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Bol quasifields

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Abstract. In the context of configurational characterisations of symmetric projective planes, a new proof of a theorem of Kallaher and Ostrom characterising planes of even order of Lenz-Barlotti type IV.a.2 via Bol conditions is given. In contrast to their proof, we need neither the Feit-Thompson theorem on solvability of groups of odd order, nor Bender's strongly embedded subgroup theorem, depending rather on Glauberman's Z^*-theorem.

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1. Introduction

Some of the most startlingly beautiful results in the foundations of projective geometry concern configurational characterisations of algebraic properties of coordinates, a field is characterised by the theorem of Pappus, a division ring is characterised by the theorem of Desargues and a non-associative alternative division ring is characterised by the little theorem of Desargues. In the presence of finiteness, all of these distinctions are collapsed and the little theorem of Desargues implies the theorem of Pappus: this is the content of the Artin-Zorn-Levi theorem. Similar, but stronger, results hold in the affine setting, where the configurational theorem is only assumed for a special position of the line at infinity: the coordinates form a field if and only if an affine version of the theorem of Pappus holds and a division ring if and only if an affine version of the theorem of Desargues holds. But the affine version of the little theorem of Desargues no longer characterises non-associative alternative division rings, but only leads to quasifields; in other words, the plane need not be a Moufang plane, but only a translation plane. The Bol condition is a remedy for this defect in the finite setting in that via its use a characterisation is obtained for a more interesting class of planes than finite translation planes. As a consequence, these are known as *Bol planes* and the algebraic structure coordinatising such planes are known as *Bol quasifields*. The Bol condition has two interesting affine specialisations - one that was pursued by Klingenberg and one that was pursued by Burn, and it is

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the latter that characterises nearfields (and hence projective planes of Lenz-Barlotti type IV.a.2) under the hypothesis of finiteness.

Perhaps surprisingly, the characterisation depends on the classification of finite simple groups and its proof appears scattered over a large number of papers.

In this paper, we give a proof that Bol quasifields of even order are nearfields by using less group theory than the original proof. We make use of theorems of Kallaher, Hering and Glauberman, with the latter two depending upon little more than some Brauer's older work on modular representation theory.

2. Extensions of the Artin-Zorn-Levi theorem

In 1937, G. Bol [5] introduced a postulate to web theory in topology, which was taken up in projective geometry by W. Klingenberg in 1952 [53].

A projective plane π with a line ℓ and distinct points P, R not on ℓ *satisfies Bol's* (P, R, ℓ)*-postulate* if, for any point C on ℓ and points $A_i, B_i (i = 1, 2, 3, 4)$ not on PR such that A_1, A_2, B_1, B_2 are on ℓ, $A_1 \neq A_2, B_1 \neq B_2$ and A_3, A_4, B_3, B_4 are not on ℓ, whenever the following triples of points are collinear C, A_3, B_3; C, A_4, B_4; R, A_1, A_4; R, A_2, A_3; R, B_1, B_4; R, B_2, B_3; P, A_1, A_3; P, A_2, A_4; P, B_1, B_3; then so is the triple P, B_2, B_4.

A projective plane π satisfying Bol's (P, R, ℓ)-postulate for all lines ℓ not on a fixed pair P, R of distinct points of π is called a *Bol plane with respect to* P *and* R, or simply, a *Bol plane.* The plane π *satisfies Bol's postulate universally* if it satisfies Bol's (P, R, ℓ)-postulate for all lines ℓ of π and all pairs P, R of points of π not on ℓ.

A *left quasifield* is a group under addition whose non-zero elements form a loop under multiplication, which satisfies the left distributive law $a(b + c) = ab + ac$ and for which the equation $ax = bx+c$ has exactly one solution x for all a, b, c with $a \neq b$. Given a left quasifield Q, the incidence structure with points the elements of $Q \times Q$ and lines the sets $\{(x, mx + b) : x \in Q\}$, for all $m, b \in Q$ and $\{(c, y) : y \in Q\}$, for all $c \in Q$, with incidence set membership is an affine plane, which we denote by $\mathrm{Aff}(Q)$, whose projective completion is a projective plane, which we denote by $\Pi(Q)$. $\Pi(Q)$ is a translation plane with a translation line the line at infinity of $\mathrm{Aff}(Q)$.

The following theorem is due to Bruck-Bose [11].

Theorem 1 (see [16, 3.1.22])**.** *A projective plane is a translation plane if and only if it is isomorphic to* $\Pi(Q)$*, where* Q *is a left quasifield.*

Desargues' theorem says that if two triangles in a projective plane are perspective from a point then they are perspective from a line. Little Desargues' theorem asserts the truth of Desargues' theorem only for the special case, where the point and line of perspectivity of the triangles are incident with one another. A projective plane π is *Desarguesian* if Desargues' theorem holds in π. In algebraic terms, π is Desarguesian if and only if it is isomorphic to $\Pi(Q)$, where Q is a skewfield [16, 3.1.22]. A projective plane π is *Moufang* if every line of π is a translation line. A Moufang plane can also be described as a projective plane in which little Desargues' theorem holds. It is a theorem of Moufang that a projective plane is Moufang if and only if it is isomorphic to $\Pi(Q)$, where Q is an alternative division ring [20, Theorem 5.8] or [16, 3.1.22]. The following result characterises Moufang planes in terms of Bol's postulate.

Theorem 2 (see [53]). *A projective plane π satisfies Bol's postulate universally if and only if π is a Moufang plane.*

Klingenberg also proved a stronger, affine version of this theorem.

Theorem 3 (see [53]). *A projective plane π satisfies Bol's (P, R, ℓ)-postulate for all pairs P, R of distinct points of π not on the fixed line ℓ if and only if π is a translation plane with translation line ℓ.*

This made it clear that special cases of Bol's postulate were made for interesting study. Rather than fixing ℓ and varying P and R, R.P. Burn fixed P and R and varied ℓ.

Theorem 4 (see [12], [55, Theorem I.4.9]). *A projective plane π that satisfies Bol's (P, R, ℓ)-postulate for all lines ℓ not on a fixed pair P, R of distinct points of π is a translation plane with translation line PR.*

A *left nearfield* is an abelian group under addition whose non-zero elements form a group under multiplication and which satisfies the left distributive law $a(b+c) = ab + ac$. A left nearfield is *planar* if the map $x \mapsto -ax + b$ is a bijection whenever $a \neq b$. All finite nearfields are planar. A planar left nearfield is a left quasifield, but not conversely. The finite nearfields were classified by Zassenhaus in [68], and, thereby, the finite sharply 2-transitive groups were classified.

If Q is a planar left nearfield, P is the point on the translation line of $\Pi(Q)$ corresponding to the set of vertical lines of $\mathrm{Aff}(Q)$ and R is the point on the translation line of $\Pi(Q)$ corresponding to the set of horizontal lines of $\mathrm{Aff}(Q)$, then $\Pi(Q)$ is a Bol plane with respect to P and R. Burn [12] conjectured the converse if the projective plane is finite:

Conjecture 1. *A finite Bol plane π is isomorphic to $\Pi(Q)$ for some planar left nearfield Q.*

Burn [12] also showed that certain infinite André planes provide counterexamples to the conjecture if the finiteness hypothesis is dropped. Later, infinite counterexamples that are not André planes were given in [15]. Karzel raised an equivalent question in the same year [48, Problem 1, p. 201].

In 1937, Bol [5] was the first to separately study the *(left) Bol law*, also known as the *left Moufang condition*:

$$a(b(ac)) = (a(ba))c.$$

A (left) quasifield Q is *Bol* if it satisfies the (left) Bol law for all $a, b, c \in Q$.

Theorem 5 (see [12]). *Let π be a finite Bol plane with respect to P and R. Then π is isomorphic to $\Pi(Q)$ for some Bol quasifield Q, and the isomorphism takes P, Q to the point on the translation line of $\Pi(Q)$ corresponding to the set of vertical and horizontal lines of $\mathrm{Aff}(Q)$.*

Thus Burn's conjecture may be equivalently stated in algebraic form as:

Conjecture 2. *A finite Bol quasifield is a nearfield.*

The above conjecture was worked on for twenty-two years before it was finally proved. The proof depends on the classification of finite simple groups. Apart from further work by Burn [13], the driving force in the proof was M. Kallaher: the proof appears scattered over the papers [42, 43, 44, 45, 46], with the coup de grâce in [26] and [38]. Kallaher had also been a pioneer in the area, as shown in [39, 40, 41].

Theorem 6 (Burn-Hanson-Johnson-Kallaher-Ostrom)**.** *A finite Bol plane π is isomorphic to $\Pi(Q)$ for some planar nearfield Q.*

Corollary 1. *A finite Bol quasifield is a nearfield.*

Remark 1. *Since a finite Bol quasifield is a nearfield if and only if it has associative multiplication, it is clear that Burn's conjecture is a stronger, affine version of the Artin-Zorn theorem. With this context, it is also clear that non-associative alternative division rings are counterexamples to the conjecture in the infinite case, and so non-Desarguesian Moufang planes are counterexamples to the other form of the conjecture in the infinite case. These and Desarguesian planes are sometimes called* improper *Bol planes.*

The labelling of theorem and corollary is appropriate, for the theorem was proved and the corollary deduced. To quote Kallaher in [44]: "a theorem about algebraic structures is proven using geometry, a reverse of the usual procedure". This quote seems appropriate for this paper, which takes a similar perspective.

Note that, although the Burn-Hanson-Johnson-Kallaher-Ostrom theorem is stronger than the Artin-Zorn-Levi theorem, it does not provide a new proof of this theorem, as the classification of finite simple groups depends upon the Artin-Zorn theorem, via results of Ostrom and Wagner, who in turn use results of Gleason depending upon Artin-Zorn; see [57, 58, 59, 34, 24, 64, 65], and [66] for historical details. This also invalidates a proof of Artin-Zorn using the characterisation of groups normalising $\mathrm{PSL}(n, q)$ as 2-transitive permutation groups by O'Nan [56], or the results of Kantor-McDonough [47]. These remarks show how fundamental the Artin-Zorn theorem is. The point of application of the classification of finite simple groups is the use of the classification of linear groups acting transitively on non-zero vectors by Hering [28, 31, 32], part of the classification of all finite 2-transitive groups; indeed, it is the cornerstone of the classification of those of affine type. By a 19th century theorem of Burnside, a 2-transitive group is of affine type or almost simple [14, Section 134].

In 1938, Witt [67] introduced the following method of studying finite projective planes. A *sharply 2-transitive set* of permutations of a set X is a set S of permutations X such that whenever $x \neq y \in X$ and $x' \neq y' \in X$, there is a unique $g \in S$ such that $x^g = x'$ and $y^g = y'$. $|X|$ is the *degree* of S.

Given a sharply 2-transitive set S of permutations of the finite set X, define the incidence structure $A(S)$ to have points elements of $X \times X$ and lines $\{(a, y) : y \in X\}$, for $a \in X$, $\{(x, a) : x \in X\}$, for $a \in X$ and $\{(x, x^g) : x \in X\}$ for $g \in S$, with incidence set membership. Then $A(S)$ is an affine plane of order $|X|$. Conversely, any affine plane of order n is isomorphic to $A(S)$ for some sharply 2-transitive set S of permutations of degree n which contains the identity.

Theorem 7 (see [25], [16, 5.1.2])**.** *The affine plane $A(S)$, where S is a sharply 2-transitive set of permutations containing the identity, is isomorphic to* $\mathrm{Aff}(Q)$ *via an isomorphism taking the vertical lines of $A(S)$ to the vertical lines of* $\mathrm{Aff}(Q)$ *and taking the horizontal lines of $A(S)$ to the horizontal lines of* $\mathrm{Aff}(Q)$*, where Q is a nearfield if and only if S is a group.*

Theorem 8 (see [13])**.** *The projective completion of the affine plane $A(S)$, where S is a sharply 2-transitive set of permutations containing the identity, is isomorphic to* $\Pi(Q)$ *for some Bol quasifield Q, if and only if $g, h \in S$ implies $ghg \in S$.*

Thus we have another equivalent statement of the Burn-Hanson-Kallaher-Ostrom theorem:

Theorem 9. *Let S be a sharply 2-transitive set of permutations of a finite set X such that $ghg \in S$ for all $g, h \in S$. Then S is a group.*

Using [54, Theorem VIII.2.4], we can also reformulate this as a purely synthetic statement about projective planes. To do this, we will need another postulate with origins from web theory in topology, and which was again taken up in projective geometry by W. Klingenberg [53], this time due to Reidemeister [61].

A projective plane π with three distinct points P, R, T *satisfies Reidemeister's* (P, R, T)*-postulate* if, for distinct points $A_1, \dots, A_6$ of π not on any of the line PR, PT or RT, whenever the following triples of points are collinear A_1, A_2, T; A_3, A_4, T; A_5, A_6, T; A_1, A_3, P; A_2, A_5, P; A_4, A_6, P; A_1, A_6, R; A_3, A_5, R; then so is the triple A_2, A_4, R.

The plane π *satisfies Reidemeister's postulate universally* if it satisfies Reidemeister's (P, R, T)-postulate for all triples P, R, T of distinct points of π.

Theorem 10 (see [53])**.** *A projective plane π is Desarguesian if and only if π satisfies Reidemeister's postulate universally if and only if there is a line ℓ of π such that π satisfies Reidemeister's (P, R, T)-postulate for all triples of points P, R, T such that $P \neq R$, and that P, R are on ℓ and T is not on ℓ.*

Thus the following result can be seen as the stronger, affine version of the immediately preceding theorem of Klingenberg in the finite case.

Theorem 11 (Burn-Hanson-Kallaher-Lüneburg-Ostrom)**.** *Let π be a finite projective plane and P, R distinct points of π. Then the following are equivalent:*

(*i*) *Bol's (P, R, ℓ)-postulate holds for all lines ℓ on neither P nor R;*

(*i*) *π is (P, ℓ)-Desarguesian for all lines ℓ on R but not on P;*

(*ii*) *Reidemeister's (P, R, T)-postulate holds for all points T not on PR.*

A projective plane π satisfies the *special perspective quadrangle condition* for some pair of distinct points P, R and line ℓ, where P, R are not on ℓ, if any two quadrangles which have diagonal line ℓ and diagonal points P, R in common are in centrally perspective position from a centre O on ℓ.

A projective plane π with two distinct points P, R *satisfies the* (P, R)*-quadrangle postulate* if, for distinct points $A_1, \dots, A_4, B_1, \dots, B_4, C_1, \dots, C_4$ of π not on the

line PR, such that $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ are quadrangles (i.e., each has no three points collinear), whenever the following triples of points are collinear: A_1, A_2, P; A_3, A_4, P; B_1, B_2, P; B_3, B_4, P; C_1, C_2, P; C_3, C_4, P; A_1, A_3, R; A_2, A_4, R; B_1, B_3, R; B_2, B_4, R; C_1, C_3, R; C_2, C_4, R; A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3; then so is the triple A_4, B_4, C_4.
A projective plane π with two distinct points P, R *satisfies the restricted* (P, R)*-quadrangle postulate* if π satisfies the (P, R)-quadrangle postulate for points such that $A_1, B_1, C_1, A_3, C_3, C_3$ are collinear. Finally, we can add another three equivalent conditions to the preceding theorem.

Theorem 12. *Let π be a finite projective plane and P, R distinct points of π. Then the following are equivalent:*

(*i*) *Bol's (P, R, ℓ)-postulate holds for all lines ℓ on neither P nor R;*

(*ii*) *π is (P, ℓ)-Desarguesian for all lines ℓ on R but not on P;*

(*iii*) *Reidemeister's (P, R, T)-postulate holds for all points T not on PR;*

(*iv*) *the special perspective quadrangle condition holds for P, R and all lines ℓ not on either P or R;*

(*v*) *π satisfies the (P, R)-quadrangle postulate;*

(*vi*) *π satisfies the restricted (P, R)-quadrangle postulate.*

Proof. (iv) is proved equivalent to (i) in [22, Theorem 6.3](without requiring finiteness).

In the light of the preceding theorems, to prove that (v) is equivalent to (i), it is sufficient to show that the (P, R)-quadrangle postulate holds for the projective completion of $A(S)$, where S is a sharply 2-transitive set of permutations containing the identity and P and R correspond to the parallel classes of vertical and horizontal lines of $A(S)$ if and only if S is a group. Let $f, g \in S$, $A_1 = (x, x)$, $B_1 = (y, y)$, $C_1 = (z, z)$, $A_2B_2C_2$ be the line $\{(t, t^f) : t \in X\}$ and $A_3B_3C_3$ the line $\{(t, t^g) : t \in X\}$. Then $A_2 = (x^{f^{-1}}, x)$, $A_3 = (x^{f^{-1}}, x^{f^{-1}g})$, $A_4 = (x, x^{f^{-1}g})$, $B_2 = (y^{f^{-1}}, y)$, $B_3 = (y^{f^{-1}}, y^{f^{-1}g})$, $B_4 = (y, y^{f^{-1}g})$, $C_2 = (z^{f^{-1}}, z)$, $C_3 = (z^{f^{-1}}, z^{f^{-1}g})$ and $C_4 = (z, z^{f^{-1}g})$. If $h \in S$ with $\{(t, t^h) : t \in X\}$ the line A_4B_4, then since A_4, B_4, C_4 are collinear for all $z \in X$, with $x \neq z \neq y \neq x$, it follows that $t^h = t^{f^{-1}g}$, for all $t \in X$. Hence $h = f^{-1}g$, giving $f^{-1}g \in S$. So S is a group. The argument is reversible to prove the converse.

In the light of the preceding theorems, to prove that (vi) is equivalent to (i), it is sufficient to show that the restricted (P, R)-quadrangle postulate holds for the projective completion of $A(S)$, where S is a sharply 2-transitive set of permutations containing the identity and P and R correspond to the parallel classes of vertical and horizontal lines of $A(S)$ if and only if $fgf \in S$, for all $f, g \in S$. Let $f, g \in S$, $A_1 = (x, x^f)$, $B_1 = (y, y^f)$, $C_1 = (z, z^f)$, $A_2B_2C_2$ be the line $\{(t, t^g) : t \in X\}$. Then $A_2 = (x^{fg^{-1}}, x^f)$, $A_3 = (x^{fg^{-1}}, x^{fg^{-1}f})$, $A_4 = (x, x^{fg^{-1}f})$, $B_2 = (y^{fg^{-1}}, y)$, $B_3 = (y^{fg^{-1}}, y^{fg^{-1}f})$, $B_4 = (y, y^{fg^{-1}f})$, $C_2 = (z^{fg^{-1}}, z)$, $C_3 = (z^{fg^{-1}}, z^{fg^{-1}f})$ and

$C_4 = (z, z^{f^{-1}g})$. If $h \in S$ with $\{(t, t^h) : t \in X\}$ the line A_3B_3, then since A_3, B_3, C_3 are collinear for all $z \in X$, with $x \neq z \neq y \neq x$, it follows that $t^h = t^{fg^{-1}f}$, for all $t \in X$. Hence $h = fg^{-1}f$, giving $fg^{-1}f \in S$. Putting $f = 1$ shows that S is closed under inverses. Hence $fgf \in S$. Again, the argument is reversible to prove the converse. We recall that Burn proved that if $g, h \in S$ implies $ghg \in S$; then S is closed under inverses [13, proof of Theorem 11]. □

In the (B)-geometry arising from S the (P, R)-quadrangle condition corresponds to the second rectangle condition; see [4, 7] for more details. Thus, there arises a similarity with the N. Durante-A. Siciliano proof in [19] of the L.Bader-G.Lunardon-J.A.Thas classification of flocks of finite hyperbolic quadrics [2, 62, 63], which was preceded by the 1992 proof of A. Bonisoli-G. Korchmaros [8], which, like that of the Burn-Hanson-Kallaher-Ostrom theorem, relied on group-theoretic methods. Moreover, Bader-Lunardon's portion of the original proof used Bol quasifields occurring there as Bol planes and relied upon results in [43] and [44]; see also [2, 6, 7, 8, 19, 62, 63]. Along these lines, the work of [49, 50, 51] is relevant: in their terms, we are showing that a finite symmetric 2-structure is double symmetric.

3. A proof of the Burn-Hanson-Johnson-Kallaher-Ostrom theorem for planes of even order

In our opinion, it is desirable to have a proof of the Burn-Hanson-Johnson-Kallaher-Ostrom theorem that does not depend on the classification of finite simple groups.

To that end, we venture the following proof of the Burn-Hanson-Johnson-Kallaher-Ostrom theorem for planes of even order. The original proof used Bender's strongly embedded subgroup theorem [3] via the work of Hering [29, 30], the Feit-Thompson theorem [21] that every group of odd order is soluble. The interested reader can find an exposition of this proof in [55, Theorem 41.10, Sections 35, 41].

For our proof we need the following theorems.

Theorem 13 (see [41, Corollary 3.2.2])**.** *Let π be a Bol plane with respect to P and R. Then, if π is non-Desarguesian, the collineation group of π fixes $\{P, R\}$.*

The following is a cleaner statement of results of [45, 46].

Theorem 14 (see [44, Theorem 2.2])**.** *Let π be a Bol plane with respect to P and R of order p^n, p prime, let T be a point of π not on PR and let H be the stabiliser of P and R in the group generated by involutory central collineations of π fixing T. Then H acts transitively on the set of points of PR, other than P or R. Hence H induces a subgroup K of $\mathrm{GL}(n, p)$ transitive on non-zero vectors on $\mathrm{GF}(p)^n$. If K is soluble, then either π is a nearfield plane or $p^n = 5^2, 7^2, 11^2$ or 3^4.*

Theorem 15 (see [33, Theorem 1])**.** *Let p be a prime, let V be a vector space of finite dimension $n > 0$ over $\mathrm{GF}(p)$, and let G be a subgroup of the general linear group $\mathrm{GL}(V)$ such that G is transitive on $V \setminus \{0\}$ and p does not divide the order of G. Then G has an irreducible normal subgroup of prime order unless $p = 2$ and $n \in \{3, 4, 6, 8, 10, 12, 20\}$, or $p = 3$ and $n \in \{4, 6\}$ or $n = 2$.*

The **core** $O(G)$ of a finite group G is the largest normal subgroup of odd order. The subgroup of G containing $O(G)$ and such that $Z^*(G)/O(G) = Z(G/O(G))$ is denoted by $Z^*(G)$. We need Glauberman's Z^*-theorem.

Theorem 16 (see [23, Corollary 1])**.** *Let S be a Sylow 2-subgroup of a finite group G. Suppose $x \in S$. A necessary and sufficient condition for $x \notin Z^*(G)$ is that there exist $y \in C_S(x)$ such that y is conjugate to x in G and $y \neq x$.*

Theorem 17. *A Bol plane π of even order is isomorphic to $\Pi(Q)$ for some planar left nearfield Q.*

Proof. By Theorem 4 the order of π is a power of 2. We may assume that $\{P, R\}$ is fixed by the collineation group of π, as otherwise π is Desarguesian by Theorem 13. It is known that in a plane of even order, any involutory central collineation is an elation [16, 4.1.9]. Choose a point T not on PR and consider the group G generated by involutory elations of π that fix T. Each such elation g must interchange P and R. As any two of them must have different axes and centres, then no two of such elations commute.

Let S be a Sylow 2-subgroup of G and $t \in S$. Then t is an involutory elation in $C_S(t)$. If $u \in C_S(t)$ is conjugate to t, then u is again an involutory elation (since t is), hence $t = u$. By Theorem 16, all elations of π that fix R lie in $Z^*(G)$, so if t, u are involutory elations of π that fix R, then $(tu)^2 = [t, u] \in O(G)$. Now the elements of the dihedral group $\langle t, u\rangle$ not in $\langle tu\rangle$ are conjugate to t or u, so are elations. If tu had even order, say $2m$, then t and $t(tu)^m$ would commute. As no two such elations commute, tu has odd order.

We also have that $\langle tu\rangle$ is the unique cyclic subgroup of index 2 in $\langle t, u\rangle$ and $\langle (tu)^2\rangle = \langle (ut)^2\rangle$. This implies that $tu \in \langle (tu)^2\rangle$, so $tu \in O(G)$. Therefore, $O(G)$ has index two in G and odd order.

Let H be the stabiliser of P and R in G and let 2^n be the order of π. Then, by Theorem 14, H induces a subgroup K of $\mathrm{GL}(n, 2)$ transitive on non-zero vectors of $\mathrm{GF}(2)^n$; indeed $H = O(G)$, so K has odd order.

Now, by Theorem 15, either K normalises an irreducible normal subgroup of prime order, or $n \in \{2, 3, 4, 6, 8, 10, 12, 20\}$.

Let M be a normal irreducible normal subgroup of K with $|M|$ a prime. By Schur's Lemma [1, p.38], $\mathrm{End}_M(V) = C_{\mathrm{GL}(V)}(M) \bigcup\{0\}$ is a division ring and thus by Wedderburn's Theorem and [52, Lemma 2.10.2], it is a finite field $\mathrm{GF}(2^m)$ with m a divisor of n. In other words, the elements of M commute with every nonzero element of $\mathrm{GF}(2^m)$, giving M as a subgroup of $\mathrm{GL}(e, 2^m)$, with $e = n/m$. Note that $\mathrm{GF}(2^m)^* = Z(\mathrm{GL}(e, 2^m))$. Since the order of M is a prime, M is cyclic and hence abelian. This implied that M is contained in its centraliser, giving M as a subgroup of $\mathrm{GF}(2^m)$. By a result of Dye [17], as a subgroup of $\mathrm{GL}(V)$, $\mathrm{GL}(e, 2^m)$ acts on V by fixing a Desarguesian m-spread ‡. In particular, $\mathrm{GF}(2^m)^*$ fixes the spread elementwise [18]. Since M is irreducible, we have $m = n$, which implies that M is a subgroup of a Singer cyclic subgroup S of $\mathrm{GL}(V)$. As K normalises

‡An m-spread of $V(n, q)$ is a set of m-subspaces which partition the nonzero vectors of $V(r, n)$; an m-spread exists if and only if $m|n$. An m-spread $\mathcal{S}$ of $V(n, q)$ is *Desarguesian* if it takes as points the elements of $\mathcal{S}$ and as lines the $(2m-1)$-subspaces generated by two elements of $\mathcal{S}$, and the inclusion inherited from $V(n, q)$ gives a projective space $\mathrm{PG}(n/m - 1, q^m)$.

M, and $N_{\mathrm{GL}(V)}(S) = S \rtimes \mathrm{Gal}(\mathrm{GF}(2^n)/\mathrm{GF}(2)) = \Gamma\mathrm{L}(1, 2^n)$ by [35, p.187], we get $K \leq \Gamma\mathrm{L}(1, 2^n)$, hence K is soluble.

The linear group $\mathrm{GL}(2, 2)$ has order 6. Therefore K is a cyclic 3-subgroup, hence solvable.

If $n = 3$, then K contains a Sylow 7-subgroup which is actually a cyclic Singer subgroup S. By Sylow Theorem, K normalises S giving that K is contained in $\Gamma\mathrm{L}(1, 2^3)$ by [35, Satz 7.3]. Therefore, K is solvable.

Let $n \in \{4, 6, 8, 10, 12, 20\}$. Since K is transitive on non-zero vectors of $V(n, 2)$, K is irreducible. Again by Schur's Lemma, $\mathrm{End}_K(V(n, 2)) = C_{\mathrm{GL}(n,2)}(K) \bigcup \{0\}$. By Wedderburn's Theorem and [52, Lemma 2.10.2], it is a finite field $\mathrm{GF}(2^m)$ with m a divisor of n. In other words, the elements of K commute with $\mathrm{GF}(2^m)^*$, giving that K is a subgroup of $\mathrm{GL}(e, 2^m)$, with $e = n/m$. Hence, $V(n, 2)$ is absolutely irreducible as a $\mathrm{GF}(2^m)K$-module.

By Theorem 15.11 in [60], K has an irreducible representation of degree e over the complex numbers. By [36, Theorem 3.11] or [37, Theorem 22.11]), it follows that e divides $|K|$. Hence, e is odd as K has odd order.

Therefore, for $n = 4, 8$, necessarily K is a subgroup of $\mathrm{GF}(2^n)^*$, giving that K solvable.

If $n = 6$ and $e = 3$, then $3^2.7$ divides $|K|$, which in turn divides $3^4.5.7$, as K has odd order. Suppose K is insolvable. By Burnside $p^a q^b$-theorem, 5 divides the order of K. Therefore K induces a subgroup of $\mathrm{PSL}(3, 4)$ with the order divisible by 35. From [27], we see that such a subgroup does not exist.

If $n = 10$ and $e = 5$, then 31 divides $|K|$. As $31 > 2.5 + 1$, by the main theorem in [10], K normalises a Sylow 31-subgroup P. In addition, 31 is a 4-primitive prime divisor of $4^5 - 1$. From [35, Satz 7.3], we have $N_{\mathrm{GL}(5,4)}(P) \simeq \mathrm{GF}(4^5)^* \rtimes \mathrm{Gal}(\mathrm{GF}(4^5)/\mathrm{GF}(4)) \leq \Gamma\mathrm{L}(1, 2^{10})$. Therefore K is solvable.

If $n = 12$ (respectively $n = 20$), we may use the same arguments to get that K is solvable. We need to notice that 13 (resp. 31) divides $|K|$, $13 > 2.3 + 1$ (resp. $31 > 2.5 + 1$) and 13 is a 2^4-primitive prime divisor of $2^{12} - 1$, (resp. 31 is a 2^4-primitive prime divisor of $2^{20} - 1$).

Theorem 14 completes the proof. $\square$

Note that the work of Glauberman [23] and that of Hering [33] depend upon little more than some Brauer's older work on modular representation theory [9, 10].

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