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Δ-RELATED FUNCTIONS AND GENERALIZED INVERSE LIMITS

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Abstract. For any continuous single-valued functions $f, g : [0,1] \to [0,1]$ we define upper semicontinuous set-valued functions $F, G : [0,1] \multimap [0,1]$ by their graphs as the unions of the diagonal Δ and the graphs of set-valued inverses of f and g respectively. We introduce when two functions are Δ-related and show that if f and g are Δ-related, then the inverse limits $\varprojlim F$ and $\varprojlim G$ are homeomorphic. We also give conditions under which $\varprojlim G$ is a quotient space of $\varprojlim F$.

1. Introduction

Given two inverse limits $\varprojlim F$ and $\varprojlim G$, it is usually a very difficult problem to see whether $\varprojlim F$ and $\varprojlim G$ are homeomorphic. That is why there are many authors researching the properties of bonding functions F and G that guarantee the existence of a homeomorphism from $\varprojlim F$ to $\varprojlim G$; for examples see [3, 4, 5, 6, 7]. In present paper we give sufficient conditions on set-valued functions F and G from a large class of upper semicontinuous functions such that their inverse limits are homeomorphic.

Our motivation in defining this class of upper semicontinuous functions is Ingram's paper [8], where the inverse limits with upper semicontinuous functions whose graphs are unions of graphs of single-valued functions are studied. In particular, we start with any continuous function $f : [0,1] \to [0,1]$ and the identity function $id : [0,1] \to [0,1]$, and define the upper semicontinuous

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function $F : [0,1] \to [0,1]$ by

$$\Gamma(F) = \{(s,t) \in [0,1] \times [0,1] \mid (t,s) \in \Gamma(f) \cup \Gamma(id)\}.$$

Our main result says that if f and g are Δ-related, then the inverse limits $\varprojlim F$ and $\varprojlim G$ (where F and G are defined as above) are homeomorphic. We also give conditions under which $\varprojlim G$ is a quotient space of $\varprojlim F$.

We proceed as follows. In Section 2, the basic definitions and notation are given. Section 3 serves as an illustrative motivation for our results and in Section 4, our main results are presented.

2. Definitions and notation

In the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_0$ the set of all nonnegative integers. A continuum is a nonempty compact and connected metric space.

For each $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ in the Hilbert cube $Q = \Pi_{i=1}^{\infty}[0,1]$ we use the standard notation for the i-th projection, i.e. $\pi_i(\mathbf{x}) = x_i$. We always use Q to denote the Hilbert cube $\Pi_{i=1}^{\infty}[0,1]$.

$2^{[0,1]}$ denotes the set of all nonempty closed subsets of $[0,1]$. A function $F : [0,1] \to 2^{[0,1]}$ is called a *set-valued function* from $[0,1]$ to $[0,1]$. We use $F : [0,1] \multimap [0,1]$ to denote such functions.

A function $F : [0,1] \multimap [0,1]$ is *upper semicontinuous at the point* $x \in [0,1]$ provided that if V is any open set in $[0,1]$ containing $F(x)$ then there is an open set U in $[0,1]$ containing x such that $F(t) \subseteq V$ for any $t \in U$. A function F is called *upper semicontinuous* if it is upper semicontinuous at each point of $[0,1]$.

The *graph* $\Gamma(F)$ *of a function* $F : [0,1] \multimap [0,1]$ is the set of all points $(x,y) \in [0,1] \times [0,1]$ such that $y \in F(x)$.

The following theorem is a well-known characterization of upper semicontinuous functions ([2, Theorem 1.2]).

Theorem 2.1. *Let* $F : [0,1] \multimap [0,1]$ *be a function. Then* F *is upper semicontinuous if and only if its graph* $\Gamma(F)$ *is closed in* $[0,1] \times [0,1]$.

In this paper we always deal with *inverse sequences* $\{X_i, F_i\}_{i=1}^{\infty}$, where $X_i = [0,1]$ and $F_i : [0,1] \multimap [0,1]$ is upper semicontinuous function for each i. We denote them by $\{[0,1], F_i\}_{i=1}^{\infty}$. The functions F_i are called the bonding functions.

The *inverse limit* of an inverse sequence $\{[0,1], F_i\}_{i=1}^{\infty}$ is defined to be the subspace of the product space $\prod_{i=1}^{\infty}[0,1]$ of all $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty}[0,1]$, such that $x_i \in F_i(x_{i+1})$ for each i. The inverse limit is denoted by $\varprojlim \{[0,1], F_i\}_{i=1}^{\infty}$. These inverse limits are a recent generalization (introduced by T. W. Ingram and W. S. Mahavier) of inverse limits of inverse sequences $\{[0,1], f_i\}_{i=1}^{\infty}$, where $f_i : [0,1] \to [0,1]$ are continuous functions. Such inverse limits are usually denoted by $\varprojlim\{[0,1], f_i\}_{i=1}^{\infty}$. Obviously, for any inverse sequence $\{[0,1], f_i\}_{i=1}^{\infty}$ of compact metric spaces and continuous bonding functions,

$$\varprojlim\{[0,1], f_i\}_{i=1}^{\infty} = \underset{\circ\!\!\longleftarrow}{\lim}\{[0,1], F_i\}_{i=1}^{\infty}$$

if $F_i(x) = \{f_i(x)\}$ for each i and each $x \in [0,1]$.

In the article we deal only with inverse sequences $\{[0,1], F_i\}_{i=1}^{\infty}$ where all the bonding functions are the same. In the case where $F_i = F$ for each i, the inverse limit $\underset{\circ\!\!\longleftarrow}{\lim}\{[0,1], F_i\}_{i=1}^{\infty}$ will be denoted by $\underset{\circ\!\!\longleftarrow}{\lim} F$.

Next we introduce some notation that is used in the paper.

For each $t \in [0,1]$ let $\overline{t} = (t, t, t, \ldots)$. Next, let $\Delta = \{(t,t) \mid t \in [0,1]\}$ and

$$L^{\infty} = \left\{\overline{t} \mid t \in [0,1]\right\}.$$

For any continuous function $f : [0,1] \to [0,1]$ we define

$$f^* = \{(f(x), x) \mid x \in [0,1]\},$$

$$L_{(n_i)_{i=0}^k}(f) = \left\{\left(\underbrace{t, \ldots, t}_{n_0}, \underbrace{f(t), \ldots, f(t)}_{n_1}, \ldots, \underbrace{f^k(t), \ldots, f^k(t)}_{n_k}, \overline{f^{k+1}(t)}\right) \middle| t \in [0,1]\right\}$$

for each $k \in \mathbb{N}_0$ and for any $(k+1)$-tuple $(n_0, n_1, n_2, \ldots, n_k) \in \mathbb{N}^{k+1}$, and

$$L_{(n_i)_{i=0}^{\infty}}(f) = \left\{\left(\underbrace{t, \ldots, t}_{n_0}, \underbrace{f(t), \ldots, f(t)}_{n_1}, \underbrace{f^2(t), \ldots, f^2(t)}_{n_2}, \ldots\right) \middle| t \in [0,1]\right\}$$

for any sequence $(n_0, n_1, n_2, \ldots)$ of positive integers.

Next, for each $n_0 \in \mathbb{N}$ we denote

$$\begin{aligned}\mathcal{L}^{n_0}(f) = &\left\{L_{(n_i)_{i=0}^k}(f) \mid k \in \mathbb{N}_0 \text{ and } n_1, n_2, \ldots, n_k \in \mathbb{N}\right\} \\ &\cup \left\{L_{(n_i)_{i=0}^{\infty}}(f) \mid n_i \in \mathbb{N} \text{ for each } i \in \mathbb{N}\right\},\end{aligned}$$

and

$$L^{n_0}(f) = \bigcup \mathcal{L}^{n_0}(f),$$

meaning that $L^{n_0}(f)$ is the union of sets from $\mathcal{L}^{n_0}(f)$.

3. Motivation examples

In this section we take three simple u.s.c. functions from $[0,1]$ to $[0,1]$ and study relationships of their inverse limits. Those functions will serve as a motivation for our main results.

We define each of the three functions by defining their graphs. The graph of each function is the union of the diagonal Δ and the set f^* for some continuous function $f : [0,1] \to [0,1]$. We define the three functions in such a way that their inverse limits are not homeomorphic, but still, they are related in the sense that there is a quotient map from one to another. First we prove the following proposition.

PROPOSITION 3.1. *Let $f : [0,1] \to [0,1]$ be a continuous function and $F : [0,1] \multimap [0,1]$ the upper semicontinuous function defined by $\Gamma(F) = \Delta \cup f^*$. Then*

$$\varprojlim F = \mathrm{Cl}\left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(f)\right) = \left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(f)\right) \cup L^\infty.$$

PROOF. The equalities

$$\varprojlim F = \left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(f)\right) \cup L^\infty$$

and

$$\mathrm{Cl}\left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(f)\right) = \left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(f)\right) \cup L^\infty$$

are obvious. We leave the details to a reader. ∎

EXAMPLE 3.2. Let $f : [0,1] \to [0,1]$ be the piecewise linear function, whose graph is the union of two straight line segments connecting the points $(0,1)$, $(\frac{1}{2}, \frac{3}{4})$ and $(1,1)$. We define $F : [0,1] \multimap [0,1]$ by $\Gamma(F) = \Delta \cup f^*$. See Figure 1.

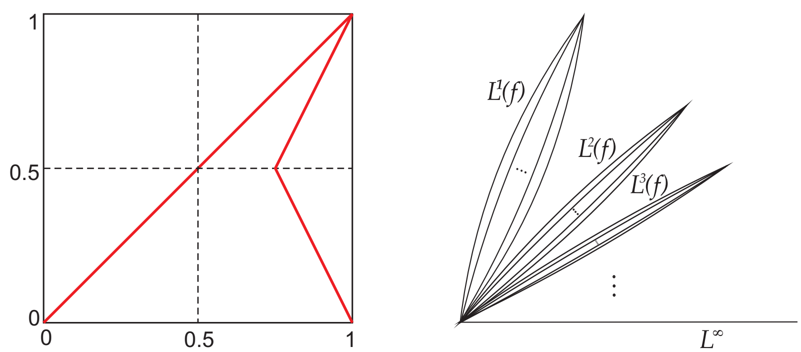

FIGURE 1. $\Gamma(F)$ (left) and a homeomorphic copy of $\varprojlim F$ (right)

Then by Proposition 3.1,

$$\varprojlim F = \mathrm{Cl}\left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(f)\right) = \left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(f)\right) \cup L^\infty.$$

For a geometrical interpretation of the inverse limit, let n_0 be arbitrarily chosen. One can easily see that each element of $\mathcal{L}^{n_0}(f)$ is an arc with endpoints $(\underbrace{0,0,\ldots,0}_{n_0},\overline{1})$ and $\overline{1}$, and that $L^{n_0}(f)\cap(\{t_0\}\times Q)$ is a Cantor set for each $t_0\in(0,1)$.

Also, note that $L^{n_0}(f)\cap L^{n_0'}(f)=\{\overline{1}\}$ if and only if $n_0\neq n_0'$, $L^{n_0}(f)\cap L^{\infty}=\{\overline{1}\}$ for each $n_0\in\mathbb{N}$ and $\lim_{n_0\to\infty}L^{n_0}(f)=L^{\infty}$; see Figure 1.

Next we define the second function of the example.

Let $g:[0,1]\to[0,1]$ be the piecewise linear function, whose graph is the union of four straight line segments connecting the points $(0,1)$, $(\frac{1}{4},\frac{7}{8})$, $(\frac{1}{2},1)$, $(\frac{3}{4},\frac{7}{8})$ and $(1,1)$.

We define $G:[0,1]\multimap[0,1]$ by $\Gamma(G)=\Delta\cup g^*$. See Figure 2.

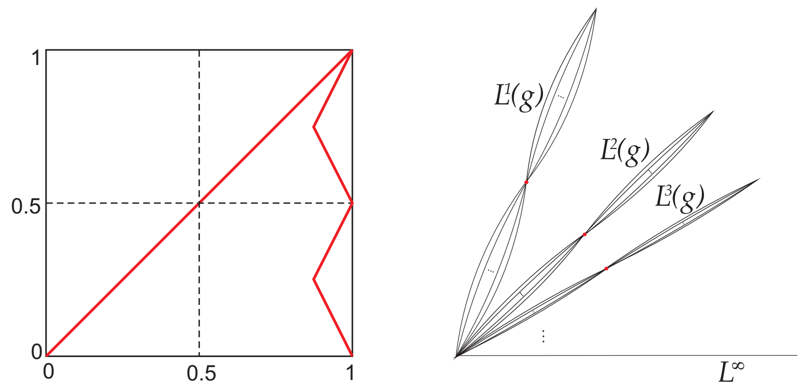

FIGURE 2. $\Gamma(G)$ (left) and a homeomorphic copy of $\varprojlim G$(right)

Then by Proposition 3.1,

$$\varprojlim G=\mathrm{Cl}\left(\bigcup_{n_0\in\mathbb{N}}L^{n_0}(g)\right)=\left(\bigcup_{n_0\in\mathbb{N}}L^{n_0}(g)\right)\cup L^{\infty}.$$

Let $\varphi:\varprojlim F\to\varprojlim G$ be defined by

$$\varphi\left(\underbrace{t,t,\ldots,t}_{n_0},\underbrace{f(t),f(t),\ldots,f(t)}_{n_1},\underbrace{f^2(t),f^2(t),\ldots,f^2(t)}_{n_2},\ldots\right)$$
$$=\left(\underbrace{t,t,\ldots,t}_{n_0},\underbrace{g(t),g(t),\ldots,g(t)}_{n_1},\underbrace{g^2(t),g^2(t),\ldots,g^2(t)}_{n_2},\ldots\right).$$

It is easy to see that φ is well defined and surjective. Since g is continuous, φ is also a continuous function. This means that φ is a quotient map from $\varprojlim F$ to the $\varprojlim G$.

Note that φ is not injective. For instance, let

$$\mathbf{x}_n = (\frac{1}{2}, \underbrace{f(\frac{1}{2}), f(\frac{1}{2}), \ldots, f(\frac{1}{2})}_{n}, \overline{f^2(\frac{1}{2})}) \in L_{(1,n)}(f)$$

for each positive integer n. Recall that $L_{(1,n)}(f)$ are arcs with

$$\bigcap_{n\in\mathbb{N}} L_{(1,n)}(f) = \left\{(0,\overline{1}), \overline{1}\right\},$$

and therefore $\mathbf{x}_n \neq \mathbf{x}_m$ for each $n \neq m$. But obviously $\varphi(\mathbf{x}_n) = (\frac{1}{2}, \overline{1})$ for each positive integer n. See Figure 2, where $\varprojlim G$ is presented.

Finally, the last function of the example is defined.

Let $h : [0,1] \to [0,1]$ be the piecewise linear function, whose graph is the union of four straight line segments connecting the points $(0,1)$, $(\frac{1}{4}, \frac{3}{8})$, $(\frac{1}{2}, 1)$, $(\frac{3}{4}, \frac{7}{8})$ and $(1,1)$.

We define $H : [0,1] \multimap [0,1]$ by $\Gamma(H) = \Delta \cup h^*$. See Figure 3.

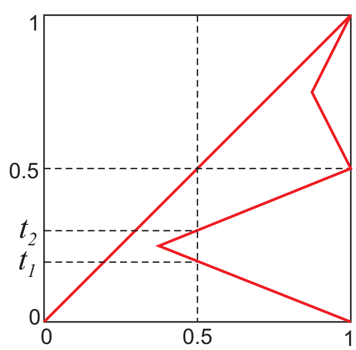

FIGURE 3. $\Gamma(H)$

Then by Proposition 3.1,

$$\varprojlim H = \mathrm{Cl}\left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(h)\right) = \left(\bigcup_{n_0\in\mathbb{N}} L^{n_0}(h)\right) \cup L^{\infty}.$$

As before, one can easily see that

$$\psi\left(\underbrace{t,t,\dots,t}_{n_0},\underbrace{g(t),g(t),\dots,g(t)}_{n_1},\underbrace{g^2(t),g^2(t),\dots,g^2(t)}_{n_2},\dots\right)$$
$$=\left(\underbrace{t,t,\dots,t}_{n_0},\underbrace{h(t),h(t),\dots,h(t)}_{n_1},\underbrace{h^2(t),h^2(t),\dots,h^2(t)}_{n_2},\dots\right)$$

defines a quotient map $\varprojlim G \to \varprojlim H$.

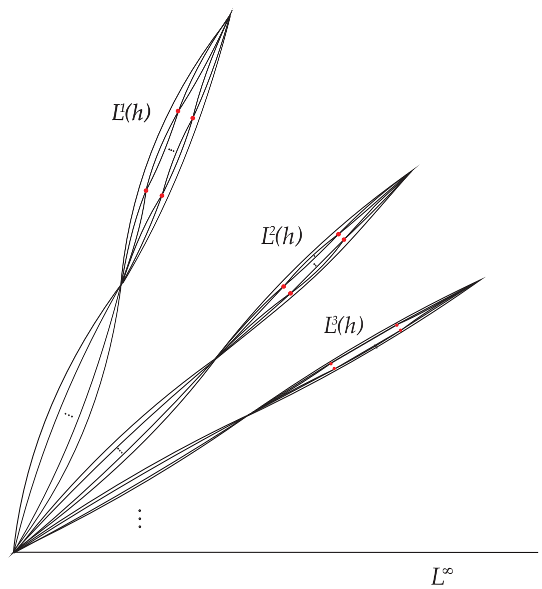

FIGURE 4. A homeomorphic copy of $\varprojlim H$

We show that ψ is not injective. From the definition of the function h it follows that there exist $t_1, t_2 \in [0,1]$ with $t_1 \neq t_2$ and $h(t_1) = h(t_2) = \frac{1}{2}$. Let

$$\mathbf{x}_n = (t_1, g(t_1), \underbrace{g^2(t_1), g^2(t_1), \dots, g^2(t_1)}_{n}, \overline{g^3(t_1)})$$

and

$$\mathbf{y}_n = (t_2, g(t_2), \underbrace{g^2(t_2), g^2(t_2), \dots, g^2(t_2)}_{n}, \overline{g^3(t_2)}).$$

Since $\mathbf{x}_n, \mathbf{y}_n \in L_{(1,1,n)}(g)$ and

$$\bigcap_{n\in\mathbb{N}} L_{(1,1,n)}(g) = \left\{(0,0,\overline{1}), \overline{1}\right\},$$

it holds that $\mathbf{x}_n \neq \mathbf{x}_m$ and $\mathbf{y}_n \neq \mathbf{y}_m$ for each $n \neq m$. But $\psi(\mathbf{x}_n) = (t_1, \frac{1}{2}, \overline{1})$ and $\psi(\mathbf{y}_n) = (t_2, \frac{1}{2}, \overline{1})$ for each positive integer n.

See Figure 4, where $\varprojlim H$ is presented. On the Figure some special points of $\varprojlim H$ which are not special for $\varprojlim G$ may be seen; i.e., within the upper loops, sets of ramification points appear in $\varprojlim H$: one set associated with t_1 (the ramification points on the left hand part of the upper loops) and the other set associated with t_2 (the ramification points on the right hand part of the upper loops).

We can define a quotient map $\varprojlim F \to \varprojlim H$ by using the same formula as before.

Note that the harmonic fan (which is homeomorphic to the inverse limit $\varprojlim \Lambda$ where $\Gamma(\Lambda) = \Delta \cup (\{1\} \times [0,1])$, see [2, p. 31] for instance) is a quotient space of each inverse limit from the example. To see this, a reader may follow a similar arguing as above.

4. Main results

In the present section we give our main results. In particular, for continuous functions $f, g : [0,1] \to [0,1]$ and the upper semicontinuous functions $F, G : [0,1] \multimap [0,1]$, where

$$\Gamma(F) = \Delta \cup f^* \text{ and } \Gamma(G) = \Delta \cup g^*,$$

we give sufficient conditions on f and g under which inverse limits $\varprojlim F$ and $\varprojlim G$ are homeomorphic. We introduce some notation first.

For a continuous function $f : [0,1] \to [0,1]$ let

$$A_0(f) = \{x \in [0,1] \mid f(x) = x\},$$
$$A_i(f) = \left\{x \in [0,1] \mid f^{i+1}(x) = f^i(x)\right\} \text{ for each } i \in \mathbb{N},$$

and let

$$A(f) = \bigcup_{i=0}^{\infty} A_i(f)\ .$$

Definition 4.1. *Let $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ be continuous functions. We say that f and g are Δ-related, if there is an increasing homeomorphism $\alpha : [0,1] \to [0,1]$ such that $\alpha(A_i(f)) = A_i(g)$ for each nonnegative integer i.*

Note that the condition "$\alpha(A_i(f)) = A_i(g)$ for each nonnegative integer i" in Definition 4.1 is equivalent to "$\alpha(A_{i+1}(f) \setminus A_i(f)) = A_{i+1}(g) \setminus A_i(g)$ for each nonnegative integer i" as well as to "$\alpha|_{A_i(f)} : A_i(f) \to A_i(g)$ is an increasing homeomorphism for each nonnegative integer i".

THEOREM 4.2. *Let $f, g : [0,1] \to [0,1]$ be Δ-related continuous functions and let $F, G : [0,1] \multimap [0,1]$ be defined by their graphs:*

$$\Gamma(F) = \Delta \cup f^*, \ \Gamma(G) = \Delta \cup g^*.$$

Then $\varprojlim F$ is homeomorphic to $\varprojlim G$.

PROOF. Let $\alpha : [0,1] \to [0,1]$ be an increasing homeomorphism, such that $\alpha(A_i(f)) = A_i(g)$ for each nonnegative integer i.

We define $\varphi : \varprojlim F \to \varprojlim G$ by

$$\varphi\left(\underbrace{t,\ldots,t}_{n_0},\underbrace{f(t),\ldots,f(t)}_{n_1},\underbrace{f^2(t),\ldots,f^2(t)}_{n_2},\ldots\right)$$
$$= \left(\underbrace{\alpha(t),\ldots,\alpha(t)}_{n_0},\underbrace{g(\alpha(t)),\ldots,g(\alpha(t))}_{n_1},\underbrace{g^2(\alpha(t)),\ldots,g^2(\alpha(t))}_{n_2},\ldots\right)$$

and show that φ is a homeomorphism. Note that $n_0 > 0$.

Obviously, φ is well defined and since α and g are both continuous, φ is also a continuous function. Since for each

$$\mathbf{y} = (\underbrace{t,\ldots,t}_{n_0},\underbrace{g(t),\ldots,g(t)}_{n_1},\underbrace{g^2(t),\ldots,g^2(t)}_{n_2},\ldots) \in \varprojlim G$$

there is

$$\mathbf{x} = (\underbrace{\alpha^{-1}(t),\ldots,\alpha^{-1}(t)}_{n_0},\underbrace{f(\alpha^{-1}(t)),\ldots,f(\alpha^{-1}(t))}_{n_1},$$
$$\underbrace{f^2(\alpha^{-1}(t)),\ldots,f^2(\alpha^{-1}(t))}_{n_2},\ldots) \in \varprojlim F$$

such that $\varphi(\mathbf{x}) = \mathbf{y}$ it follows that φ is surjective. To show that φ is injective let $\varphi(\mathbf{x}_1) = \varphi(\mathbf{x}_2)$. We already know that there exist $t, s \in [0,1]$ and $n_i, m_i \in \mathbb{N} \cup \{\infty\}$ for each nonnegative integer i such that

$$\varphi(\mathbf{x}_1) = \varphi(\underbrace{t,\ldots,t}_{n_0},\underbrace{f(t),\ldots,f(t)}_{n_1},\ldots)$$
$$= (\underbrace{\alpha(t),\ldots,\alpha(t)}_{n_0},\underbrace{g(\alpha(t)),\ldots,g(\alpha(t))}_{n_1},\ldots)$$

and

$$\begin{aligned}\varphi(\mathbf{x}_2) &= \varphi(\underbrace{s,\dots,s}_{m_0},\underbrace{f(s),\dots,f(s)}_{m_1},\dots)\\ &= (\underbrace{\alpha(s),\dots,\alpha(s)}_{m_0},\underbrace{g(\alpha(s)),\dots,g(\alpha(s))}_{m_1},\dots).\end{aligned}$$

It follows that $\alpha(t)=\alpha(s)$ and therefore $t=s$ since α is bijective. We have

$$\begin{aligned}\varphi(\mathbf{x}_1) =&(\underbrace{\alpha(t),\dots,\alpha(t)}_{n_0},\underbrace{g(\alpha(t)),\dots,g(\alpha(t))}_{n_1},\underbrace{g^2(\alpha(t)),\dots,g^2(\alpha(t))}_{n_2},\dots)\\ =&(\underbrace{\alpha(t),\dots,\alpha(t)}_{m_0},\underbrace{g(\alpha(t)),\dots,g(\alpha(t))}_{m_1},\underbrace{g^2(\alpha(t)),\dots,g^2(\alpha(t))}_{m_2},\dots)\\ =&\varphi(\mathbf{x}_2).\end{aligned}$$

Suppose that $n_0 \neq m_0$. Then $g(\alpha(t)) = \alpha(t)$ and therefore $\alpha(t) \in A_0(g)$. It follows that $t \in A_0(f)$ and $\mathbf{x}_1 = \overline{t} = \mathbf{x}_2$.

Next, suppose that there exists $k \in \mathbb{N}$ such that $n_i = m_i$ for each $i < k$ and $n_k \neq m_k$.

Then $g^{k+1}(\alpha(t)) = g^k(\alpha(t))$ and therefore $\alpha(t) \in A_k(g)$ and $t \in A_k(f)$. Thus $\pi_i(\mathbf{x}_1) = t = \pi_i(\mathbf{x}_2)$ for each $i \geq k$ and since (by the assumption) $\pi_i(\mathbf{x}_1) = \pi_i(\mathbf{x}_2)$ for each $i < k$, it follows that $\mathbf{x}_1 = \mathbf{x}_2$. Therefore φ is a homeomorphism. □

Next we interpret the relation “to be Δ-related functions” for the class of continuous functions $f : [0,1] \to [0,1]$, such that $A(f)$ is finite. This interpretation gives an easy tool to detect Δ-related functions.

Note that if $A(f)$ is finite, then also $A_i(f)$ is finite for each i and since $A_i(f) \subseteq A_{i+1}(f)$ for each i, there is $k \in \mathbb{N}$ such that $A_i(f) = A_{i+1}(f)$ for each $i \geq k$.

THEOREM 4.3. *Let $f, g : [0,1] \to [0,1]$ be continuous functions such that $A(f)$ and $A(g)$ are finite and let for each $i \in \mathbb{N}_0$, $A_i(f) = \{a^i_1, a^i_2, \dots, a^i_{n_i}\}$, where $a^i_j < a^i_{j+1}$ for each $j \in \{1,2,\dots,n_i-1\}$, and $A_i(g) = \{b^i_1, b^i_2, \dots, b^i_{m_i}\}$, where $b^i_j < b^i_{j+1}$ for each $j \in \{1,2,\dots,m_i-1\}$. Then f and g are Δ-related if and only if the following hold true*

1. $|A_i(f)| = |A_i(g)|$ *for each* $i \in \mathbb{N}_0$,
2. $0 \in A_i(f)$ *if and only if* $0 \in A_i(g)$ *for each* $i \in \mathbb{N}_0$,
   $1 \in A_i(f)$ *if and only if* $1 \in A_i(g)$ *for each* $i \in \mathbb{N}_0$,
3. $|A_{i+1}(f) \cap [0, a^i_j]| = |A_{i+1}(g) \cap [0, b^i_j]|$ *for each* $j \in \{1,2,\dots,n_i\}$.

Note that *(3)* of Theorem 4.3 is equivalent to *(3')* and *(3")* below

(3') $|A_{i+1}(f) \cap [a^i_j, 1]| = |A_{i+1}(g) \cap [b^i_j, 1]|$ for each $j \in \{1,2,\dots,n_i\}$.

(3") $|A_{i+1}(f)\cap[0,a_1^i]| = |A_{i+1}(g)\cap[0,b_1^i]|$, $|A_{i+1}(f)\cap[a_{n_i}^i,1]| = |A_{i+1}(g)\cap[b_{n_i}^i,1]|$ and $|A_{i+1}(f)\cap[a_j^i,a_{j+1}^i]| = |A_{i+1}(g)\cap[b_j^i,b_{j+1}^i]|$ for each $j\in\{1,2,\dots,n_i-1\}$.

Proof. First we prove that if f and g are Δ-related, then *(1)*, *(2)* and *(3)* follows. Let $\alpha:[0,1]\to[0,1]$ be an increasing homeomorphism with $\alpha(A_i(f)) = A_i(g)$ for each nonnegative integer i. Since $0\le a_1^i < a_2^i < \dots < a_{n_i}^i \le 1$ for each nonnegative integer i, it follows that $\alpha(0)=0, \alpha(1)=1$ and $\alpha(a_j^i)=b_j^i$ for each $j\in\{1,2,\dots,n_i\}$ and therefore *(1)* and *(2)* obviously hold true. Suppose that $A_{i+1}(f)\cap[0,a_j^i] = \{a_1^{i+1},a_2^{i+1},\dots,a_k^{i+1}\}$ for some $k\le n_{i+1}$. Since $\alpha([0,a_j^i]) = [0,b_j^i]$ and $\alpha(A_{i+1}(f)) = A_{i+1}(g)$ it follows that (note that α is an increasing homeomorphism and $A_{i+1}(f), A_{i+1}(g)$ are finite) $\alpha(A_{i+1}(f)\cap[0,a_j^i]) = A_{i+1}(g)\cap[0,b_j^i]$ and therefore *(3)* follows.

To prove the other implication, suppose that *(1)*, *(2)* and *(3)* hold true. Let $\alpha:[0,1]\to[0,1]$ be the increasing piecewise linear function, such that $\alpha(0)=0$, $\alpha(1)=1$ and $\alpha(a_j^i)=b_j^i$ for each $i\in\mathbb{N}_0$ and each $j\in\{1,2,\dots,n_i\}$. Obviously α is an increasing homeomorphism with $\alpha(A_i(f))=A_i(g)$ for each nonnegative integer i and therefore f and g are Δ-related. □

The following is an easy corollary of Theorem 4.2 and Theorem 4.3.

Corollary 4.4. *Let $f,g:[0,1]\to[0,1]$ be continuous functions such that $A(f)$ and $A(g)$ are finite. Further, suppose that the following hold true*

1. $|A_i(f)| = |A_i(g)|$ *for each* $i\in\mathbb{N}_0$,
2. $0\in A_i(f)$ *if and only if* $0\in A_i(g)$ *for each* $i\in\mathbb{N}_0$,
   $1\in A_i(f)$ *if and only if* $1\in A_i(g)$ *for each* $i\in\mathbb{N}_0$,
3. $|A_{i+1}(f)\cap[0,a_j^i]| = |A_{i+1}(g)\cap[0,b_j^i]|$ *for each* $j\in\{1,2,\dots,n_i\}$.

Then $\varprojlim F$ *is homeomorphic to* $\varprojlim G$.

Example 4.5. Let $F,G:[0,1]\multimap[0,1]$ be defined by their graphs $\Gamma(F)$ and $\Gamma(G)$, as shown on Figure 5. Then by Corollary 4.4 $\varprojlim F$ is homeomorphic to $\varprojlim G$.

In Theorem 4.6, conditions on f and g are presented, under which the existence of a quotient map (and not necessarily a homeomorphism) from $\varprojlim F$ to $\varprojlim G$ is accomplished.

Theorem 4.6. *Let $f,g:[0,1]\to[0,1]$ be continuous functions such that $A_i(f)\subseteq A_i(g)$ for each nonnegative integer i. Further, let $F,G:[0,1]\multimap[0,1]$ be defined by $\Gamma(F)=\Delta\cup f^*$, $\Gamma(G)=\Delta\cup g^*$. Then $\varprojlim G$ is a quotient space of $\varprojlim F$.*

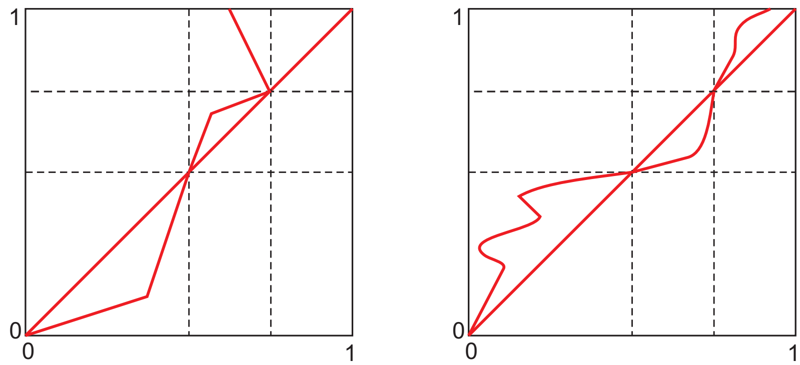

FIGURE 5. $\Gamma(F)$ (left) and $\Gamma(G)$ (right)

PROOF. We define $\varphi : \varprojlim F \to \varprojlim G$ by

$$\begin{aligned}\varphi\left(\underbrace{t,\ldots,t}_{n_0},\underbrace{f(t),\ldots,f(t)}_{n_1},\underbrace{f^2(t),\ldots,f^2(t)}_{n_2},\ldots\right)\\ = \left(\underbrace{t,\ldots,t}_{n_0},\underbrace{g(t),\ldots,g(t)}_{n_1},\underbrace{g^2(t),\ldots,g^2(t)}_{n_2},\ldots\right)\end{aligned}$$

and show that φ is a quotient map. It is easy to see that φ is well defined and surjective since $A_i(f) \subseteq A_i(g)$ for each nonnegative integer i. Since g is continuous, φ is also a continuous function. ∎

REMARK 4.7. Let φ be the function from the proof of Theorem 4.6 and suppose that $A(f)$ and $A(g)$ are finite. The quotient map φ helps picturing the inverse limit $\varprojlim G$ by gluing some points from $\varprojlim F$ together. We continue by listing such points.

We show that φ is injective if and only $A_i(f) = A_i(g)$ for each nonnegative integer i. If $A_i(f) = A_i(g)$ for each i, then by taking $\alpha : [0,1] \to [0,1]$, $\alpha(t) = t$ we can see that f and g are Δ-related and φ is a homeomorphism. Suppose that there exists $i \in \mathbb{N}_0$ such that $A_i(f)$ is a proper subset of $A_i(g)$ and let

$$B_i(f) = \Bigg\{ \Big(\underbrace{t, \dots, t}_{n_0}, \underbrace{f(t), \dots, f(t)}_{n_1}, \dots, \underbrace{f^i(t), \dots, f^i(t)}_{n_i}, f^{i+1}(t), \dots \Big) \;\Big|\; t \in A_i(g) \setminus A_i(f) \Bigg\} \subseteq \varprojlim F,$$

$$B_i(g) = \Bigg\{ \Big(\underbrace{t, \dots, t}_{n_0}, \underbrace{g(t), \dots, g(t)}_{n_1}, \dots, \underbrace{g^i(t), \dots, g^i(t)}_{n_i}, g^{i+1}(t), \dots \Big) \;\Big|\; t \in A_i(g) \setminus A_i(f) \Bigg\} \subseteq \varprojlim G.$$

Then $\varphi(B_i(f)) = B_i(g)$ and since $f^{i+1}(t) \neq f^i(t)$ and $g^{i+1}(t) = g^i(t)$ it holds that $|B_i(f)| > |B_i(g)|$, since both, $B_i(f)$ and $B_i(g)$ are finite. Therefore φ is not injective.

We conclude the paper with the following illustrative example.

Example 4.8. Let $F, G : [0,1] \multimap [0,1]$ be defined by their graphs $\Gamma(F)$ and $\Gamma(G)$, as shown on Figure 6. Then by Theorem 4.6 $\varprojlim G$ is a quotient

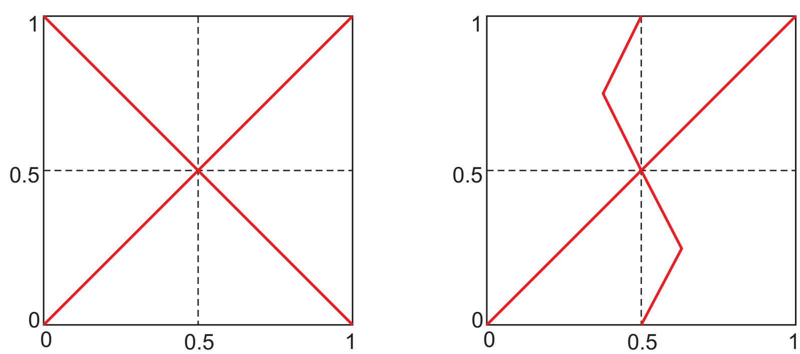

Figure 6. $\Gamma(F)$ (left) and $\Gamma(G)$ (right)

space of $\varprojlim F$. Note that $\varprojlim F$ is a Cantor fan (see [2, p. 22]), while a homeomorphic copy of the quotient space $\varprojlim G$ is seen on Figure 7.

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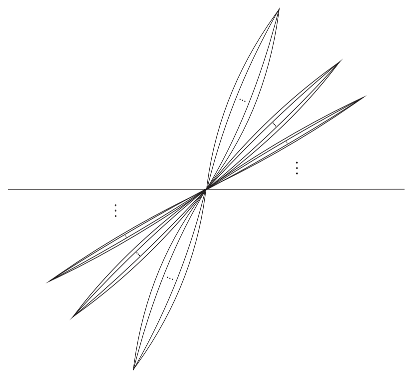

FIGURE 7. A homeomorphic copy of $\varprojlim G$ from Example 4.8

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