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Bouvaist Cubic of a Triangle in an Isotropic Plane

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ABSTRACT

The cubic in an isotropic plane which passes through the intersections of the sides of an orthic triangle with the sides of a complementary triangle of a given triangle, and through the point which is complementary to the Steiner point of triangle is studied in this paper. It is proved that its non-isotropic asymptote is parallel to Lemoine line of a given triangle.

Key words: isotropic plane, Bouvaist cubic, point complementary to the Steiner point

MSC2010: 51N25

Bouvaistova kubika trokuta u izotropnoj ravnini

SAŽETAK

U članku se proučava kubika koja prolazi kroz sjecišta stranica ortotrokuta i komplementarnog trokuta danog trokuta i kroz točku komplementarnu Steinerovoj točki tog trokuta. Dokazuje se da je neizotropna asimptota kubike paralelna s Lemoineovim pravcem danog trokuta.

Ključne riječi: izotropna ravnina, Bouvaistova kubika, komplementarna točka Steinerovoj točki

In [1], Bouvaist showed the existence of a cubic in Euclidean geometry, which passes through all nine intersections of the sides of an orthic triangle and a complementary triangle of a given triangle and through a point complementary to the Steiner point of that triangle. He proved that this cubic is circular and its real asymptote is parallel to the Lemoine line of a given triangle.

It will be shown in this paper that some analogous statement holds in the isotropic plane as well.

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, i.e., the absolute line ω, and one point on that line, i.e., the absolute point Ω. The lines through the point Ω are isotropic lines, and the points on the line ω are isotropic points (the points at infinity). Two points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ with $x_1 = x_2$ are said to be *parallel*, and we shall say they are on the same *isotropic* line. Any isotropic line is perpendicular to any non-isotropic line.

A triangle is said to be *allowable* if none of its sides is isotropic. Each allowable triangle ABC can be set by a suitable choice of the coordinate system in the *standard position*, in which its circumscribed circle has the equation $y = x^2$, and its vertices are the points $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where $a + b + c = 0$. We shall say then that ABC is a *standard triangle*. To prove geometric facts for each allowable triangle it is sufficient to give a proof for the standard triangle (see [3]).
With the labels

$$p = abc \quad \text{and} \quad q = bc + ca + ab$$

a number of useful equalities are proved in [3], as e.g.

$$a^2 = bc - q,$$
$$(b-c)^2 = -(q + 3bc),$$
$$(c-a)(a-b) = 2q - 3bc.$$

In [3], it is proved that the sides B_hC_h and B_mC_m of the orthic triangle $A_hB_hC_h$ and the complementary triangle

$A_mB_mC_m$ of the standard triangle have the equations:

$$y-2ax+q-2bc=0,$$

$$y+ax+q-\frac{bc}{2}=0,$$

and the equations of their other sides are obtained by a cyclic permutation $a \to b \to c \to a$. That is why every cubic through all nine intersections of the sides of these two triangles has the equation of the form:

$$\prod(y-2ax+q-2bc)-\lambda\prod\left(y+ax+q-\frac{bc}{2}\right)=0, \quad (1)$$

where $\prod$ denotes the product of three factors, the first of which is written, and the other two arise from the first one by cyclic permutations $a \to b \to c \to a$.
In [4], it is shown that the point

$$S=\left(\frac{3p}{2q},-\frac{9p^2}{2q^2}-q\right)$$

is complementary to the Steiner point of the standard triangle ABC. For that point we obtain

$$\begin{aligned}
y-2ax+q-2bc &= -\frac{9p^2}{2q^2}-\frac{3ap}{q}-2bc\\
&= -\frac{bc}{2q^2}(9a^2bc+6a^2q+4q^2)\\
&= -\frac{bc}{2q^2}[9bc(bc-q)+6q(bc-q)+4q^2]\\
&= \frac{bc}{2q^2}(2q^2+3bcq-9b^2c^2)\\
&= \frac{bc}{2q^2}(q+3bc)(2q-3bc)\\
&= -\frac{bc}{2q^2}(b-c)^2(c-a)(a-b),
\end{aligned}$$

$$\begin{aligned}
y+ax+q-\frac{bc}{2} &= -\frac{9p^2}{2q^2}+\frac{3ap}{2q}-\frac{bc}{2}\\
&= -\frac{bc}{2q^2}(9a^2bc-3a^2q+q^2)\\
&= -\frac{bc}{2q^2}[9bc(bc-q)-3q(bc-q)+q^2]\\
&= -\frac{bc}{2q^2}(4q^2-12bcq+9b^2c^2),\\
&= -\frac{bc}{2q^2}(2q-3bc)^2\\
&= -\frac{bc}{2q^2}(c-a)^2(a-b)^2
\end{aligned}$$

and then

$$\prod(y-2ax+q-2bc)=-\frac{a^2b^2c^2}{8q^6}(b-c)^4(c-a)^4(a-b)^4,$$

$$\prod\left(y+ax+q-\frac{bc}{2}\right)=-\frac{a^2b^2c^2}{8q^6}(b-c)^4(c-a)^4(a-b)^4.$$

Thus, the cubic of the pencil of the cubics with equation (1) passes through the point S if one takes $\lambda=1$ (Figure 1).

If that cubic of the allowable triangle ABC, which passes through the intersections of the sides of its orthic triangle with the sides of its complementary triangle, and through the point S complementary to the Steiner point of the triangle ABC (Figure 1), is called the *Bouvaist cubic* of that triangle, then we have:

Theorem 1 *The Bouvaist cubic $\mathcal{B}$ of the standard triangle ABC has the equation:*

$$\begin{aligned}
&(y-2ax+q-2bc)(y-2bx+q-2ca)\\
&(y-2cx+q-2ab)-\left(y+ax+q-\frac{bc}{2}\right)\\
&\left(y+bx+q-\frac{ca}{2}\right)\left(y+cx+q-\frac{ab}{2}\right)=0.
\end{aligned} \quad (2)$$

Let us now find the intersection points of the cubic (2) and the absolute line. We have to solve the equation

$$(y-2ax)(y-2bx)(y-2cx)-(y+ax)(y+bx)(y+cx)=0,$$

which can also be written in the following form:

$$-3(a+b+c)xy^2+3(bc+ca+ab)x^2y-9abcx^3=0,$$

and finally as $3qx^2y-9px^3=0$. We have the double solution $x=0$ and the solution $y=\frac{3p}{q}x$, which means that the cubic has an asymptote with a slope $\frac{3p}{q}$, which is by [2] a slope of the Lemoine line $\mathcal{L}$ of the triangle ABC. We obtained:

Theorem 2 *The non-isotropic asymptote of Bouvist cubic of an allowable triangle is parallel to the Lemoine line of a given triangle. Absolute point is an intersection point of the Bouvaist cubic and absolute line with intersection multiplicity 2.*

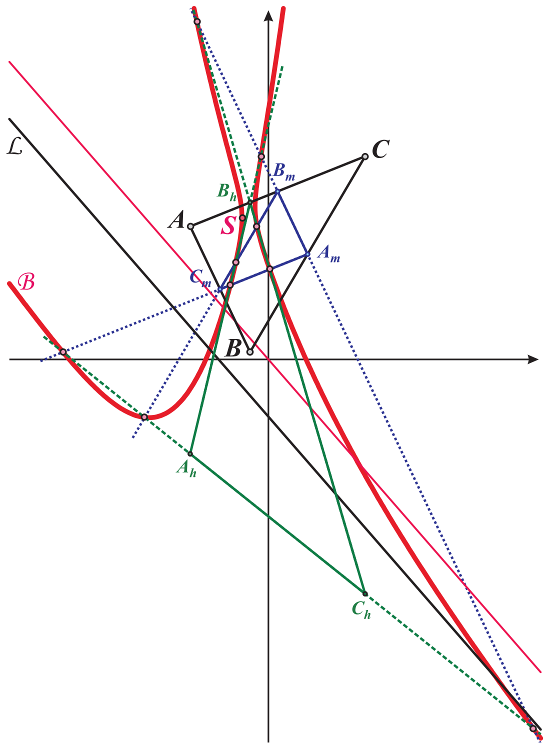

Figure 1: *Bouvaist cubic of a triangle ABC in isotropic plane*

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