations; see e.g. [33, §21]. To compute the semi-simplifications, we use the well known formula for m^* (see (JM1) in section 2.5) and an analogous formula for $m^*((a,b)$. Using this, we find that the semi-simplifications of the Jacquet modules with respect to $P_{k-1,1}, P_{k-2,2}$ and $P_{k-2,1,1}$ are given by direct sums of the following representations:
$P_{k-1,1}$:

$$|\cdot|^d \times \delta([c,d]) \otimes |\cdot|^{d+1} \tag{A}$$

$$\zeta(d,d+1) \times \delta([c+1,d]) \otimes |\cdot|^c \tag{B}$$

$P_{k-2,2}$:

$$\delta([c,d]) \otimes \zeta(d,d+1) \tag{1}$$

$$\zeta(d,d+1) \times \delta([c+2,d]) \otimes \delta([c,c+1]) \tag{2}$$

$$|\cdot|^d \times \delta([c+1,d]) \otimes |\cdot|^{d+1} \times |\cdot|^c \tag{3}$$

$P_{k-2,1,1}$:

$$\delta([c,d]) \otimes |\cdot|^d \otimes |\cdot|^{d+1} \tag{A1}$$

$$\zeta(d,d+1) \times \delta([c+2,d]) \otimes |\cdot|^{c+1} \otimes |\cdot|^c \tag{B2}$$

$$|\cdot|^d \times \delta([c+1,d]) \otimes |\cdot|^{d+1} \otimes |\cdot|^c \tag{B3}$$

$$|\cdot|^d \times \delta([c+1,d]) \otimes |\cdot|^c \otimes |\cdot|^{d+1} \tag{A3}$$

Notice that all of the above representations are irreducible by the induction hypothesis. Furthermore, $P_{k-1,1}$ shows that the length of $\zeta(d,d+1) \times \delta([c,d])$ is at most 2; if it equals 2, then one subquotient accounts for (A), and the other for (B). On the other hand, from the fact that (3) splits into (A3) and (B3) wee see that (A) and (B) come from the same subquotient as (3). In particular, (A) and (B) come from the same subquotient, which shows that the length of $\zeta(d,d+1) \times \delta([c,d])$ is 1, not 2.

This proves the lemma; the fact that $\zeta(d,d+1) \times \delta([c,d]) \cong \delta([c,d]) \times \zeta(d,d+1)$ follows from the irreducibility. □

We are now ready to prove Lemma 6.1.

Proof. First, we claim that

$$\Pi = |\cdot|^b \times |\cdot|^{b-1} \times \cdots \times |\cdot|^a \times \delta([c,d])$$

has a unique irreducible quotient.

We reduce this to the corresponding claim about standard representations. Set $s = \frac{c+d}{2}$, the midpoint of $[c,d]$. Recall that $c \leqslant a \leqslant d < b$. In particular, this means that $b > \frac{c+d}{2}$. Therefore, we can find the smallest element of $[a,b]$ which is greater than s; denote it by b_0.

We then have the following:

$$|\cdot|^b \times |\cdot|^{b-1} \times |\cdot|^{b_0} \times \delta([c,d]) \times |\cdot|^{b_0-1} \cdots \times |\cdot|^a$$

possesses a unique (Langlands) irreducible quotient (which appears with multiplicity 1). Since $a, a+1, \ldots, b_0-1$ are contained in $[c,d]$, $\delta([c,d])$ can switch places with $|\cdot|^i$, for all $i = a, \ldots, b_0-1$. Therefore, the above standard representation in isomorphic to

$$|\cdot|^b \times |\cdot|^{b-1} \times |\cdot|^{b_0} \times |\cdot|^{b_0-1} \times \cdots \times |\cdot|^a \times \delta([c,d]) = \Pi.$$

It follows that Π has a unique irreducible quotient as well – we denote this unique quotient by π. We have thus shown that Π possesses a unique irreducible quotient. Furthermore, we obviously have a surjective map $\Pi \twoheadrightarrow \zeta(a,b) \times \delta([c,d])$, so that π is a unique irreducible quotient of $\zeta(a,b) \times \delta([c,d])$. However, we also have $\Pi \twoheadrightarrow \Pi'$, where

$$\Pi' = |\cdot|^b \times |\cdot|^{b-1} \times \cdots \times |\cdot|^{d+2} \times \zeta(d,d+1) \times |\cdot|^{d-1} \times \cdots \times |\cdot|^a \times \delta([c,d])$$

(note that the segment $[d+2,b]$ and $[a,d-1]$ can be empty, but this does not change the argument). It follows that π is also a unique irreducible quotient of Π'.

As before, we have

$$\Pi' \cong |\cdot|^b \times |\cdot|^{b-1} \times \cdots \times |\cdot|^{d+2} \times \zeta(d,d+1) \times \delta([c,d]) \times |\cdot|^{d-1} \times \cdots \times |\cdot|^a$$

because $a, \ldots, d-1$ are contained in $\delta([c,d])$. By Lemma 6.2 we know that $\zeta(d,d+1) \times \delta([c,d])$ is irreducible, so that $\zeta(d,d+1) \times \delta([c,d]) \cong \delta([c,d]) \times \zeta(d,d+1)$. Thus

$$\Pi' \cong |\cdot|^b \times |\cdot|^{b-1} \times \cdots \times |\cdot|^{d+2} \times \delta([c,d]) \times \zeta(d,d+1) \times |\cdot|^{d-1} \times \cdots \times |\cdot|^a.$$

Finally, none of the numbers $d+2, \ldots, b$ are linked to $[c,d]$ so we can move them as well:

$$\Pi' \cong \delta([c,d]) \times |\cdot|^b \times |\cdot|^{b-1} \times \cdots \times |\cdot|^{d+2} \times \zeta(d,d+1) \times |\cdot|^{d-1} \times \cdots \times |\cdot|^a.$$

Since π is the unique irreducible quotient of the above representation, which maps onto $\delta([c,d]) \times \zeta(a,b)$, we deduce that π is the unique irreducible quotient

$$\delta([c,d]) \times \zeta(a,b).$$

This shows that both $\delta([c,d]) \times \zeta(a,b)$ and $\zeta(a,b) \times \delta([c,d])$ have π as an irreducible quotient which appears with multiplicity one. It follows that the two representations are irreducible and isomorphic. ∎

Remark 6.3. In a similar way (but easier, because Lemma 6.2 isn't necessary) one shows that

$$\zeta(a,b) \times \delta([c,d]) \quad \text{and} \quad \delta([c,d]) \times \zeta(a,b)$$

are irreducible and isomorphic when $[a,b]$ and $[c,d]$ are not linked.

Remark 6.4. A similar argument shows that, when the two segments are linked ($a = d + 1$), then $\delta([c, d]) \times \zeta(d + 1, b)$ has exactly two irreducible subquotients:

$$L(|\cdot|^b \times \cdots \times |\cdot|^{d+1} \times \delta([c, d])) \quad \text{and} \quad L(|\cdot|^b \times \cdots \times |\cdot|^{d+2} \times \delta([c, d + 1])).$$

The above remark will often be combined with the following lemma.

Lemma 6.5. *Denote by L the representation $L(|\cdot|^b \times \ldots \times |\cdot|^{d+1} \times \delta([c, d]))$ which appears in the above remark – note that it is a unique irreducible quotient of $\zeta(b, d + 1) \times \delta([c, d])$. Then*

$$L \times \delta([c, d]) \quad \textit{and} \quad \delta([c, d]) \times L$$

are irreducible and isomorphic.

Proof. We first prove $L \times \delta([c, d]) \cong \delta([c, d]) \times L$. Notice that

$$L \times \delta([c, d]) \hookrightarrow \delta([c, d]) \times \zeta(d + 1, b) \times \delta(c, d).$$

On the other hand, we have an intertwining map

$$T\colon \delta([c, d]) \times \zeta(d + 1, b) \times \delta(c, d) \to \delta([c, d]) \times L$$

with kernel $\ker(T)$ isomorphic to $\delta(c, d) \times L(|\cdot|^b \times \ldots \times |\cdot|^{d+2} \times \delta([c, d + 1]))$ (Remark 6.4).

Restricting T to $L \times \delta([c, d])$ we get an intertwining $\tilde{T}\colon L \times \delta([c, d]) \to \delta([c, d]) \times L$. We want to show that it is injective; to prove this, it suffices to check that $L \times \delta([c, d]) \cap \ker(T) = \{0\}$.

Notice that $\ker(T)$ has a unique irreducible subrepresentation τ – it is the Langlands quotient of

$$|\cdot|^b \times \ldots \times |\cdot|^{d+2} \times \delta([c, d + 1]) \times \delta([c, d]).$$

This uniqueness implies the following: if $L \times \delta([c, d]) \cap \ker(T)$ is non-trivial, it contains τ.

We now look at Jacquet modules again. It is easy to see that the Jacquet module of τ with respect to the appropriate standard parabolic subgroup P contains a subquotient of the form

$$\delta([c, d]) \otimes \delta([c, d + 1]) \otimes \zeta(d + 2, b).$$

If we can show that $R_P\,(L \times \delta([c, d]))$ does not have a subquotient of this form, $L \times \delta([c, d]) \cap \ker(T) = \{0\}$ will follow.

By Remark 6.4 the representation

$$A := \delta([c, d]) \times \zeta(d + 1, b) \times \delta([c, d])$$

has only two subquotients:

$$L \times \delta([c, d]) \quad \text{and} \quad L' \times \delta([c, d]),$$

where $L' = L(|\cdot|^b \times \ldots \times |\cdot|^{d+2} \times \delta([c, d+1]))$. A simple application of the fact that $m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \times m^*(\pi_2)$ shows that $R_P(A)$ contains

$$\delta([c, d]) \otimes \delta([c, d+1]) \otimes \zeta(d+2, b)$$

with multiplicity 2. It suffices to prove that both of those subquotients are accounted for by $R_P(L' \times \delta([c, d]))$.

We have $L' \hookrightarrow \delta([c, d+1]) \times \zeta(d+2, b)$, but also $L' \hookrightarrow |\cdot|^{d+1} \times \delta([c, d]) \times \zeta(d+2, b)$. From here, we easily deduce that $m^*(L')$ contains $\delta([c, d+1]) \otimes \zeta(d+2, b)$ and $|\cdot|^{d+1} \otimes \delta([c, d]) \times \zeta(d+2, b)$. This shows (using the multiplicativity of m^* again) that $m^*(L' \times \delta([c, d]))$ contains

$$\delta([c, d+1]) \times \delta([c, d]) \otimes \zeta(d+2, b) \oplus |\cdot|^{d+1} \times \delta([c, d]) \otimes \delta([c, d]) \times \zeta(d+2, b)$$

Applying the Jacquet functor (with respect to the appropriate parabolic subgroup) we see that the Jacquet module of both summands contains a subquotient of the form $\delta([c, d]) \otimes \delta([c, d+1]) \otimes \zeta(d+2, b)$. This shows that both appearances of this subquotient come from $R_P(L' \times \delta([c, d]))$.

We have now shown that $L \times \delta([c, d]) \cap \ker(T) = \{0\}$, i.e. that $\tilde{T}$ is injective. Since $L \times \delta([c, d])$ and $\delta([c, d]) \times L$ are of equal length, if follows that $\tilde{T}$ is an isomorphism, so that

$$L \times \delta([c, d]) \cong \delta([c, d]) \times L.$$

It is now easy to show that these representations are isomorphic. Let π be the unique irreducible quotient of s$L \times \delta([c, d])$. Then $\pi \hookrightarrow \delta([c, d]) \times L$. From here, we get

$$\pi \hookrightarrow \delta([c, d]) \times L \cong L \times \delta([c, d]) \twoheadrightarrow \pi.$$

Note that π appears in $L \times \delta([c, d])$ with multiplicity 1, because it is in fact the Langlands quotient of

$$|\cdot|^b \times \ldots \times |\cdot|^{d+1} \times \delta([c, d]) \times \delta([c, d]).$$

This shows that the above sequence of intertwining maps is possible only if $\delta([c, d]) \times L \cong L \times \delta([c, d])$ are irreducible. ∎

Finally, we point out another consequence of the results of [35, §9] which finds its use in determining the standard modules of higher lifts:

Remark 6.6. Let Δ_1 and Δ_2 be segments of cuspidal representations. Following [35, §7], we may consider a so-called elementary operation $\{\Delta_1, \Delta_2\} \mapsto \{\Delta^\cup, \Delta^\cap\}$, where

$$\Delta^\cup = \Delta_1 \cup \Delta_2, \quad \Delta^\cap = \Delta_1 \cap \Delta_2.$$

We apply this to draw conclusions about standard modules of $G(W_n)$-representations: let σ and σ_0 be irreducible, such that

$$\delta_k \nu^{s_k} \times \cdots \times \delta_1 \nu^{s_1} \rtimes \sigma_0 \twoheadrightarrow \sigma,$$

where $\delta_1, \dots, \delta_k$ are irreducible discrete series representations, and $s_k \geqslant \dots \geqslant s_1 > 0$. If σ_0 is tempered, the left-hand side is the standard module for σ. Otherwise, we have $\sigma_0 = L(\delta'_l\nu^{s'_l}, \dots, \delta'_1\nu^{s'_1}; \tau)$ for some $\delta'_1, \dots, \delta'_l$ in discrete series and some tempered τ. Setting $\Pi = \delta_k\nu^{s_k} \times \dots \times \delta_1\nu^{s_1} \times \delta'_l\nu^{s'_l} \times \dots \times \delta'_1\nu^{s'_1}$ we thus get

$$\Pi \rtimes \tau \twoheadrightarrow \sigma.$$

It follows that Π possesses an irreducible subquotient, say π, such that $\pi \rtimes \tau \twoheadrightarrow \sigma$. Furthermore, by [35, §9], we know that π is the quotient of a standard representation obtained from Π by a sequence of elementary operations.

We thus have the following conclusion on the shape of the standard module of σ:

- the tempered part is equal to the tempered representation τ which appears in the standard module of σ_0;
- the GL-part is obtained by performing a sequence of elementary operations on the segments defining $\delta'_1\nu^{s'_1}, \dots, \delta'_l\nu^{s'_l}$ and $\delta_1\nu^{s_1}, \dots, \delta_k\nu^{s_k}$.

We use this standard module "mixing" on more than one occasion in the following section.

Before proceeding to the proof of Theorem 5.1, we address one more technical question. The results of this section concern representations of $\mathrm{GL}_n(\mathbb{F})$; we apply them in the following section to various GL-factors which appear in induced representations of orthogonal and metaplectic groups. In case of $\mathrm{Mp}(W)$-representations, this requires some justification, as the Levi subgroups of the metaplectic group are not isomorphic to a product of a (smaller) metaplectic group and some GL-factors. However, the representation theory of GL is the same as that of genuine representations of the double cover $\widetilde{\mathrm{GL}}$ (see Section 4.1 [13]). This, combined with the exactness of the induction functor defined in Section 4.1 [31] allows us to apply the results of this section to GL-factors in induced representations of the metaplectic group.

7. Higher lifts

We are now ready to prove the rest of Theorem 5.1. Recall that we have already settled the cases in which $\theta_l(\pi_0)$ is tempered. In all the remaining cases $l = m - \epsilon - n$ is negative (and even), so we adjust the notation: letting $l > 0$ be an arbitrary even integer, we want to determine $\theta_{-l}(\pi)$.

7.1. *Subquotients of* $\Theta(\pi_0)$. We fix $l > 0$ even and set $\sigma = \theta_{-l}(\pi)$; our goal is to determine σ. Since $\pi \in \mathrm{Irr}(\mathrm{O}(V_m))$ is generic, it is isomorphic to its standard module:

$$\pi \cong \ \delta_r\nu^{s_r} \times \dots \times \ \delta_1\nu^{s_1} \rtimes \pi_0.$$

Applying Corollary 3.6 just like in Section 5, we get

$$\chi_V\delta_r\nu^{s_r} \times \dots \times \chi_V\delta_1\nu^{s_1} \rtimes \Theta_{-l}(\pi_0) \twoheadrightarrow \Theta_{-l}(\pi) \twoheadrightarrow \theta_{-l}(\pi) = \sigma. \tag{0}$$

Our main task is to determine the irreducible subquotient of $\Theta_{-l}(\pi_0)$ which participates in the above epimorphism. To describe it, we need to further analyze π_0. Using the tempered support of π_0 we can write

$$\delta_1' \times \cdots \times \ \delta_k' \rtimes \pi_{00} \twoheadrightarrow \pi_0,$$

where $\delta_1', \ldots, \delta_k', \pi_{00}$ are irreducible discrete series representations. Setting $\Delta = \delta_1' \times \cdots \times \delta_k'$ and applying Corollary 3.6 again, we get

$$\chi_V \Delta \rtimes \Theta_{-l}(\pi_{00}) \twoheadrightarrow \Theta_{-l}(\pi_0).$$

Thus

$$\chi_V \delta_r \nu^{s_r} \times \cdots \times \chi_V \delta_1 \nu^{s_1} \times \chi_V \Delta \rtimes \Theta_{-l}(\pi_{00}) \twoheadrightarrow \sigma. \tag{1}$$

We would now like to identify the irreducible subquotient (call it σ_0) of $\Theta_{-l}(\pi_{00})$ which participates in the above epimorphism. To be precise, we say that an irreducible subquotient σ_0 of T participates in an epimorphism $f\colon A \rtimes T \to \sigma$ if there is a filtration of T

$$0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_l = T$$

such that T_i/T_{i-1} is irreducible for all $i = 1, \ldots, l$, and there is an index $j \in \{1, \ldots, l\}$ such that $f(T_{j-1}) = 0$, $f(T_j) \neq 0$ and $T_j/T_{j-1} \cong \sigma_0$. In that case, we have a map $A \rtimes \sigma_0 \to \sigma$. We will show the following:

Proposition 7.1. *There is a unique irreducible subquotient of* $\Theta_{-l}(\pi_{00})$ *which participates in* (1)*; it is equal to* $\theta_{-l}(\pi_{00})$.

Remark 7.2. We recall the results of [24] and [2] (see Theorem 3.10): $\theta_{-l}(\pi_{00})$ is the Langlands quotient of

$$\chi_V|\cdot|^{\frac{l-1}{2}} \times \chi_V|\cdot|^{\frac{l-3}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1+l_0}{2}} \rtimes \theta_{-l_0}(\pi_{00}) \tag{$*$}$$

where $l_0 = \min\{l \geqslant 0 : \theta_{-l}(\pi_{00}) \neq 0\}$. As π_{00} is generic (by the hereditary property), we have $l_0 \in \{0, 2\}$ (see discussion after statement of Theorem 4.1).

Any other irreducible subquotient of $\Theta_{-l}(\pi_{00})$ is either:

- tempered; or
- the Langlands quotient of

$$\chi_V|\cdot|^{\frac{l-1}{2}} \times \chi_V|\cdot|^{\frac{l-3}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1+l'}{2}} \rtimes \sigma_0',$$

  where σ_0' is a tempered subquotient of $\Theta_{-l'}(\pi_{00})$ for some $l' \geqslant l_0$.

Note that the Langlands quotient described here is also the unique quotient of $\chi_V \zeta\left(\frac{1+l'}{2}, \frac{l-1}{2}\right) \rtimes \sigma_0'$ (using the notation of Section 6).

We are now ready to prove the proposition.

Proof. Assume, with the above remark in mind, that the subquotient of $\Theta_{-l}(\pi_{00})$ we want to find (and which we denote by σ_0) is isomorphic to the unique irreducible quotient of

$$\chi_V\zeta\left(\frac{1+l'}{2},\frac{l-1}{2}\right)\rtimes\sigma_0'.$$

Here we allow the segment $[\frac{1+l'}{2},\frac{l-1}{2}]$ to be empty, i.e. that $\sigma_0=\sigma_0'$ is tempered. We want to prove that $l'=l_0$, so that σ_0 is given by the quotient of $(*)$ in the above remark.

Since σ_0 participates in (1), we have

$$\chi_V\Pi\times\chi_V\Delta\times\chi_V\zeta\left(\frac{1+l'}{2},\frac{l-1}{2}\right)\rtimes\sigma_0'\twoheadrightarrow\sigma,$$

where we used Π to denote $\delta_r\nu^{s_r}\times\cdots\times\delta_1\nu^{s_1}$. We are now in a situation which matches the requirements of Lemma 6.1: $\zeta\left(\frac{1+l'}{2},\frac{l-1}{2}\right)$ can switch places with (almost) all of δ_i' which define Δ. This allows us to write

$$\text{(I)}\qquad \chi_V\Pi\times\chi_V\zeta\left(\frac{1+l'}{2},\frac{l-1}{2}\right)\times\chi_V\Delta\times\sigma_0'\twoheadrightarrow\sigma.$$

The only case in which we cannot proceed as above is the one in which $[\frac{1+l'}{2},\frac{l-1}{2}]$ is adjacent to the segment defining δ_i' for some $i\in\{1,\dots,k\}$, that is, when

$$\delta_i'=\delta([|\cdot|^{\frac{1-l'}{2}},|\cdot|^{\frac{l'-1}{2}}])=\mathrm{St}_{l'}.$$

This does not cause severe complications: without loss of generality we may assume that $\delta_1',\delta_2',\dots,\delta_i',\dots,\delta_k'$ are ordered increasingly with respect to the length of the defining segments. We can apply Lemma 6.1 to swap $\zeta(\frac{1+l'}{2},\frac{l-1}{2})$ with $\delta_{i+1}',\dots,\delta_k'$. After this, we arrive at the following situation:

$$\cdots\times\chi_V\delta_i'\times\chi_V\zeta\left(\frac{1+l'}{2},\frac{l-1}{2}\right)\times\dots\twoheadrightarrow\sigma.$$

Now, Remark 6.4 implies that we have

$$\cdots\times\chi_V\zeta\left(\frac{1+l'}{2},\frac{l-1}{2}\right)\times\chi_V\delta_i'\times\cdots\twoheadrightarrow\sigma$$

or

$$\cdots\times\chi_V\zeta\left(\frac{3+l'}{2},\frac{l-1}{2}\right)\times\chi_V\delta([|\cdot|^{\frac{1-l'}{2}},|\cdot|^{\frac{l'+1}{2}}])\times\cdots\twoheadrightarrow\sigma.$$

The first case leads us to the same conclusion as in (I), whereas the second—having in mind that we can now swap $\zeta(\frac{3+l'}{2},\frac{l-1}{2})$ with all the $\delta_1',\dots,\delta_{i-1}'$—leads to

$$\text{(II)}\quad \chi_V\Pi\times\chi_V\zeta\left(\frac{3+l'}{2},\frac{l-1}{2}\right)\times\chi_V\delta([|\cdot|^{\frac{1-l'}{2}},|\cdot|^{\frac{l'+1}{2}}])\times\chi_V\Delta'\rtimes\sigma_0'\twoheadrightarrow\sigma,$$

where $\Delta' \cong \delta_1' \times \cdots \times \widehat{\delta_i'} \times \cdots \times \delta_k'$ (here $\widehat{\delta_i'}$ signifies that we omit δ_i' from the product).

In both cases I and II we can do the same: using Remark 6.6 (more generally, the results of [35]), $\Pi \times \zeta\left(\frac{1+l'}{2}, \frac{l-1}{2}\right)$ (resp. $\Pi \times \zeta\left(\frac{3+l'}{2}, \frac{l-1}{2}\right) \times \delta([|\cdot|^{\frac{1-l'}{2}}, |\cdot|^{\frac{l'+1}{2}}])$) can be rearranged into

$$\Pi' = \overline{\delta}_t \nu^{e_t} \times \cdots \times \overline{\delta}_1 \nu^{e_1},$$

where $\overline{\delta}_i$ are irreducible discrete series representations and $e_t \geqslant \cdots \geqslant e_1 > 0$). In other words, we get a standard module

$$\chi_V \Pi' \rtimes \tau \twoheadrightarrow \sigma$$

for σ, where τ is an irreducible (and obviously tempered) subquotient of $\chi_V \Delta \rtimes \sigma_0'$ (in case I), or $\chi_V \Delta' \rtimes \sigma_0'$ (in case II).

This shows the following:

(I) In case (I), the cuspidal support of Π' consists of $|\cdot|^{\frac{l-1}{2}}, |\cdot|^{\frac{l-3}{2}}, \ldots, |\cdot|^{\frac{l'+1}{2}}$ in addition to the cuspidal support of Π.

(II) In case (II), the cuspidal support of Π' consists of $|\cdot|^{\frac{l-1}{2}}, |\cdot|^{\frac{l-3}{2}}, \ldots, |\cdot|^{\frac{l'+1}{2}}$ and the segment $[\frac{1-l'}{2}, \frac{l'-1}{2}]$ in addition to Π.

We now use Kudla's filtration to return to the (V_m) tower: we want to get $\theta_l(\sigma)$ (while knowing that $\theta_l(\sigma) = \pi$) by repeated use of Corollary 3.8 on

$$\chi_V \overline{\delta}_t \nu^{e_t} \times \cdots \times \chi_V \overline{\delta}_1 \nu^{e_1} \rtimes \tau \twoheadrightarrow \sigma.$$

If we apply the corollary exactly t times, we get

$$(\overline{\delta}_t \nu^{e_t}) \times \cdots \times (\overline{\delta}_1 \nu^{e_1}) \rtimes \Theta_{l-2k}(\tau) \twoheadrightarrow \pi.$$

Here we use the notation $(\overline{\delta}\nu^e)$ introduced in Remark 3.9. Furthermore, k denotes the number of segments on which option (ii) of Corollary 3.8 is used, which is why Θ_l becomes Θ_{l-2k}.

Again we rearrange the representations $(\overline{\delta}_i \nu^{e_i})$ in order to get a standard module (i.e. so that the midpoints of the corresponding segments form a decreasing sequence).

We need to show that this is actually possible. Each $\overline{\delta}_i \nu^{e_i} = \delta([\rho_i \nu^{-a_i+e_i}, \rho_i \nu^{a_i+e_i}])$ is defined by a segment with midpoint $e_i > 0$ (here a_i is a non-negative half-integer). After applying Corollary 3.8 and bringing them in front of $\Theta_{l-2k}(\tau)$, some of the $(\overline{\delta}_i \nu^{e_i})$ (namely, those obtained via option (ii)) are defined by slightly modified segments of the form $[\rho_i \nu^{-a_i+e_i}, \rho_i \nu^{a_i+e_i-1}]$, with midpoint $(e_i - \frac{1}{2})$. We have the following possibilities:

- If $a_i = 0$, then $[\rho_i \nu^{-a_i+e_i}, \rho_i \nu^{a_i+e_i-1}]$ is empty; therefore $(\overline{\delta}_i \nu^{e_i})$ doesn't exist.
- It is possible that $e_i - \frac{1}{2} = 0$, i.e. that 0 is the midpoint of the new segment.

- All the other segments satisfy $e_i - \frac{1}{2} > 0$: if we use option (ii) in Corollary 3.8, this implies (among other things) that e_i is a half-integer; in particular, $e_i \geqslant \frac{1}{2}$.

Furthermore, note that we can really reorder the $(\overline{\delta}_i\nu^{e_i})$ to obtain a decreasing sequence of exponents. Namely, if this requires us to swap $(\overline{\delta}_{i+1}\nu^{e_{i+1}})$ and $(\overline{\delta}_i\nu^{e_i})$, this means the following: the ordering has changed because $\overline{\delta}_{i+1}\nu^{e_{i+1}}$ was obtained by means of option (ii), whereas option (i) was used on $\overline{\delta}_i\nu^{e_i}$ – otherwise, they would still be ordered correctly. This implies

$$\begin{aligned}(\overline{\delta}_{i+1}\nu^{e_{i+1}}) &= \delta([|\cdot|^{a_{i+1}+e_{i+1}}, |\cdot|^{-a_{i+1}+e_{i+1}-1}]),\\ (\overline{\delta}_i\nu^{e_i}) &= \delta([\rho\nu^{-a_i+e_i}, \rho\nu^{a_i+e_i}]).\end{aligned}$$

If we assume that these segments are linked, then $\rho = \mathbb{1}$ and the following holds:

- the segments are linked, so we have $e_i - e_{i+1} \in \frac{1}{2}\mathbb{Z}$;
- they need to be swapped, so $e_{i+1} - \frac{1}{2} < e_i$;
- the original ordering implies $e_{i+1} \geqslant e_i$.

This is only possible if $e_i = e_{i+1}$. From here we easily deduce that the segments cannot really be linked, so they can freely switch places.

We have thus shown that the desired rearrangement is indeed possible. In short, we can write

$$\Pi'' \times \ \Delta'' \rtimes \Theta_{l-2k}(\tau) \twoheadrightarrow \pi.$$

Here $\Pi'' \times \Delta''$ denotes the product of $(\overline{\delta}_i\nu^{e_i})$ (in decreasing order of e_i); here we have grouped all the segments of the form $\delta([|\cdot|^{-a}, |\cdot|^{a}])$ into Δ''.

We know that all the subquotients of $\Theta_{l-2k}(\tau)$ are tempered (obviously $l-2k>0$ so this follows from Proposition 3.11), we see that the standard module of π is equal to

$$\Pi'' \rtimes \pi_0'',$$

where π_0'' is a (tempered) irreducible subquotient of $\Delta'' \rtimes \Theta_{l-2k}(\tau)$. The uniqueness of the standard module now forces

$$\Pi'' = \ \Pi \quad \text{and} \quad \pi_0'' \cong \pi_0.$$

In particular, Π'' and Π have the same cuspidal support. We have already compared the cuspidal supports of Π and Π'. On the other hand, since option (ii) of Corollary 3.8 was applied exactly k times, we see that, compared to Π', the cuspidal support of Π'' is missng

$$|\cdot|^{\frac{l-1}{2}}, |\cdot|^{\frac{l-3}{2}}, \dots, |\cdot|^{\frac{l+1}{2}-k},$$

along with all the segments grouped into Δ''.

In both cases I and II this comparison of the cuspidal supports easily leads to the conclusion $l' = l-2k$.

Remark 7.3. It is easy to see that, in case (I), none of the representations can end up in Δ'', while in case (II) Δ'' can (and must) only contain $[\frac{1-l'}{2}, \frac{l'-1}{2}]$, which arrived from the tempered part in the first place.

We now use the other condition: $\Delta'' \rtimes \Theta_{l'}(\tau)$ has an irreducible subquotient isomorphic to π_0. The following lemma shows that this is only possible if $l' = l_0$.

Lemma 7.4. *If $l' > l_0$ then $\Delta'' \rtimes \Theta_{l'}(\tau)$ does not contain a subquotient isomorphic to π_0.*

Proof. Recall that l_0 is the smallest even non-negative integer such that $\theta_{-l_0}(\pi_{00})$ is non-zero. Recall that $\theta_{-l_0}(\pi_{00})$ is tempered, whereas all the lifts $\theta_{-l'}(\pi_{00})$ for $l' > l_0$ are non-tempered (see Theorem 3.10). We also know that $\theta_{-l_0}(\pi_0)$ is tempered whenever it is non-zero. We first lay out the proof assuming Δ'' is empty (case (I)).

Let τ be an irreducible tempered representation of $Mp(W_n)$ and $l' > l_0$ such that $\Theta_{l'}(\tau)$ contains a subquotient isomorphic to π_0 (in particular, such that $\Theta_{l'}(\tau) \neq 0$). We then have two possibilities, depending on whether or not τ contains $\chi_V \mathrm{St}_{l'}$ in its tempered support.

If τ does not contain $\chi_V \mathrm{St}_{l'}$ in its tempered support, then $\Theta_{l'}(\tau)$ is easily shown to be irreducible (e.g. [2, Proposition 5.4]). This would imply that $\theta_{l'}(\tau) = \pi_0$, i.e. $\theta_{-l'}(\pi_0) = \tau$. On the other hand, we know that this is not possible because $\theta_{-l'}(\pi_0)$ is not tempered for $l' > l_0$.

Thus τ contains $\chi_V \mathrm{St}_{l'}$ in the tempered support and we may use a very similar argument. However, we do not know if $\Theta_{l'}(\tau)$ is irreducible, so further analysis is required. We can represent τ as a direct summand of

$$(*) \qquad \chi_V \delta_1 \times \cdots \times \chi_V \delta_i \times \chi_V(\mathrm{St}_{l'}, h) \rtimes \tau_d.$$

here $\delta_1, \ldots, \delta_i, \tau_d$ are discrete series representations, and h denotes the number of occurrences of $\chi_V \mathrm{St}_{l'}$ in the tempered support (also, $(\mathrm{St}_{l'}, h) = \mathrm{St}_{l'} \times \cdots \times \mathrm{St}_{l'}$ h times). Again we differentiate two cases: $\Theta_{l'}(\tau_d) \neq 0$ and $\Theta_{l'}(\tau_d) = 0$.

a) Let $\Theta_{l'}(\tau_d) \neq 0$. We have

$$\chi_V \Delta \times \chi_V(\mathrm{St}_{l'}, h) \rtimes \tau_d \twoheadrightarrow \tau$$

where Δ is temporarily used to denote $\delta_1 \times \cdots \times \delta_i$. Since we are looking for subquotients of $\Theta_{l'}(\tau)$, we can use Proposition 3.5 and the same arguments as in the proof of Corollary 3.7. We get that π_0 is a subquotient of one of the following:

- $\Delta \times (\mathrm{St}_{l'}, h) \rtimes \Theta_{l'}(\tau_d)$;
- $\Delta \times (\mathrm{St}_{l'}, h-1) \times \delta([|\cdot|^{\frac{1-l'}{2}}, |\cdot|^{\frac{l'-3}{2}}]) \rtimes \Theta_{l'-2}(\tau_d)$.

Both $\Theta_{l'}(\tau_d)$ and $\Theta_{l'-2}(\tau_d)$ are irreducible discrete series representations by Theorem 3.10; furthermore, $\Theta_{l'-2}(\tau_d)$ is a subquotient of $|\cdot|^{\frac{l'-1}{2}} \rtimes \Theta_{l'}(\tau_d)$

so that π_0 is in fact a subquotient of

(i) $\Delta \times (\mathrm{St}_{l'}, h) \rtimes \Theta_{l'}(\tau_d)$

(ii) or $\Delta \times (\mathrm{St}_{l'}, h-1) \times \delta([|\cdot|^{\frac{1-l'}{2}}, |\cdot|^{\frac{l'-3}{2}}]) \times |\cdot|^{\frac{l'-1}{2}} \rtimes \Theta_{l'}(\tau_d)$.

Consider the representation $\delta([|\cdot|^{\frac{1-l'}{2}}, |\cdot|^{\frac{l'-3}{2}}]) \times |\cdot|^{\frac{l'-1}{2}}$ which appears in (ii). It has two irreducible subquotients, namely $\mathrm{St}_{l'}$ and the corresponding Langlands subrepresentation which we denote by L (see §6). Therefore, any irreducible subquotient of (ii) is either a subquotient of (i), or a subquotient of $\Delta \times (\mathrm{St}_{l'}, h-1) \times L \rtimes \Theta_{l'}(\tau_d)$. This representation, however, cannot contain π_0 (by heredity) since L is not generic. This shows that π_0 is necessarily a subquotient of (i).

By the uniqueness of the tempered support, we now conclude that $\Theta_{l'}(\tau_d) = \pi_{00}$. However, this implies that $\theta_{-l'}(\pi_{00}) = \tau_d$ is in discrete series despite $l' > l_0$. This contradicts the remarks at the beginning of this proof and shows that $l' > l_0$ is impossible.

b) It remains to see what happens when $\Theta_{l'}(\tau_d) = 0$. Recall that τ is a direct summand of its tempered support, $(*)$. Choosing the appropriate irreducible (tempered) subquotient τ_1 of $\mathrm{St}_{l'} \rtimes \tau_d$, we define $\tau_{j+1} = \mathrm{St}_{l'} \rtimes \tau_j$ for $j = 1, 2, \dots, h-1$ and see that τ is a direct summand of $\chi_V \Delta \rtimes \tau_h$. As before, we use Corollary 3.6 to get

$$\Delta \rtimes \Theta_{l'}(\tau_h) \twoheadrightarrow \Theta_{l'}(\tau).$$

Using an inductive argument, we now show that the left-hand side of the above epimorphism cannot possess an irreducible subquotient isomorphic to π_0 if $l' > l_0$.

Note that we have $\Theta_{l'}(\tau_d) = 0$, but are working with $\Theta_{l'}(\tau_h) \neq 0$. This implies that $\Theta_{l'}(\tau_j) \neq 0$ for $j = 1, \dots, h-1$ as well, and that the L-parameter of τ_d does not contain $\chi_V S'_l$. If it did contain $\chi_V S'_l$, this would mean that $\Theta_{l'}(\tau_d)$ is equal to zero because the alternating condition of [2, Theorem 4.1] fails. This would also imply that it fails for all the τ_j, further implying $\Theta_{l'}(\tau_j) = 0$.

Assume the contrary, i.e. that π_0 appears as a subquotient in the above representation. Just as in case a), this means that π_0 is a subquotient of

(i) $\Delta \times \mathrm{St}_{l'} \rtimes \Theta_{l'}(\tau_{h-1})$

(ii) or $\Delta \times \delta([|\cdot|^{\frac{1-l'}{2}}, |\cdot|^{\frac{l'-3}{2}}]) \times |\cdot|^{\frac{l'-1}{2}} \rtimes \theta_{l'}(\tau_{h-1})$

(it is easy to see that $\Theta_{l'-2}(\tau_{h-1})$ is irreducible, so we can write it as a subquotient of $|\cdot|^{\frac{l'-1}{2}} \rtimes \theta_{l'}(\tau_{h-1})$). Again, the reasoning from case a) shows that π_0 has to be a subquotient of (i).

Repeating this argument $h-1$ times, we get that π_0 is a subquotient of

$$\Delta \times (\mathrm{St}_{l'}, h-1) \rtimes \Theta_{l'}(\tau_1).$$

But now all the irreducible subquotients of $\Theta_{l'}(\tau_1)$ are in discrete series—this follows from the fact that the L-parameter of τ_d does not contain $\chi_V S'_l$ (as discussed above)—so that (by Howe duality) $\Theta_{l'}(\tau_1)$ is an irreducible discrete series representation. This means that the above representation is in fact the tempered support of π_0. In particular, we have $\theta_{l'}(\tau_1) = \pi_{00}$. This forces $\theta_{-l'}(\pi_{00}) = \tau_1$ to be tempered, which is again impossible for $l' > l_0$.

We point out that, in all the cases, our proof boils down to the fact that $\Theta_{l'}(\tau_d)$ has to be (and cannot be, for $l' > l_0$) isomorphic to π_{00}; this shows that the same proof works even when Δ'' is non-empty (see Remark 7.3).

□

This completes our proof of 7.1: we have shown that $l' = l_0$, which implies that the subquotient which participates in (1) is equal to $\theta_{-l}(\pi_{00})$. Therefore, we have

(2) $$\chi_V \delta_r \nu^{s_r} \times \cdots \times \chi_V \delta_1 \nu^{s_1} \times \chi_V \Delta \rtimes \theta_{-l}(\pi_{00}) \twoheadrightarrow \sigma.$$

□

7.2. *Determining the standard modules.* The above epimorphism (2) provides valuable information, but is not sufficient to uniquely determine σ. To do this, we will have to find the standard module of σ; we do so in this section. Before we start, let us return for a moment to (0), section 7.1. Our goal is to show two things (see Theorem 5.1):

- the subquotient of $\Theta_{-l}(\pi_0)$ which participates in that epimorphism is $\theta_{-l}(\pi_0)$;
- the standard module of σ is obtained by adding $\chi_V \delta_1 \nu^{s_1}, \dots, \chi_V \delta_r \nu^{s_r}$ to the standard module of $\theta_{-l}(\pi_0)$ (and sorting the representations decreasingly with respect to the exponents).

The shape of $\theta_{-l}(\pi_0)$ is completely determined by Theorems 4.3 and 4.5 of [2]; as it is useful to have in mind during the ensuing calculations, we compile the results of these theorems in the following proposition.

PROPOSITION 7.5. *Let $\pi_0 \in \mathrm{Irr}(O(V_m))$ be tempered and generic; let (ϕ, η) be its L-parameter and let $l \geqslant 0$ be even. We have two cases*

(i) On the going-down tower, $\theta_0(\pi_0)$ is the first lift of π_0; it is tempered. For $l > 0$ we have
$$\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1}{2}} \rtimes \theta_0(\pi_0) \twoheadrightarrow \theta_{-l}(\pi_0).$$

(ii) On the going-up tower, $\theta_{-2}(\pi_0)$ is the first lift of π_0; it is tempered. For $l > 2$ we have
$$\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{3}{2}} \rtimes \theta_{-2}(\pi_0) \twoheadrightarrow \theta_{-l}(\pi_0).$$

Our proof starts by analyzing the map (2) established by Proposition 7.1. We have a few cases depending on the shape of π_0, each of them corresponding to one of the cases of the previous Proposition. All the cases share the same basic approach and result in analogous conclusions. However, we do have to treat them separately, mainly because of the exceptional cases which arise in some of them. The first case contains all the key ideas (and no tricky exceptions), so we present it full detail.

<u>CASE 1: the going-down tower</u>

In this case we know that $\theta_{-l}(\pi_{00})$ is the Langlands quotient of

$$\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1}{2}} \rtimes \theta_0(\pi_{00}).$$

This also implies that $\theta_{-l}(\pi_{00})$ is the unique quotient of

$$\chi_V\zeta(\frac{1}{2}, \frac{l-1}{2}) \rtimes \theta_0(\pi_{00})$$

(see the notation of Section 6). Combining this with the epimorphism in (2) we get

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\Delta \times \chi_V\zeta(\frac{1}{2}, \frac{l-1}{2}) \rtimes \theta_0(\pi_{00}) \twoheadrightarrow \sigma.$$

We now use Lemma 6.1: $\chi_V\zeta(\frac{1}{2}, \frac{l-1}{2})$ can switch places with all the δ_i' appearing in Δ. This means that we can write

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\zeta(\frac{1}{2}, \frac{l-1}{2}) \times \chi_V\Delta \rtimes \theta_0(\pi_{00}) \twoheadrightarrow \sigma. \tag{3}$$

Finally, we observe that there is an irreducible subquotient τ of $\chi_V\Delta \rtimes \theta_0(\pi_{00})$ such that

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\zeta(\frac{1}{2}, \frac{l-1}{2}) \rtimes \tau \twoheadrightarrow \sigma. \tag{4}$$

Note that τ is tempered, because $\theta_0(\pi_{00})$ is, too (moreover, in this case, $\theta_0(\pi_{00})$ is in discrete series), as are all the irreducible subquotients of Δ. We now claim the following:

LEMMA 7.6. *The representation appearing on the left-hand side of* (4) *has a unique irreducible quotient.*

PROOF. We will show that the representation in question is itself a quotient of a standard module, and the conclusion will follow. We use Lemma 6.1. Let $[\rho\nu^c, \rho\nu^d]$ be the segment which defines $\delta_1\nu^{s_1}$ (in particular, we have $s_1 = \frac{c+d}{2}$). Assume that ρ is equal to the trivial character $\mathbb{1}$ of $\mathrm{GL}_1(\mathbb{F}) = \mathbb{F}^\times$ and that $c, d \in \frac{1}{2} + \mathbb{Z}$ are half-integers. If these conditions are not met, the proof is the same, only simpler, because Lemma 6.1 is not needed.

If $s_1 \geqslant \frac{l-1}{2}$ then the representation in question is a quotient of the standard module

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \chi_V|\cdot|^{\frac{1}{2}} \rtimes \tau,$$

and we are done. If $s_1 < \frac{l-1}{2}$ we use the following technical observation, based on Lemma 6.1 (we use the notation of §6).

LEMMA 7.7. *Let σ be an irreducible representation of $Mp(W_n)$ and assume that*

$$A \times \delta([a,b]) \times \zeta(c,d) \rtimes \sigma_0 \twoheadrightarrow \sigma$$

for $\frac{a+b}{2} \leqslant d$ and some representations A and σ_0 . If

$$\text{(i)} \qquad c \neq b+1$$

then setting $s = \min\{s'' \in [c,d] : s'' \geq \frac{a+b}{2}\}$ we have

$$A \times \zeta(s,d) \times \delta([a,b]) \times \zeta(c,s-1) \rtimes \sigma_0 \twoheadrightarrow \sigma.$$

Here, the segment $[c, s-1]$ can be empty (i.e., $s = c$ can happen). Assume, a fortiori, that $[c,d]$ and $[a,b]$ satisfy

$$\text{(ii)} \qquad c \lneqq \frac{a+b}{2} + 1$$

(notice that this implies (i)*). Then, for any $[a',b']$ with $\frac{a'+b'}{2} \geqslant \frac{a+b}{2}$ the segments $[s,d]$ and $[a',b']$ also satisfy the above condition* (ii) *(i.e. we have $s \lneqq \frac{a'+b'}{2} + 1$).*

PROOF. We know that $\zeta(c,d)$ is a quotient of $\zeta(s,d) \times \zeta(c,s-1)$, so we have

$$A \times \delta([a,b]) \times \zeta(s,d) \times \zeta(c,s-1) \rtimes \sigma_0 \twoheadrightarrow \sigma.$$

If (i) holds, then Lemma 6.1 (along with Remark 6.3) shows that $\delta([a,b])$ and $\zeta(s,d)$ can switch places. We thus get

$$A \times \zeta(s,d) \times \delta([a,b]) \times \zeta(c,s-1) \rtimes \sigma_0 \twoheadrightarrow \sigma,$$

as required.

For the second part of the claim, assume that $[c,d]$ and $[a,b]$ satisfy condition (ii). Then $s \geqslant \frac{a'+b'}{2} + 1$ would imply

$$s \geqslant \frac{a'+b'}{2} + 1 \geqslant \frac{a+b}{2} + 1 > c.$$

Now $s > c$ implies $s-1 \in [c,d]$, but we also have $s-1 \geqslant \frac{a+b}{2}$, contradicting our choice of s. ∎

We apply Lemma 7.7 inductively—first with $\delta([a,b]) = \delta_1\nu^{s_1}$ and $\zeta(c,d) = \zeta(\frac{1}{2}, \frac{l-1}{2})$, then $\delta([a,b]) = \delta_2\nu^{s_2}$ and $\zeta(c,d) = \zeta(s, \frac{l-1}{2})$, etc.—we show that the representation appearing in (4) is indeed a quotient of a standard representation. Notice that condition (ii) is fulfilled already in the first step (this ensures that we can proceed with the induction): we have $\frac{a+b}{2} > 0$ and $c = \frac{1}{2}$. This proves Lemma 7.6. ∎

Note that the Lemma 7.6 determines the appearance of the standard module for σ: the representations $\chi_V|\cdot|^{\frac{l-1}{2}}, \dots, \chi_V|\cdot|^{\frac{1}{2}}$ are simply inserted among $\chi_V\delta_r\nu^{s_r}, \dots, \chi_V\delta_1\nu^{s_1}$ so that the exponents form a decreasing sequence. The only thing that remains to be determined is the tempered part, i.e. τ.

We have shown that $\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\zeta(\frac{1}{2}, \frac{l-1}{2}) \rtimes \tau$ appearing in (4) has a unique irreducible quotient. Therefore, we have

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \rtimes \tau' \twoheadrightarrow \sigma \tag{5}$$

where τ' is the unique irreducible quotient of $\chi_V\zeta(\frac{1}{2}, \frac{l-1}{2}) \rtimes \tau$ (that is, the Langlands quotient of $\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1}{2}} \rtimes \tau$). It is now important to note the following:

Lemma 7.8. *The representation τ' is a subquotient of $\Theta_{-l}(\pi_0)$.*

Proof. We revisit the maps we have used so far: (0), (1), (2), (4) and (5). Let Π denote $\delta_r\nu^{s_r} \times \cdots \times \delta_1\nu^{s_1}$. Starting from

$$T\colon \chi_V\Delta \rtimes \Theta_{-l}(\pi_{00}) \twoheadrightarrow \Theta_{-l}(\pi_0)$$

we induce to obtain

$$\mathrm{Ind}(T)\colon \chi_V\Pi \times \chi_V\Delta \rtimes \Theta_{-l}(\pi_{00}) \twoheadrightarrow \chi_V\Pi \rtimes \Theta_{-l}(\pi_0).$$

Composing this with (0) (which is given by $S\colon \chi_V\Pi \rtimes \Theta_{-l}(\pi_0) \twoheadrightarrow \sigma$) we get epimorphism (1):

$$S \circ \mathrm{Ind}(T) : \chi_V\Pi \times \chi_V\Delta \rtimes \Theta_{-l}(\pi_{00}) \twoheadrightarrow \sigma.$$

Proposition 7.1 shows that no subquotient of $\Theta_{-l}(\pi_{00})$ except $\theta_{-l}(\pi_{00})$ can participate in the above epimorphism; in other words, we have $\chi_V\Pi \times \chi_V\Delta \rtimes \Theta^0 \subseteq \ker S \circ \mathrm{Ind}(T)$ where we have used Θ^0 to denote the maximal proper subrepresentation of $\Theta_{-l}(\pi_{00})$.

Taking the quotient of $S \circ \mathrm{Ind}(T)$ by $\chi_V\Pi \times \chi_V\Delta \rtimes \Theta^0$ we get a new map, (2):

$$\widetilde{S \circ \mathrm{Ind}}(T) : \chi_V\Pi \times \chi_V\Delta \rtimes \theta_{-l}(\pi_{00}) \twoheadrightarrow \sigma.$$

By the construction of this map it is obvious that any subquotient τ' of $\chi_V\Delta \rtimes \theta_{-l}(\pi_{00})$ participating in the above epimorphism must be a subquotient of $\Theta_{-l}(\pi_0)$, so we get (5). This subquotient is written as a subquotient of $\chi_V\zeta(\frac{1}{2}, \frac{l-1}{2}) \rtimes \tau$ in (4), and Lemma 7.6 shows that τ' is in fact a quotient of $\chi_V\zeta(\frac{1}{2}, \frac{l-1}{2}) \rtimes \tau$. ∎

Finally, it remains to verify the following.

Lemma 7.9. *The only subquotient of $\Theta_{-l}(\pi_0)$ with standard module of the form $\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1}{2}} \rtimes \tau$ is $\theta_{-l}(\pi_0)$.*

Proof. Let τ' be a subquotient of $\Theta_{-l}(\pi_0)$ such that

$$\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1}{2}} \rtimes \tau \twoheadrightarrow \tau'$$

for some tempered τ. Denote by τ_1 the Langlands quotient of $\chi_V|\cdot|^{\frac{l-3}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1}{2}} \rtimes \tau$, so that $\chi_V|\cdot|^{\frac{l-1}{2}} \rtimes \tau_1 \twoheadrightarrow \tau'$, i.e. $\tau' \hookrightarrow \chi_V|\cdot|^{\frac{1-l}{2}} \rtimes \tau_1$.

We now use Kudla's filtration: the map we've just obtained shows that $\mathrm{Hom}(\tau', \chi_V|\cdot|^{\frac{1-l}{2}} \rtimes \tau_1) \neq 0$. Using Frobenius reciprocity, this means that $\mathrm{Hom}(R_{Q_1}(\tau'), \chi_V|\cdot|^{\frac{1-l}{2}} \otimes \tau_1) \neq 0$, where Q_1 denotes the appropriate standard maximal parabolic subgroup of $Mp(W_n)$. From here, we deduce that $\chi_V|\cdot|^{\frac{1-l}{2}} \otimes \tau_1$ is a quotient of $R_{Q_1}(\tau')_{\chi_V|\cdot|^{\frac{1-l}{2}}}$, which implies that it is also a subquotient of $R_{Q_1}(\Theta_{-l}(\pi_0))_{\chi_V|\cdot|^{\frac{1-l}{2}}}$.

On the other hand, $\pi_0 \otimes R_{Q_1}(\Theta_{-l}(\pi_0))_{\chi_V|\cdot|^{\frac{1-l}{2}}}$ is obviously a quotient of $R_{Q_1}(\omega_{m_0,n_0})$—here m_0 is defined by $\pi_0 \in \mathrm{Irr}(\mathrm{O}(V_{m_0}))$, $n_0 = m_0 - 1 + l$, and ω_{m_0,n_0} is the corresponding Weil representation. Kudla's filtration of $R_{Q_1}(\omega_{m_0,n_0})$ is

$$\begin{aligned} J^0 &= \chi_V|\cdot|^{\frac{1-l}{2}} \otimes \omega_{m_0,n_0-2} \quad \text{(the quotient)} \\ J^1 &= \mathrm{Ind}(\Sigma_1 \otimes \omega_{m_0-2,n_0-2}) \quad \text{(the subrepresentation).} \end{aligned}$$

It is now easy to show that J^1 cannot participate in the epimorphism $R_{Q_1}(\omega_{m_0,n_0}) \twoheadrightarrow \pi_0 \otimes R_{P_1'}(\Theta_{-l}(\pi_0))_{\chi_V|\cdot|^{\frac{1-l}{2}}}$. Otherwise, an application of the second Frobenius reciprocity would show that $R_{\overline{P}_1}(\pi_0)$ (where $\overline{P}_1$ denotes the parabolic subgroup opposite to P_1) has a quotient of the form $|\cdot|^{\frac{l-1}{2}} \otimes \pi_1$. As π_0 is tempered, and $\frac{l-1}{2} > 0$, Casselman's criterion shows that this is impossible.

This means that $\pi_0 \otimes R_{Q_1}(\Theta_{-l}(\pi_0))_{\chi_V|\cdot|^{\frac{1-l}{2}}}$ is a quotient of J^0, which immediately implies that τ_1 is a subquotient of $\Theta_{2-l}(\pi_0)$.

Inductively repeating this argument shows that τ is a subquotient of $\Theta_0(\pi_0)$; however, $\Theta_0(\pi_0)$ is irreducible, so we must have $\tau = \Theta_0(\pi_0) = \theta_0(\pi_0)$. This proves that τ' is the Langlands quotient of

$$\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{1}{2}} \rtimes \theta_0(\pi_0).$$

By Proposition 7.5 (i), we conclude that $\tau' = \theta_{-l}(\pi_0)$. ∎

This completes case (1). Let us summarize: we have shown that

$$\chi_V \Pi \rtimes \theta_{-l}(\pi_0) \twoheadrightarrow \theta_{-l}(\pi),$$

and we have determined the standard module of $\theta_{-l}(\pi)$:

$$\theta_{-l}(\pi) = L(\chi_V \delta_r \nu^{s_r}, \ldots, \chi_V \delta_1 \nu^{s_1}, \chi_V|\cdot|^{\frac{l-1}{2}}, \ldots, \chi_V|\cdot|^{\frac{1}{2}}; \theta_{-1}(\pi_0)).$$

Case 2: the going-up tower

Recall that the going-up tower for π_0 is the same as the going-up tower for π_{00}. In other words, π_{00} first appears on this tower when $l = -2$. Furthermore, Proposition 7.5 shows that $\theta_{-2}(\pi_{00})$ is tempered, whereas for $l > 2$ the representation $\theta_{-l}(\pi_{00})$ is the Langlands quotient of

$$\chi_V \nu^{\frac{l-1}{2}} \times \cdots \times \chi_V \nu^{\frac{3}{2}} \rtimes \theta_{-2}(\pi_{00}),$$

that is, the unique quotient of

$$\chi_V \zeta(\frac{3}{2}, \frac{l-1}{2}) \rtimes \theta_{-2}(\pi_{00}).$$

Using this in (2) we get

$$\chi_V \delta_r \nu^{s_r} \times \cdots \times \chi_V \delta_1 \nu^{s_1} \times \chi_V \Delta \times \chi_V \zeta(\frac{3}{2}, \frac{l-1}{2}) \rtimes \theta_{-2}(\pi_{00}) \twoheadrightarrow \sigma. \tag{6}$$

We proceed like in Case 1: according to Lemma 6.1 and Remark 6.3, $\chi_V \zeta(\frac{3}{2}, \frac{l-1}{2})$ can switch places with all the representations which define $\chi_V \Delta$, except $\chi_V \mathrm{St}_2 = \chi_V \delta([|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{1}{2}}])$. Thus, we initially assume that St_2 does not appear in the definition of Δ or, equivalently:

Case 2.1: St_2 does not appear in the tempered support of π_0.
By the discussion above, in this case we have

$$\chi_V \delta_r \nu^{s_r} \times \cdots \times \chi_V \delta_1 \nu^{s_1} \times \chi_V \zeta(\frac{3}{2}, \frac{l-1}{2}) \times \chi_V \Delta \rtimes \theta_{-2}(\pi_{00}) \twoheadrightarrow \sigma.$$

This implies that there is an irreducible tempered subquotient τ of $\chi_V \Delta \rtimes \theta_{-2}(\pi_{00})$ such that

$$\chi_V \delta_r \nu^{s_r} \times \cdots \times \chi_V \delta_1 \nu^{s_1} \times \chi_V \zeta(\frac{3}{2}, \frac{l-1}{2}) \rtimes \tau \twoheadrightarrow \sigma.$$

We can now repeat the arguments of Case 1—we apply Lemmas 7.6, 7.7, 7.8 and 7.9 the same way to show that

$$\chi_V \delta_r \nu^{s_r} \times \cdots \times \chi_V \delta_1 \nu^{s_1} \rtimes \theta_{-l}(\pi_0) \twoheadrightarrow \sigma$$

and that

$$\theta_{-l}(\pi) = L(\chi_V \delta_r \nu^{s_r}, \ldots, \chi_V \delta_1 \nu^{s_1}, \chi_V |\cdot|^{\frac{l-1}{2}}, \ldots, \chi_V |\cdot|^{\frac{3}{2}}; \theta_{-2}(\pi_0)).$$

The only part we need to check are the conditions of Lemma 7.7. In Case 1 we had (using the notation of 7.7) $c = \frac{1}{2}$, so that condition (ii) of the Lemma was automatically satisfied. In this case, we have a different situation: since $c = \frac{3}{2}$, condition (i) of the Lemma can be violated if $\delta_1 \nu^{s_1} = |\cdot|^{\frac{1}{2}}$.

First, we note that (by Lemma 6.1) we can swap $\zeta(\frac{3}{2}, \frac{l-1}{2})$ with any $\delta_i \nu_i^s$ such that $s_i = \frac{1}{2}$, except maybe $|\cdot|^{\frac{1}{2}}$. However, it is easy to show that $|\cdot|^{\frac{1}{2}}$ cannot appear in the standard module for π: it would force the standard module to reduce (cf. Propositions 4.5 and 4.8 of [12]), and this is impossible by the standard module conjecture.

Any other $\delta_1\nu^{s_1} = \delta([a,b])$ with $a, b \in \frac{1}{2}+\mathbb{Z}$ satisfies $s_i = \frac{a+b}{2} > \frac{1}{2}$. Therefore, condition (ii) of Lemma 7.7 is fulfilled, and we can proceed with the inductive procedure of Lemma 7.6.

Case 2.2: St_2 appears in the tempered support of π_0.
Let h be the number of times St_2 appears in the tempered support. Setting $(\mathrm{St}_2, h) = \mathrm{St}_2 \times \cdots \times \mathrm{St}_2$ (h times), we write $\Delta = \Delta' \times (\mathrm{St}_2, h)$ where Δ' is the representation induced from all the other representations which appear in the tempered support. Rewriting (6), we thus get

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\Delta' \times \chi_V(\mathrm{St}_2, h) \times \chi_V\zeta(\frac{3}{2}, \frac{l-1}{2}) \rtimes \theta_{-2}(\pi_{00}) \twoheadrightarrow \sigma.$$

This brings us to the technical difficulty we pointed out above: since St_2 cannot simply switch places with $\zeta(\frac{3}{2}, \frac{l-1}{2})$, we cannot immediately conclude that

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\zeta(\frac{3}{2}, \frac{l-1}{2}) \times \chi_V\Delta' \times \chi_V(\mathrm{St}_2, h) \rtimes \theta_{-2}(\pi_{00}) \twoheadrightarrow \sigma$$

holds, as we did in Case 2.1. To remedy this problem, we divide Case 2.2 into two more subcases:

Subcase 2.2.1: the parameter of π_{00} contains S_2.
Under this assumption, $\theta_{-2}(\pi_{00})$ is no longer in discrete series; rather, the multiplicity of $\chi_V S_2$ in its parameter is two (see Theorem 4.5 in [2]). This means that there is an irreducible tempered representation σ_{00} such that $\theta_{-2}(\pi_{00}) \hookrightarrow \chi_V\mathrm{St}_2 \rtimes \sigma_{00}$. This implies that $\theta_{-l}(\pi_{00})$ is a quotient of

$$\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{3}{2}} \times \chi_V\mathrm{St}_2 \rtimes \sigma_{00}.$$

From here, one can easily show that $\theta_{-l}(\pi_{00})$ is in fact a quotient of $\chi_V L \rtimes \sigma_{00}$ where we have used L to denote the Langlands quotient of $|\cdot|^{\frac{l-1}{2}} \times \cdots \times |\cdot|^{\frac{3}{2}} \times \mathrm{St}_2$.

Using this in (6), we get

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\Delta' \times \chi_V(\mathrm{St}_2, h) \times \chi_V L \rtimes \sigma_{00} \twoheadrightarrow \sigma.$$

Now L can switch places with all the St_2, so we have

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\Delta' \times \chi_V L \times \chi_V(\mathrm{St}_2, h) \rtimes \sigma_{00} \twoheadrightarrow \sigma.$$

Recall that L is a quotient of $\zeta(\frac{3}{2}, \frac{l-1}{2}) \times \mathrm{St}_2$, and that $\zeta(\frac{3}{2}, \frac{l-1}{2})$ can freely switch places with all the representations in Δ'. Taking this into account, we may write

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\zeta(\frac{3}{2}, \frac{l-1}{2}) \times \chi_V\Delta' \times \chi_V(\mathrm{St}_2, h+1) \rtimes \sigma_{00} \twoheadrightarrow \sigma.$$

The above map implies that there is an irreducible (and necessarily tempered) subquotient τ of $\chi_V\Delta' \times \chi_V(\mathrm{St}_2, h+1) \rtimes \sigma_{00}$ such that

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\zeta(\frac{3}{2}, \frac{l-1}{2}) \rtimes \tau \twoheadrightarrow \sigma.$$

We can now finish the proof just like we did in Case 2.1.

Subcase 2.2.2: the parameter of π_{00} does not contain S_2.
Keeping the notation of Subcase 2.2.1. and returning to (6) again, we have

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V(\mathrm{St}_2, h) \times \chi_V\Delta' \times \chi_V\zeta(\frac{3}{2}, \frac{l-1}{2}) \rtimes \theta_{-2}(\pi_{00}) \twoheadrightarrow \sigma.$$

Again, $\zeta(\frac{3}{2}, \frac{l-1}{2})$ can switch places with all the representations in Δ'. As in the previous case, we cannot immediately swap $\zeta(\frac{3}{2}, \frac{l-1}{2})$ and (St_2, h). Moreover, since $\theta_{-2}(\pi_{00})$ is now a discrete series representation, we do not have a workaround like in Subcase 2.2.1. In fact, Remark 6.4 shows that we have two distinct options:

(i) there exists an irreducible (tempered) subquotient τ_1 of $\chi_V\Delta \rtimes \theta_{-2}(\pi_{00})$ such that

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V\zeta(\frac{3}{2}, \frac{l-1}{2}) \rtimes \tau_1 \twoheadrightarrow \sigma$$

(ii) there exists an irreducible (tempered) subquotient τ_2 of $\chi_V(\mathrm{St}_2, h-1) \times \chi_V\Delta' \rtimes \theta_{-2}(\pi_{00})$ such that

$$\chi_V\delta_r\nu^{s_r} \times \cdots \times \chi_V\delta_1\nu^{s_1} \times \chi_V L' \rtimes \tau_2 \twoheadrightarrow \sigma,$$

where L' denotes the Langlands quotient of $|\cdot|^{\frac{l-1}{2}} \times \cdots \times |\cdot|^{\frac{5}{2}} \times \mathrm{St}_3\nu^{\frac{1}{2}}$, that is, the unique irreducible quotient of $\zeta(\frac{5}{2}, \frac{l-1}{2}) \times \mathrm{St}_3\nu^{\frac{1}{2}}$. If we can show that (i) always holds, then we can finish the proof like we did in Case 2.1 (or Subcase 2.2.1). Let us therefore show that option (ii) is impossible.

Assume the contrary. Then we can adjust the proof of Lemma 7.6 to show that the representation on the left-hand side of (ii) has a unique irreducible quotient. First, by Lemma 6.1, $\zeta(\frac{5}{2}, \frac{l-1}{2})$ can switch places with all the $\delta_i\nu^{s_i}$ for which $s_i \leqslant \frac{3}{2}$; the only exceptions are $|\cdot|^{\frac{3}{2}}$, $\delta([|\cdot|^{\frac{1}{2}}, |\cdot|^{\frac{3}{2}}])$ and $\delta([|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{3}{2}}])$. However, $|\cdot|^{\frac{3}{2}}$ and $\delta([|\cdot|^{\frac{1}{2}}, |\cdot|^{\frac{3}{2}}])$ cannot appear among $\delta_1\nu^{s_1}, \dots, \delta_r\nu^{s_r}$—since π_0 contains St_2 in its tempered support, this would cause the standard module of π to reduce. On the other hand, Lemma 6.5 shows that $\delta([|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{3}{2}}])$ is not problematic, since $L' \times \delta([|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{3}{2}}]) \cong \delta([|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{3}{2}}]) \times L'$. All the other s_i satisfy $s_i > \frac{3}{2}$, which means that condition (ii) of Lemma 7.7 is satisfied. We can thus apply Lemma 7.7 inductively to complete the argument.

We can now use the arguments of Case 1 to arrive at a contradiction. In particular, we now know that the irreducible subquotient of $\chi_V L' \rtimes \tau_2$ which participates in the above map (ii) is in fact its (unique) irreducible quotient, i.e. $L(\chi_V|\cdot|^{\frac{l-1}{2}} \times \cdots \times \chi_V|\cdot|^{\frac{5}{2}} \times \chi_V\mathrm{St}_3\nu^{\frac{1}{2}} \rtimes \tau_2)$. Repeating the arguments of

Lemma 7.8, we show that this irreducible quotient must also be a subquotient of $\Theta_{-l}(\pi_0)$. This implies, using the inductive procedure from Lemma 7.9, that $L(\chi_V \mathrm{St}_3\nu^{\frac{1}{2}} \rtimes \tau_2)$ is an irreducible subquotient of $\Theta_{-4}(\pi_0)$.

We complete the proof by showing that this is not possible. Recall that π_0 is a subquotient of $(\mathrm{St}_2, h) \rtimes \pi_0'$, where π_0' is an irreducible subquotient of $\Delta' \rtimes \pi_{00}$. In fact, since the parameter of π_0' no longer contains S_2, Lemma 2.3 shows that $(\mathrm{St}_2, h) \rtimes \pi_0'$ decomposes as a direct sum of two irreducible tempered representations. This means that there is an irreducible tempered representation π_1 such that

$$(\mathrm{St}_2, h) \rtimes \pi_0' \cong \pi_0 \oplus \pi_1.$$

This implies

$$\begin{aligned}\mathrm{Hom}(\omega,\ (\mathrm{St}_2, h) \rtimes \pi_0')_\infty &= \mathrm{Hom}(\omega, \pi_0)_\infty \oplus \mathrm{Hom}(\omega, \pi_1)_\infty \\ &= \Theta_{-4}^\vee(\pi_0) \oplus \Theta_{-4}^\vee(\pi_1),\end{aligned}$$

where ω denotes the appropriate Weil representation. We now repeat the computations of Corollary 3.7 to show that $\mathrm{Hom}(\omega,\ (\mathrm{St}_2, h)) \hookrightarrow \chi_V(\mathrm{St}_2, h) \rtimes \Theta_{-4}^\vee(\pi_0')$. Taking contragredients we thus arrive at

$$(*) \qquad \chi_V(\mathrm{St}_2, h) \rtimes \Theta_{-4}(\pi_0') \twoheadrightarrow \Theta_{-4}(\pi_0) \oplus \Theta_{-4}(\pi_1).$$

We now need

Lemma 7.10. *The only non-tempered irreducible subquotient of* $\Theta_{-4}(\pi_0')$ *is* $\theta_{-4}(\pi_0')$, *i.e.,* $L(\chi_V|\cdot|^{\frac{3}{2}} \rtimes \theta_{-2}(\pi_0'))$.

Proof. Any non-tempered subquotient of $\Theta_{-4}(\pi_0')$ is easily shown to be a subquotient of $\chi_V|\cdot|^{\frac{3}{2}} \rtimes \theta_{-2}(\pi_0'))$. However, π_0' does not contain S_2, so $\theta_{-2}(\pi_0'))$ is a discrete series representation. A simple application of Casselman's criterion now shows that $\chi_V|\cdot|^{\frac{3}{2}} \rtimes \theta_{-2}(\pi_0'))$ has no non-tempered subquotients apart from its Langlands quotient. ◻

As a consequence of this lemma, any non-tempered irreducible subquotient of $\Theta_{-4}(\pi_0)$ must also be a subquotient $\chi_V(\mathrm{St}_2, h) \times \chi_V|\cdot|^{\frac{3}{2}} \rtimes \Theta_{-2}(\pi_0')$. Furthermore, we have

Lemma 7.11. *The representation* $\chi_V(St_2, h) \times \chi_V|\cdot|^{\frac{3}{2}} \rtimes \Theta_{-2}(\pi_0')$ *contains a unique irreducible subquotient whose standard module is of the form* $\chi_V\delta([|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{3}{2}}]) \rtimes \tau_2$.

Proof. Let ξ be such a subquotient. Then $\chi_V\delta([|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{3}{2}}]) \rtimes \tau_2 \twoheadrightarrow \xi$ is equivalent to $\xi \hookrightarrow \chi_V\delta([|\cdot|^{-\frac{3}{2}}, |\cdot|^{\frac{1}{2}}]) \rtimes \tau_2$, which implies $R_{Q_3}(\xi) \twoheadrightarrow \chi_V\delta([|\cdot|^{-\frac{3}{2}}, |\cdot|^{\frac{1}{2}}]) \otimes \tau_2$. An easy application of Tadić's formula for μ^* (cf. Section 2.5) and Casselman's criterion now show that the only irreducible subquotient of $R_{Q_3}(\chi_V(\mathrm{St}_2, h) \times \chi_V|\cdot|^{\frac{3}{2}} \rtimes \Theta_{-2}(\pi_0'))$ in which $GL_3(F)$ acts by $\chi_V\delta([|\cdot|^{-\frac{3}{2}}, |\cdot|^{\frac{1}{2}}])$

is in fact $\chi_V\delta([|\cdot|^{-\frac{3}{2}},|\cdot|^{\frac{1}{2}}])\otimes\chi_V(\mathrm{St}_2,h)\rtimes\Theta_{-2}(\pi_0')$. Thus $\tau_2=\chi_V(\mathrm{St}_2,h)\rtimes\Theta_{-2}(\pi_0')$.

On the other hand, ξ is either a subquotient of $A=\chi_V(\mathrm{St}_2,h-1)\times\chi_V\delta([|\cdot|^{-\frac{1}{2}},|\cdot|^{\frac{3}{2}}])\rtimes\Theta_{-2}(\pi_0')$, or of $B=\chi_V(\mathrm{St}_2,h-1)\times\chi_V L\rtimes\Theta_{-2}(\pi_0')$, where we have used L to denote the Langlands quotient of $|\cdot|^{\frac{3}{2}}\times St_2$.

In the first case, we have $A\cong\chi_V\delta([|\cdot|^{-\frac{1}{2}},|\cdot|^{\frac{3}{2}}])\times\chi_V(\mathrm{St}_2,h-1)\rtimes\Theta_{-2}(\pi_0')$, so that ξ is in fact the Langlands quotient of A (note that the parameter of $\Theta_{-2}(\pi_0')$ contains $\chi_V S_2$, so that $\chi_V(\mathrm{St}_2,h-1)\rtimes\Theta_{-2}(\pi_0')$ is irreducible). As the Langlands quotient appears with multiplicity one, it remains to show that ξ cannot appear in B.

Note that we have $B\cong\chi_V L\times\chi_V(\mathrm{St}_2,h-1)\rtimes\Theta_{-2}(\pi_0')$, so B has a unique irreducible quotient, which is $L(\chi_V\nu^{\frac{3}{2}};(\chi_V\mathrm{St}_2,h)\rtimes\Theta_{-2}(\pi_0'))$. We claim that this is the only non-tempered subquotient of B. To prove this, observe that this Langlands quotient is equal to the image of the intertwining operator

$$T:\chi_V(\mathrm{St}_2,h-1)\times\chi_V L\rtimes\Theta_{-2}(\pi_0')\to\chi_V(\mathrm{St}_2,h-1)\times\chi_V L^\vee\rtimes\Theta_{-2}(\pi_0').$$

Therefore, any other non-tempered subquotient of B must lie in the kernel of this intertwining operator. However, this operator is a restriction of the intertwining operator

$$\chi_V(\mathrm{St}_2,h)\times\chi_V\nu^{\frac{3}{2}}\rtimes\Theta_{-2}(\pi_0')\to\chi_V(\mathrm{St}_2,h)\times\chi_V\nu^{-\frac{3}{2}}\rtimes\Theta_{-2}(\pi_0'),$$

which is induced from

$$T_0:\chi_V\nu^{\frac{3}{2}}\rtimes\Theta_{-2}(\pi_0')\to\chi_V\nu^{-\frac{3}{2}}\rtimes\Theta_{-2}(\pi_0').$$

From here we see that $\ker T\subseteq\chi_V(\mathrm{St}_2,h)\rtimes\ker T_0$. On the other hand, we have already seen (cf. Lemma 7.10) that $\chi_V\nu^{\frac{3}{2}}\rtimes\Theta_{-2}(\pi_0')$ has no non-tempered subquotients apart from its Langlands quotient. In particular, $\ker T_0$ contains only tempered subquotients, which implies the same for $\ker T$. We have thus shown that there are no non-tempered subquotients of B apart from its unique quotient. This completes the proof of the lemma. ∎

Returning to the discussion before Lemma 7.10, we notice that $\theta_{-4}(\pi_1)$ has a standard module of the form described in Lemma 7.11 (cf. Theorem 4.5 (3) of [2]). However, we have just proved that $\chi_V(\mathrm{St}_2,h)\rtimes\Theta_{-4}(\pi_0')$ contains only one irreducible subquotient which satisfies this property. In other words, the unique irreducible subquotient of $\chi_V(\mathrm{St}_2,h)\rtimes\Theta_{-4}(\pi_0')$ with a standard module of the prescribed form belongs to $\Theta_{-4}(\pi_1)$. Taking $(*)$ into account, this means that it cannot appear in $\theta_{-4}(\pi_0)$. This completes the proof of our Claim, and with it, the final step of our proof in Subcase 2.2.2.

We have thus analyzed all the cases obtained by considering different possibilities with respect to the target tower and the L-parameter of π_0. Along with the first lifts determined in Section 5 this provides a comprehensive description of all the lifts we have considered. The results are summarized in Theorem 5.1.

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