Hermiteov identitet i primjene

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Poznati francuski matematičar Charles Hermite (1822. – 1901.) je davne 1884. dokazao identitet iz teorije brojeva (vidi [4]), vezan za funkciju najveće cijelo, koji ima primjene, među ostalim, u brojnim zadacima koji se pojavljuju na matematičkim natjecanjima.

Neka je $x \in \mathbf{R}$. Tada s $\lfloor x \rfloor$ označavamo najveći cijeli broj koji nije veći od x, a s $\{x\} = x - \lfloor x \rfloor$ razlomljeni dio od x, $0 \le \{x\} < 1$.

Teorem 1 (Hermite). *Za svaki realan broj x i prirodan broj n vrijedi:*

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor.$$

Prvi dokaz. Ako je $\{x\} = 0$, onda je $x \in \mathbf{Z}$ i tvrdnja očigledno vrijedi jer $\left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor x \rfloor$, $k = 0, \ldots n-1$.

Pretpostavimo $0 < \{x\} < 1$. Tada zbog

$$\lfloor x \rfloor \le \lfloor x + \frac{1}{n} \rfloor \le \cdots \le \lfloor x + \frac{n-1}{n} \rfloor \le \lfloor x \rfloor + 1$$

postoji i, $1 \le i \le n-1$ takav da je

$$\lfloor x \rfloor = \left\lfloor x + \frac{1}{n} \right\rfloor = \cdots = \left\lfloor x + \frac{i-1}{n} \right\rfloor$$

i

$$\left\lfloor x + \frac{i}{n} \right\rfloor = \left\lfloor x + \frac{i+1}{n} \right\rfloor = \cdots = \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor x \rfloor + 1.$$

Sada je

$$\begin{aligned} \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor &= i\lfloor x \rfloor + (n-i)(\lfloor x \rfloor + 1) \\ &= n\lfloor x \rfloor + n - i. \end{aligned} \tag{1}$$

Dalje, i zadovoljava

$$\{x\} + \frac{i-1}{n} < 1 \quad \text{i} \quad \{x\} + \frac{i}{n} \ge 1,$$

sve zajedno

$$\frac{n-i}{n} \le \{x\} < \frac{n-i+1}{n}.$$

Donja i gornja granica za nx sada glasi

$$n\lfloor x \rfloor + n - i \le n\lfloor x \rfloor + n\{x\} = nx < n\lfloor x \rfloor + n - i + 1,$$

i iz (1) imamo tvrdnju. $\square$

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Japanski matematičar Matsuoka, u [5], je dao elegantan dokaz Hermiteovog identiteta:

Drugi dokaz. Neka je $f(x)$ razlika između lijeve i desne strane našeg identiteta. Tada vrijedi:

$$\begin{aligned} f\left(x+\frac{1}{n}\right) &= \left\lfloor x+\frac{1}{n}\right\rfloor + \ldots + \left\lfloor x+\frac{1}{n}+\frac{n-1}{n}\right\rfloor - \left\lfloor n\cdot\left(x+\frac{1}{n}\right)\right\rfloor \\ &= \left\lfloor x+\frac{1}{n}\right\rfloor + \ldots + \left\lfloor x+\frac{n-1}{n}\right\rfloor + \lfloor x+1\rfloor - \lfloor nx+1\rfloor = f(x),\ \forall x\in\mathbf{R}. \end{aligned}$$

Dakle, funkcija f je periodička s periodom $\frac{1}{n}$. Prema tome, dovoljno ju je promatrati na intervalu $\left[0,\frac{1}{n}\right)$. No, za sve te vrijednosti ona je jednaka nuli, pa je $f(x)=0$, $\forall x\in\mathbf{R}$. □

Pokažimo sada primjenu Hermiteovog identiteta na nekim zadacima. Sljedeći zadatak je zadao poznati indijski matematičar Ramanujan (vidi [6]), a pojavio se i kao problem u MFL-u broju $205/1$ iz 2001. godine.

Zadatak 1. *Dokažite da za svaki $n\in\mathbf{N}$ vrijedi*

$$\left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{n+2}{6}\right\rfloor + \left\lfloor\frac{n+4}{6}\right\rfloor = \left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n+3}{6}\right\rfloor.$$

Prvo rješenje.

$$\begin{aligned} \left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{n+2}{6}\right\rfloor + \left\lfloor\frac{n+4}{6}\right\rfloor &= \left\lfloor\frac{n}{3}\right\rfloor - \left\lfloor\frac{n}{6}\right\rfloor + \left(\left\lfloor\frac{n}{6}\right\rfloor + \left\lfloor\frac{n}{6}+\frac{1}{3}\right\rfloor + \left\lfloor\frac{n}{6}+\frac{2}{3}\right\rfloor\right) \\ (\text{Hermite}) &= \left\lfloor\frac{n}{3}\right\rfloor - \left\lfloor\frac{n}{6}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor = \left\lfloor 2\cdot\frac{n}{6}\right\rfloor - \left\lfloor\frac{n}{6}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor \\ (\text{Hermite}) &= \left\lfloor\frac{n}{6}+\frac{1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor = \left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n+3}{6}\right\rfloor. \end{aligned}$$ □

Drugo rješnje. Za drugo rješenje zadatka koristimo gornju Matsuokinu ideju: funkcija $f:\mathbf{N}\to\mathbf{Z}$

$$f(n) = \left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{n+2}{6}\right\rfloor + \left\lfloor\frac{n+4}{6}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor - \left\lfloor\frac{n+3}{6}\right\rfloor$$

je perioda 6 jer $f(n)=f(n+6)$, pa je dovoljno provjeriti da je $f(n)=0$, za $n=0,1,2,3,4,5$. □

Ponekad trik s periodičosti nije očit, što pokazuje sljedeći primjer. U tom slučaju je zgodnije koristiti Hermiteov identet.

Zadatak 2. *Dokažite da za svaki $n\in\mathbf{Z}$ vrijedi*

$$\left|\left\lfloor -\frac{n}{2}\right\rfloor\right| = \left|\left\lfloor\frac{n}{2}+\frac{1}{2}\right\rfloor\right|.$$

Rješenje.

$$\left|\left\lfloor -\frac{n}{2}\right\rfloor\right| = (\text{Hermite}) = \left| -n - \left\lfloor -\frac{n}{2}+\frac{1}{2}\right\rfloor\right| = \left| n + \left\lfloor -\frac{n}{2}+\frac{1}{2}\right\rfloor\right| = \left|\left\lfloor\frac{n}{2}+\frac{1}{2}\right\rfloor\right|.$$ □

Zadatak 3. *Dokažite da za sve realne brojeve x vrijedi*

$$\left\lfloor x+\frac{1}{3}\right\rfloor+\left\lfloor x+\frac{2}{3}\right\rfloor+\left\lfloor 3x+\frac{1}{3}\right\rfloor+\left\lfloor 3x+\frac{2}{3}\right\rfloor+\cdots+\left\lfloor 3^n x+\frac{1}{3}\right\rfloor+\left\lfloor 3^n x+\frac{2}{3}\right\rfloor=\lfloor 3^{n+1}x\rfloor-\lfloor x\rfloor.$$

Rješenje. Uzastopnom primjenom Hermiteova identiteta za $n = 3$ imamo:

$$\left.\begin{aligned}\left\lfloor x+\frac{1}{3}\right\rfloor+\left\lfloor x+\frac{2}{3}\right\rfloor&=\lfloor 3x\rfloor-\lfloor x\rfloor\\ \left\lfloor 3x+\frac{1}{3}\right\rfloor+\left\lfloor 3x+\frac{2}{3}\right\rfloor&=\lfloor 3^2x\rfloor-\lfloor 3x\rfloor\\ &\vdots\\ \left\lfloor 3^n x+\frac{1}{3}\right\rfloor+\left\lfloor 3^n x+\frac{2}{3}\right\rfloor&=\lfloor 3^{n+1}x\rfloor-\lfloor 3^n x\rfloor\end{aligned}\right\}$$

Zbrajanjem ovih jednakosti slijedi tvrdnja zadatka. □

Zadatak 4. *Izračunajte sumu*

$$\sum_{0\le i<j\le n}\left\lfloor\frac{x+i}{j}\right\rfloor.$$

Rješenje.

$$\sum_{0\le i<j\le n}\left\lfloor\frac{x+i}{j}\right\rfloor=\sum_{j=1}^{n}\left(\sum_{i=1}^{j-1}\left\lfloor\frac{x+i}{j}\right\rfloor\right)=(\text{Hermite})=\sum_{j=1}^{n}\left\lfloor j\cdot\frac{x}{j}\right\rfloor=n\lfloor x\rfloor$$ □

Zadatak 5. (10. MMO) *Izračunajte:*

$$\left\lfloor\frac{n+1}{2}\right\rfloor+\left\lfloor\frac{n+2}{4}\right\rfloor+\left\lfloor\frac{n+4}{8}\right\rfloor+\ldots+\left\lfloor\frac{n+2^k}{2^{k+1}}\right\rfloor+\ldots$$

gdje je n nenegativan cijeli broj.

Rješenje.

$$\begin{aligned}S_k&=\left\lfloor\frac{n}{2}+\frac{1}{2}\right\rfloor+\left\lfloor\frac{n}{4}+\frac{1}{2}\right\rfloor+\left\lfloor\frac{n}{8}+\frac{1}{2}\right\rfloor+\ldots+\left\lfloor\frac{n}{2^k}+\frac{1}{2}\right\rfloor=(\text{Hermite},\ n=2)\\ &=\left(\lfloor n\rfloor-\left\lfloor\frac{n}{2}\right\rfloor\right)+\left(\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{n}{4}\right\rfloor\right)+\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n}{8}\right\rfloor\right)+\cdots+\left(\left\lfloor\frac{n}{2^{k-1}}\right\rfloor-\left\lfloor\frac{n}{2^k}\right\rfloor\right)\\ &=n-\left\lfloor\frac{n}{2^k}\right\rfloor.\end{aligned}$$

Sada je

$$S=\lim_{k\to\infty}S_k=\lim_{k\to\infty}\left(n-\left\lfloor\frac{n}{2^k}\right\rfloor\right)=n.$$ □

Zadatak 6. (Srbija i Crna Gora 2004.) *Niz* $\{a_n\}$ *zadan je uvjetima* $a_1 = x \in \mathbf{R}$ *i* $3a_{n+1} = a_n + 1$, *za* $n \geq 1$. *Neka je:*

$$A=\sum_{n=1}^{\infty}\left\lfloor a_n-\frac{1}{6}\right\rfloor,\quad B=\sum_{n=1}^{\infty}\left\lfloor a_n+\frac{1}{6}\right\rfloor.$$

Izračunajte zbroj $A + B$ *u ovisnosti o* x.

Rješenje. Označimo k-tu parcijalnu sumu

$$S_k = \sum_{n=1}^{k} \left(\left\lfloor a_n - \frac{1}{6} \right\rfloor + \left\lfloor a_n + \frac{1}{6} \right\rfloor \right).$$

Iz rekurzije $3a_{n+1} = a_n + 1$ imamo

$$\begin{aligned} 3a_2 &= a_1 + 1 \\ 9a_3 &= 3a_2 + 3 \\ &\vdots \\ 3^{n-2}a_{n-1} &= 3^{n-3}a_{n-2} + 3^{n-3} \\ 3^{n-1}a_n &= 3^{n-2}a_{n-1} + 3^{n-2}. \end{aligned}$$

Teleskopiranjem gornjih jednakosti:

$$a_n = \frac{1}{3^{n-1}} \left(x - \frac{1}{2} \right) + \frac{1}{2}.$$

Računamo:

$$\left\lfloor a_n - \frac{1}{6} \right\rfloor = \left\lfloor \frac{1}{3^{n-1}} \left(x - \frac{1}{2} \right) + \frac{1}{3} \right\rfloor, \quad \left\lfloor a_n + \frac{1}{6} \right\rfloor = \left\lfloor \frac{1}{3^{n-1}} \left(x - \frac{1}{2} \right) + \frac{2}{3} \right\rfloor.$$

Sada je, korištenjem Hermiteovog identiteta,

$$\begin{aligned} S_k &= \sum_{n=1}^{k} \left(\left\lfloor \frac{1}{3^{n-2}} \left(x - \frac{1}{2} \right) \right\rfloor - \left\lfloor \frac{1}{3^{n-1}} \left(x - \frac{1}{2} \right) \right\rfloor \right) \\ &= \left\lfloor 3 \left(x - \frac{1}{2} \right) \right\rfloor - \left\lfloor \frac{1}{3^{k-1}} \left(x - \frac{1}{2} \right) \right\rfloor. \end{aligned}$$

Ako je $x \geq \frac{1}{2}$, za dovoljno velike k je $\left\lfloor \frac{1}{3^{k-1}} \left(x - \frac{1}{2} \right) \right\rfloor = 0$, a ako je $x < \frac{1}{2}$, za dovoljno velike k je $\left\lfloor \frac{1}{3^{k-1}} \left(x - \frac{1}{2} \right) \right\rfloor = -1$. Sada je

$$A + B = \left\lfloor 3 \left(x - \frac{1}{2} \right) \right\rfloor - \lim_{k \to \infty} \left\lfloor \frac{1}{3^{k-1}} \left(x - \frac{1}{2} \right) \right\rfloor = \begin{cases} \left\lfloor 3 \left(x - \frac{1}{2} \right) \right\rfloor, & x \geq \frac{1}{2} \\ \left\lfloor 3 \left(x - \frac{1}{2} \right) \right\rfloor + 1, & x < \frac{1}{2}. \end{cases}$$

□

Zadatak 7. *Dokaži da za svako* $n \in \mathbf{N}$ *vrijedi*

$$\left\lfloor \frac{3n+3}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} \right\rfloor + \left\lfloor \frac{n+5}{13} \right\rfloor + \left\lfloor \frac{n+9}{13} \right\rfloor. \tag{2}$$

Rješenje. Prema Hermiteovom identetu vrijedi

$$\left\lfloor \frac{3n+3}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} \right\rfloor + \left\lfloor \frac{n+1}{13} + \frac{1}{3} \right\rfloor + \left\lfloor \frac{n+1}{13} + \frac{2}{3} \right\rfloor .$$

Pokazat ćemo da vrijedi

$$\left\lfloor \frac{n+5}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} + \frac{1}{3} \right\rfloor , \qquad \left\lfloor \frac{n+9}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} + \frac{2}{3} \right\rfloor .$$

Prvo $\dfrac{n+5}{13} = \dfrac{n+1}{13} + \dfrac{1}{3} - \dfrac{1}{39}$, pa je

$$\left\lfloor \frac{n+5}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} + \frac{1}{3} \right\rfloor \iff \frac{1}{39} \leq \left\{ \frac{n+1}{13} + \frac{1}{3} \right\} < 1. \tag{3}$$

Kako za svaki $n \in \mathbf{N}$ vrijedi $n = 13k + r$, $r = 0, 1, \ldots, 12$, sada je dovoljno provjeriti (3) za $n = 0, 1, \ldots, 12$.

Slično, $\dfrac{n+5}{13} = \dfrac{n+1}{13} + \dfrac{2}{3} - \dfrac{2}{39} \implies$

$$\left\lfloor \frac{n+9}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} + \frac{2}{3} \right\rfloor \iff \frac{2}{39} \leq \left\{ \frac{n+1}{13} + \frac{2}{3} \right\} < 1. \tag{4}$$

Kako (4) vrijedi za $n = 0, 1, \ldots, 12$, zaključujemo da (2) vrijedi za sve $n \in \mathbf{N}$. $\square$

Napomena. Zadatak se može riješiti i na drugi način, koristeći gore spomenutu tehniku:

Definirajmo funkciju

$$f(n) = \left\lfloor \frac{3n+3}{13} \right\rfloor - \left\lfloor \frac{n+1}{13} \right\rfloor - \left\lfloor \frac{n+5}{13} \right\rfloor - \left\lfloor \frac{n+9}{13} \right\rfloor .$$

Kako je $f(n) = f(n+13)$, dovoljno je samo provjeriti $f(n) = 0$, $n = 0, 1, \ldots, 12$. $\square$

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Jedno zanimljivo pitanje: može li se u (2) izraz $\dfrac{3n+3}{13}$ zamijeniti s $\dfrac{3n+2}{13}$, tj. vrijedi li

$$\left\lfloor \frac{3n+2}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} \right\rfloor + \left\lfloor \frac{n+5}{13} \right\rfloor + \left\lfloor \frac{n+9}{13} \right\rfloor ?$$

Da bi odgovorili na ovo pitanje koristimo Hermiteov identitet:

$$\left\lfloor \frac{3n+2}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} \right\rfloor + \left\lfloor \frac{n+\frac{2}{3}}{13} + \frac{1}{3} \right\rfloor + \left\lfloor \frac{n+\frac{2}{3}}{13} + \frac{2}{3} \right\rfloor$$

i slijedeći gornju metodu, treba provjeriti,

$$\left\lfloor \frac{n+1}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} \right\rfloor , \quad \left\lfloor \frac{n+5}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} + \frac{1}{3} \right\rfloor , \quad \left\lfloor \frac{n+9}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} + \frac{2}{3} \right\rfloor .$$

Međutim, zbog $\dfrac{n+1}{13} = \dfrac{n+\frac{2}{3}}{13} + \dfrac{1}{39}$ zaključujemo

$$\left\lfloor \frac{n+1}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} \right\rfloor \iff \left\{ \frac{n+\frac{2}{3}}{13} \right\} + \frac{1}{39} < 1$$

a to ne vrijedi za $n = 12, 24, 36, \ldots$

O generiranju zadatka 6 uočimo da vrijedi $\dfrac{4}{13} \approx \dfrac{1}{3}$, $\dfrac{8}{13} \approx \dfrac{2}{3}$, a to su razlomci koji se javljaju u Hermiteovom identitetu za $n = 3$.

U zaključku rada ističemo da korištenjem Hermiteovog identiteta imamo bolji uvid u problem, njegovu analizu i metodu generiranja sličnih zadataka.

Zadaci za vježbu

1. Dokažite da za svako $x \in \mathbf{R}$ i $n \in \mathbf{N}$ vrijedi
$$\{x\} + \left\{x + \frac{1}{n}\right\} + \left\{x + \frac{2}{n}\right\} + \ldots + \left\{x + \frac{n-1}{n}\right\} = \{nx\} + \frac{n-1}{2}.$$

2. Dokažite da za svaki realan broj x vrijedi
$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor + \left\lfloor 2x + \frac{1}{2} \right\rfloor + \ldots + \left\lfloor 2^{2015}x + \frac{1}{2} \right\rfloor = \lfloor 2^{2016}x \rfloor.$$

3. Neka je r realan broj za koji vrijedi
$$\left\lfloor r + \frac{19}{100} \right\rfloor + \left\lfloor r + \frac{20}{100} \right\rfloor + \ldots + \left\lfloor r + \frac{91}{100} \right\rfloor = 546.$$
Nađite $\lfloor 100r \rfloor$. (Rezultat: $\lfloor 100r \rfloor = 743$.)

4. Dokažite da za svaki $n \in \mathbf{N}$ vrijedi
$$\left\lfloor \frac{n-2^0}{2} \right\rfloor + \left\lfloor \frac{n-2^1}{2^2} \right\rfloor + \left\lfloor \frac{n-2^2}{2^3} \right\rfloor + \ldots + \left\lfloor \frac{n-2^{n-1}}{2^n} \right\rfloor = 0.$$

5. Dokažite da se u jednakosti
$$n = \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \ldots + \frac{n}{2^k} + \ldots$$
svi razlomci mogu zamijeniti s njima najbližim cijelim brojem:
$$n = \left\langle \frac{n}{2} \right\rangle + \left\langle \frac{n}{4} \right\rangle + \left\langle \frac{n}{8} \right\rangle + \ldots + \left\langle \frac{n}{2^k} \right\rangle + \ldots$$
gdje je $\langle r \rangle = \left\lfloor r + \dfrac{1}{2} \right\rfloor$.

6. Izračunajte sumu
$$\sum_{k=1}^{\infty} \left\lfloor \frac{x+3^k}{3^{k+1}} + \left\lfloor \frac{x+2\cdot 3^k}{3^{k+1}} \right\rfloor \right\rfloor.$$
(Rezultat: $\left\lfloor \dfrac{x}{3} \right\rfloor$.)

7. Dokažite da za svaki realan broj x i prirodan broj m vrijedi:
$$\sum_{n=1}^{m}\left(\left\{2^n x+\frac{1}{2}\right\}-\frac{1}{2}\right)\leq 1.$$
8. Dokažite da za svako $n\in\mathbf{N}$ vrijedi
$$\left\lfloor\frac{3n+8}{25}\right\rfloor=\left\lfloor\frac{n+2}{25}\right\rfloor+\left\lfloor\frac{n+11}{25}\right\rfloor+\left\lfloor\frac{n+19}{25}\right\rfloor.$$
9. Ako je:
$$A=\sum_{k=0}^{2000}\left\lfloor\frac{3^k+2000}{3^{k+1}}\right\rfloor \quad \text{i} \quad B=\sum_{k=0}^{2000}\left\lfloor\frac{3^k-2000}{3^{k+1}}\right\rfloor,$$
izračunajte $A-B$. (Rezultat: $A-B=2000$.)

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