Singular BGG complexes for the symplectic case[*]

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Abstract. Using the Penrose transform, we construct analogues of the BGG (Bernstein-Gelfand-Gelfand) resolutions in certain singular infinitesimal characters in the holomorphic geometric setting over the Lagrangian Grassmannian. We prove the exactness of the constructed complex over the big affine cell.

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1. Introduction and preliminaries

The BGG complexes were introduced in [2] by Bernstein, Gelfand and Gelfand. For a semisimple Lie algebra $\mathfrak{g}$ (complex, finite-dimensional), they constructed for each finite-dimensional irreducible $\mathfrak{g}$-module F a resolution consisting of direct sums of Verma modules. This construction was generalized by Lepowsky in [15], from the Borel case to the case of generalized Verma modules for any parabolic subalgebra $\mathfrak{p}$.

It is well known that homomorphisms of generalized Verma modules correspond to invariant differential operators acting between sheaves of sections of homogeneous vector bundles over the generalized flag manifold G/P. On the geometric side, BGG complexes were studied by Čap, Slovák and Souček in [5]. They constructed BGG complexes in the more general theory of parabolic geometries, for which our G/P is a special case – the flat model. In the flat model, their construction yields a locally exact resolution of the constant sheaf over G/P defined by F, by direct sums of homogeneous vector bundles and invariant differential operators. In the case when the parabolic $\mathfrak{p}$ is |1|-graded, which is equivalent to G/P having a structure of a Hermitian symmetric space, the BGG resolution in trivial infinitesimal character coincides with the holomorphic de Rham complex.

The BGG complexes are special cases of Cousin complexes of D-modules, which follows from an unpublished result by Miličić.

Many important operators live in singular infinitesimal character (e.g. the scalar wave operator on the Minkowski space, Dirac-Weyl operators on conformal manifolds, Dirac-Feuter operators on quaternionic manifolds, etc.), and there are no general constructions of resolutions as above in these cases. Several problems emerge here, one of which is a lack of the so-called standard operators. So, in order to make a resolution out of the singular orbit, one must construct many non-standard operators. Non-standard operators have not yet been classified in general. It turned out that the Penrose transform is a particularly useful tool for the construction of such operators. In [18], Pandžić and Souček constructed singular BGG resolutions over the big affine cell in type A, for all maximal parabolics. It is visible there that singular BGG resolutions cover the whole singular orbit, and moreover, they have the same shape as certain regular resolutions in a lower rank.

This paper deals with type C. Here, G is the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. There is just one $|1|$-graded standard parabolic subgroup P, and G/P is the Lagrangian Grassmannian. The difference here is that we have two types of singularities: singularity of the first kind, involving only short simple roots, and of the second kind, involving also the long simple root. Moreover, the Weyl group here has a more complicated structure. It also contains the sign changes of the coordinates, rather that just permutations, as in type A. In the construction, we assume that the infinitesimal character is semi-regular, i.e., orthogonal to only one simple root. In the first kind, the constructed BGG complex (Theorem 3 and Definition 1) covers the whole singular orbit. But in the second kind, the orbit decomposes into two complexes (Theorem 5 and Definition 2), in agreement with Enright-Shelton's theory [9]. This feature was not present in type A. In case of singularity of the second kind, construction of non-standard operators is technically more involved. The main property of our singular BGG complexes, exactness over the big affine cell, is proved in Theorem 7.

For some results in a higher grading, see [12, 14, 19, 20]. Recent algebraic constructions in singular blocks for the Borel case are available in [16].

1.1. Parabolic subalgebras

Let G be a semisimple complex Lie group, connected and simply connected, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ its fixed Cartan subalgebra, and $\Delta^+(\mathfrak{g}, \mathfrak{h})$ a fixed set of positive roots. The half sum of all positive roots will be denoted by ρ. For an element $w \in W_{\mathfrak{g}}$ in the Weyl group, denote by $l(w)$ the minimal number of simple reflections required to obtain w as a product. Denote also $\Phi_w := \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) : w^{-1}\alpha < 0\}$. A subset $S \subseteq \Delta^+(\mathfrak{g}, \mathfrak{h})$ is said to be *saturated* if for any $\alpha, \beta \in S$ such that $\alpha + \beta$ is a root, we have $\alpha + \beta \in S$. A subset $S \subseteq \Delta^+(\mathfrak{g}, \mathfrak{h})$ is said to be *admissible* if both S and $\Delta^+(\mathfrak{g}, \mathfrak{h}) \setminus S$ are saturated. For $w, w' \in W_{\mathfrak{g}}$ we write $w \xrightarrow{\alpha} w'$ if $l(w') = l(w) + 1$ and $w' = \sigma_\alpha \circ w$, for some $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})$, not necessarily simple. We often write only $w \longrightarrow w'$. In this way, $W_{\mathfrak{g}}$ becomes a directed graph. Besides the standard action of $W_{\mathfrak{g}}$ on $\mathfrak{h}^*$, we also use the affine action: $w \cdot \lambda = w(\lambda + \rho) - \rho$.

Fix a standard parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$. It will be denoted by crossing the nodes in the Dynkin diagram for $\mathfrak{g}$ that are not in the Levi factor $\mathfrak{l}$. Denote by $\Delta(\mathfrak{u})$ the set of positive roots whose root subspaces lie in the nilpotent radial $\mathfrak{u}$. We write $\mathfrak{u}^-$ for the opposite nilpotent radical. The *(regular) Hasse diagram* of $\mathfrak{p}$ is the full subgraph of $W_{\mathfrak{g}}$ with the following nodes:

$$W^{\mathfrak{p}} := \{w \in W_{\mathfrak{g}} \ : \ \Phi_w \subseteq \Delta(\mathfrak{u})\} .$$

It consists of all elements in $W_{\mathfrak{g}}$ that map $\mathfrak{g}$-dominant weights to $\mathfrak{p}$-dominant ones. We will mostly be interested in parabolics with abelian nilpotent radical. These are said to be $|1|$-*graded* (and also of *Hermitian type*).

For a weight $\lambda \in \mathfrak{h}^*$ integral and dominant for $\mathfrak{g}$, we write $F(\lambda)$ for the finite-dimensional, irreducible representation of $\mathfrak{g}$ with highest weight λ, and $E(\lambda)$ for its dual. If λ is $\mathfrak{p}$-dominant, we write $F_{\mathfrak{p}}(\lambda)$ for the finite-dimensional, irreducible representation of $\mathfrak{l}$ with highest weight λ, and with $\mathfrak{u}$ acting by 0. We write $E_{\mathfrak{p}}(\lambda)$ for its dual. The same notation will be used for the group representations. In a $|1|$-graded case, the one-dimensional center of $\mathfrak{l}$ acts by the scalar $\lambda(E) = \frac{2\langle\lambda,\omega\rangle}{\langle\alpha,\alpha\rangle}$, where E is the *grading element* (the unique element from the center of $\mathfrak{l}$ acting as 1 on $\mathfrak{u}$), α is the crossed simple root, and ω the corresponding fundamental weight. This scalar is called the *generalized conformal weight*.

1.2. Geometric setup

The Dynkin notation for $\mathfrak{p}$ will also denote the corresponding parabolic subgroup $P \subseteq G$, and the (complex) generalized flag manifold G/P. For two standard parabolic subgroups $Q \subseteq P$, the *relative Hasse diagram* $W_{\mathfrak{p}}^{\mathfrak{q}}$ of the fibration $G/Q \to G/P$ is the Hasse diagram of the parabolic $\mathfrak{l}_{ss} \cap \mathfrak{q}$ in $\mathfrak{l}_{ss}$, where $\mathfrak{l}_{ss}$ is the semisimple part of the Levi factor of $\mathfrak{p}$.

Given a finite-dimensional holomorphic representation $\pi \colon P \to \mathrm{End}(V)$, we can form the homogeneous holomorphic vector bundle $G \times_P V \to G/P$. Its holomorphic sections correspond to V-valued holomorphic functions that are defined on open subsets of G and are P-equivariant. For $V = E_{\mathfrak{p}}(\lambda)$, this sheaf is denoted by $\mathcal{O}_{\mathfrak{p}}(\lambda)$.

Remark 1. *Recall the relative Bott-Borel-Weil theorem. Let $\tau \colon G/Q \to G/P$ be the obvious fibration, and let $\lambda \in \mathfrak{h}^*$ be a $\mathfrak{g}$-integral and $\mathfrak{p}$-dominant weight. If $\lambda + \rho$ is $\mathfrak{p}$-singular, all the higher direct images $\tau_*^q \mathcal{O}_{\mathfrak{q}}(\lambda)$ are 0. Otherwise, there is a unique $w \in W_{\mathfrak{p}} \subseteq W_{\mathfrak{g}}$, such that $w \cdot \lambda$ is $\mathfrak{p}$-dominant (and necessarily $w^{-1} \in W_{\mathfrak{p}}^{\mathfrak{q}}$). Then, $\tau_*^{l(w)} \mathcal{O}_{\mathfrak{q}}(\lambda) \cong \mathcal{O}_{\mathfrak{p}}(w \cdot \lambda)$, and all other higher direct images are 0. See [1, 5.3.].*

By an *invariant differential operator* we will mean a $\mathbb{C}$-linear differential operator $\mathcal{O}_{\mathfrak{p}}(\lambda) \to \mathcal{O}_{\mathfrak{p}}(\mu)$, equivariant with respect to the left translation of sections.

Remark 2. *Peetre's theorem states that any local map between the sections (where "local" means that the support of a section is not increased) of vector bundles is necessarily a differential operator. See [13, V.19.].*

Remark 3. *In the $|1|$-graded situation, the order of a non-zero invariant differential operator is equal to the difference between the generalized conformal weights in the*

domain and the codomain. Such an operator is unique up to a non-zero scalar. See e.g. [10].

Consider the Borel subgroup $B \subseteq P$. If there exists a non-zero invariant differential operator $\mathcal{O}_{\mathfrak{b}}(\lambda) \to \mathcal{O}_{\mathfrak{b}}(\mu)$, then it is unique up to a scalar (see [1, 11.]). The direct image of such a map via $G/B \to G/P$ is again an invariant differential operator called the *standard* operator $\mathcal{O}_{\mathfrak{p}}(\lambda) \to \mathcal{O}_{\mathfrak{p}}(\mu)$. It may be zero, and there may exist invariant differential operators which are *non-standard*, for $P \neq B$. Standard operators are in principle completely known, but non-standard ones have not yet been classified. Here is the theorem that we want to find analogues of:

Theorem 1 (Bernstein-Gelfand-Gelfand-Lepowsky [15, 2], Čap-Slovák-Souček [5]). *For any $\mathfrak{g}$-integral and $\mathfrak{g}$-dominant weight λ, there is a locally exact sequence on G/P resolving the constant sheaf defined by $E(\lambda)$, called the (regular) BGG resolution*

$$0 \to E(\lambda) \to \Delta^{\bullet}(\lambda), \quad where \quad \Delta^k(\lambda) = \bigoplus_{w \in W^{\mathfrak{p}}, \ l(w)=k} \mathcal{O}_{\mathfrak{p}}(w \cdot \lambda). \tag{1}$$

The morphisms are the direct sums of the standard operators $\mathcal{O}_{\mathfrak{p}}(w \cdot \lambda) \to \mathcal{O}_{\mathfrak{p}}(w' \cdot \lambda)$ for $w \to w'$ in $W^{\mathfrak{p}}$, all of which are non-zero.

1.3. Duality

There is a contravariant correspondence between the sheaves $\mathcal{O}_{\mathfrak{p}}(\lambda)$ and the generalized Verma modules $M_{\mathfrak{p}}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_{\mathfrak{p}}(\lambda)$. See [1, 11.] or [5, the appendix of preprint]:
$$\mathrm{Diff}_G(\mathcal{O}_{\mathfrak{p}}(\lambda), \mathcal{O}_{\mathfrak{p}}(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(\mu), M_{\mathfrak{p}}(\lambda)).$$

1.4. Algebraic setup

Recall the decomposition $\mathcal{O}^{\mathfrak{p}} = \bigoplus_{\lambda \in \mathfrak{h}^*/W_{\mathfrak{g}}} \mathcal{O}^{\mathfrak{p}}_{\lambda}$ of the parabolic category $\mathcal{O}^{\mathfrak{p}}$, where $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ denotes the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ consisting of the modules with generalized infinitesimal character λ. These subcategories are called the *(infinitesimal) blocks* (even though they may be decomposable, as we will see later). Any two blocks with regular generalized infinitesimal characters are mutually equivalent (Jantzen-Zuckerman translation functors), so one usually works only with the so-called *principal block* $\mathcal{O}^{\mathfrak{p}}_{\rho}$. The Hasse diagram $W^{\mathfrak{p}}$ parametrizes the $\mathfrak{p}$-dominant elements of the affine $W_{\mathfrak{g}}$-orbit of a dominant weight. So, $W^{\mathfrak{p}}$ parametrizes both the generalized Verma modules and the simple modules in $\mathcal{O}^{\mathfrak{p}}_{\rho}$. For details, see [11].

One can do similarly in singular blocks. Take an integral weight $\lambda \in \mathfrak{h}^*$ such that $\lambda + \rho$ is dominant, and denote by Σ the set of *simple singular roots* for λ:

$$\Sigma = \{\alpha \in \Pi \colon \langle \lambda + \rho, \check{\alpha} \rangle = 0\}.$$

The subgroup of $W_{\mathfrak{g}}$ generated by $\{\sigma_{\alpha} \colon \alpha \in \Sigma\}$, denoted by W_{Σ}, is equal to the stabilizer $\{z \in W_{\mathfrak{g}} \colon z \cdot \lambda = \lambda\}$. So, $\lambda + \rho$ is regular if and only if $\Sigma = \emptyset$. In general, one can define the *singular Hasse diagram* of the pair $(\mathfrak{p}, \Sigma)$ by the following formula:

$$W^{\mathfrak{p}, \Sigma} = \{w \in W^{\mathfrak{p}} \colon w\sigma_{\alpha} \in W^{\mathfrak{p}} \text{ and } w < w\sigma_{\alpha}, \text{ for all } \alpha \in \Sigma\}. \tag{2}$$

The set $W^{\mathfrak{p},\Sigma}$ is precisely the set of unique minimal length representatives of the left cosets wW_Σ that are contained in $W^\mathfrak{p}$. It parametrizes simple modules and generalized Verma modules in the singular block $\mathcal{O}^\mathfrak{p}_{\lambda+\rho}$. For more details and proofs, see [3].

In the $|1|$-graded case, there are certain equivalences of categories between a singular block and some regular blocks of some other type, called the *Enright-Shelton equivalences*. See [9] and [8, 5.5].

1.5. The Penrose transform

A standard reference is the book [1]. Choose standard parabolic subgroups $P, R \subseteq G$. Their intersection $Q = P \cap R$ is also a standard parabolic subgroup. Denote by η and τ the canonical surjections $G/Q \to G/R$ and $G/Q \to G/P$, respectively. Choose an open subset $X \subseteq G/P$, and define $Y := \tau^{-1}(X)$ and $Z := \eta(Y)$. The subsets Y and Z are open submanifolds of G/Q and G/R, respectively. We have the *double fibration* and the *restricted double fibration*:

$$
\begin{array}{ccc}
& G/Q & \\
{}^{\eta}\swarrow & & \searrow^{\tau} \\
G/R & & G/P,
\end{array}
\qquad
\begin{array}{ccc}
& Y & \\
{}^{\eta}\swarrow & & \searrow^{\tau} \\
Z & & X.
\end{array}
$$

The spaces G/R and Z are usually called the *twistor spaces* of G/P and X, respectively. Start with a weight λ, $\mathfrak{g}$-integral and $\mathfrak{r}$-dominant, and form the sheaf $\mathcal{O}_\mathfrak{r}(\lambda)$ on $Z \subseteq G/R$. Consider the topological inverse image sheaf $\eta^{-1}\mathcal{O}_r(\lambda)$ on Y, whose sections correspond to the sections of the pull-back bundle that are constant on the fibers of η. The weight λ remains dominant on the fibers of η, which themselves are generalized flag manifolds. By resolving $\eta^{-1}\mathcal{O}_r(\lambda)|_{\eta^{-1}(x)}$ over each fiber, one obtains an exact sequence of sheaves on G/Q and standard invariant differential operators, called the *relative BGG resolution*:

$$
0 \to \eta^{-1}\mathcal{O}_\mathfrak{r}(\lambda) \to \Delta^\bullet_\eta(\lambda), \quad \text{where} \quad \Delta^k_\eta(\lambda) = \bigoplus_{w \in W^\mathfrak{q}_\mathfrak{r}, \; l(w)=k} \mathcal{O}_\mathfrak{q}(w \cdot \lambda). \tag{3}
$$

For a full treatment of the relative BGG sequences, see [6, 7]. The *hypercohomology spectral sequence* applied to the exact sequence (3) has the form

$$
E^{pq}_1 = H^p(Y, \Delta^q_\eta(\lambda)) \implies H^{p+q}(Y, \eta^{-1}\mathcal{O}_\mathfrak{r}(\lambda)). \tag{4}
$$

Consider the higher direct images along τ of the sequence (3). Let us assume that $X \subseteq G/P$ is an open *Stein subset*, for example the big affine cell, an open ball or a polydisc inside the big affine cell. By the Bott-Borel-Weil theorem, the sheaves $\tau^q_* \Delta^k_\eta(\lambda)$ are locally free, and therefore coherent. Cartan's theorem B implies that for each $k \geq 0$ the Leray spectral sequence for τ^q_* collapses, and gives isomorphisms $H^q(Y, \Delta^k_\eta(\lambda)) \cong \Gamma(X, \tau^q_* \Delta^k_\eta(\lambda))$, for $k \geq 0$. This settles the left-hand side of (4). For the right-hand side, if the fibers of $\eta \colon Y \to Z$ are smoothly contractible, then there are isomorphims $H^r(Y, \eta^{-1}\mathcal{O}_\mathfrak{r}(\lambda)) \cong H^r(Z, \mathcal{O}_\mathfrak{r}(\lambda))$ for $r \geq 0$. In conclusion, we have:

Theorem 2 (Baston-Eastwood)**.** *If $X \subseteq G/P$ is an open Stein subset, and the fibers of $\eta\colon Y \to Z$ are smoothly contractible, there is a first quadrant spectral sequence:*

$$E_1^{pq} = \Gamma(X, \tau_*^q \Delta_\eta^p(\lambda)) \implies H^{p+q}(Z, \mathcal{O}_{\mathfrak{r}}(\lambda)). \tag{5}$$

On the first page, the differentials are standard operators (induced from the relative BGG), but on the other pages, we have non-standard invariant differential operators.

2. Structure of the Hasse diagrams

2.1. Type C

We specialize to $G = \mathrm{Sp}(2n, \mathbb{C}) = \underset{\alpha_1 \quad \alpha_2 \qquad \alpha_{n-2} \quad \alpha_{n-1} \quad \alpha_n}{\circ\!\!-\!\!\circ\cdots\circ\!\!-\!\!\circ\!\Leftarrow\!\circ}$, the complex symplectic group. Choose the Cartan subalgebra consisting of diagonal matrices $\mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. The positive roots are:

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{a_{ij} := \epsilon_i - \epsilon_j, \quad b_i := 2\epsilon_i, \quad c_{ij} := \epsilon_i + \epsilon_j \; : \; 1 \leq i < j \leq n\},$$

where ϵ_i denotes the projection to the i-th coordinate. The simple roots are $\alpha_i = a_{i,i+1}$ for $i = 1, \ldots, n-1$, and $\alpha_n = b_n$. A weight $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \in \mathfrak{h}^*$ is integral if all $\lambda_i \in \mathbb{Z}$, and dominant if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. A weight is regular if and only if it does not have two coordinates with the same absolute value, and if all the coordinates are non-zero. The half sum of all positive roots is $\rho = [n, n-1, \ldots, 1]$. The fundamental weights are $\omega_i = \epsilon_1 + \ldots + \epsilon_i$, $1 \leq i \leq n$. The Weyl group acts by permutations and sign changes of the coordinates.

Weights for the Levi subalgebra of a standard parabolic subalgebra can be written as n-tuples again, but for every crossed node α_i in the Dynkin diagram for the parabolic subalgebra, we will put a bar after the i-th coordinate of the weight.

2.2. $|1|$-graded parabolic subalgebra

Fix $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} = \circ\!\!-\!\!\circ\cdots\circ\!\!-\!\!\circ\!\Leftarrow\!\times$, which has

$$\Delta^+(\mathfrak{l}, \mathfrak{h}) = \{a_{ij} \colon i < j\}, \quad \Delta(\mathfrak{u}) = \{b_i\} \cup \{c_{ij} \colon i < j\}.$$

Moreover, $\mathfrak{l} \cong \mathfrak{gl}(n, \mathbb{C})$, and $\mathfrak{u}$ consists of the matrices of the form $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$, where C is an $n \times n$ symmetric matrix. The grading element is $\frac{1}{2} \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, -1, \ldots, -1)$.

A weight $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \in \mathfrak{h}^*$ is $\mathfrak{p}$-dominant if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and strictly $\mathfrak{p}$-dominant if all the inequalities are strict.

The generalized flag manifold corresponding to this parabolic subalgebra is known as the (complex) *Lagrangian Grassmannian*, denoted by $\mathrm{iGr}(n, 2n)$. It can be realized as the space of all maximal isotropic (Lagrangian) subspaces in a fixed $2n$-dimensional symplectic vector space.

2.3. Generalized Young diagrams

Elements of the Hasse diagram $W^{\mathfrak{p}}$ will be represented using the map Φ from Subsection 1.1 as admissible subsets of $\Delta(\mathfrak{u})$. The point of this identification is to get a simple criterion for the arrow relation.
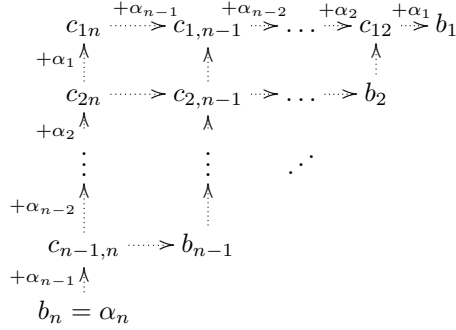
$$
\begin{array}{ccccccc}
c_{1n} & \xrightarrow{+\alpha_{n-1}} & c_{1,n-1} & \xrightarrow{+\alpha_{n-2}} & \ldots & \xrightarrow{+\alpha_2} & c_{12} & \xrightarrow{+\alpha_1} & b_1 \\
{\scriptstyle +\alpha_1} \uparrow & & \uparrow & & & & \uparrow \\
c_{2n} & \dashrightarrow & c_{2,n-1} & \dashrightarrow & \ldots & \dashrightarrow & b_2 \\
{\scriptstyle +\alpha_2} \uparrow & & \uparrow \\
\vdots & & \vdots & & & \ddots \\
{\scriptstyle +\alpha_{n-2}} \uparrow & & \uparrow \\
c_{n-1,n} & \dashrightarrow & b_{n-1} \\
{\scriptstyle +\alpha_{n-1}} \uparrow \\
b_n = \alpha_n
\end{array}
$$

Figure 1: $\Delta(\mathfrak{u})$ *for* $\mathfrak{p} = \circ\!\!-\!\!-\!\!\circ \cdots \circ\!\!-\!\!-\!\!\circ\!\!\Leftarrow\!\!\times$

Proposition 1. *Denote* $b_i = c_{ii}$. *A subset* $S \subseteq \Delta(\mathfrak{u})$ *is admissible if and only if*

$$c_{ij} \in S, \ i \leq j, \quad \Longrightarrow \quad c_{kl} \in S \quad \text{for all} \quad k \geq i, \ l \geq j, \ k \leq l \leq n. \qquad (6)$$

Proof. Direct checking. □

An admissible subset S will be represented in the following way: for each element $c_{ij} \in S$, we put a box $\square$ on the position c_{ij} in Figure 1. The diagram obtained in this way will be called the *generalized Young diagram* of the corresponding Hasse diagram element (see [8]).
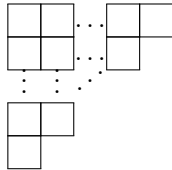
The maximal admissible subset is $\Delta(\mathfrak{u})$, and we denote it by the figure on the left. Condition (6) translated into the generalized Young diagram setting is: for each box in S, all possible boxes bellow, and left of it are again contained in S. The notion of the length and the arrow relation transfer very nicely to the generalized Young diagram setting. Namely, the length of an element in $W^{\mathfrak{p}}$ is equal to the number of boxes in the generalized Young diagram. Furthermore, an arrow between elements in $W^{\mathfrak{p}}$ corresponds to the *"adding one box"* *operation* on the generalized Young diagrams; the label of the arrow is the same as the label of the added box (follows from [4, 3.2.]). See Figure 2.

2.4. Lascoux-Schützenberger (LS) notation

Note that a generalized Young diagram is completely determined by a zig-zag line from the top left point to the diagonal. We can easily reconstruct the generalized Young diagram from the zig-zag line, by removing the part above this line from the maximal generalized Young diagram. For each move to the right along this line, we write 1, and for each move down, we write 0. This way we get a binary sequence of

length n, called the *LS notation* of the generalized Young diagram (see [8]), written with overline. See Figure 2. This way of denoting elements in $W^{\mathfrak{p}}$ does not follow usual conventions in [1], but it turns out to be very convenient for describing singular orbits, and the Penrose transform in our case. The reason is that the LS notation fits well with the inductive structure of $W^{\mathfrak{p}}$.

Proposition 2. *As directed graphs,*

$$W^{\mathfrak{p}} \cong \left\{ \overline{d_1 d_2 \ldots d_n} \ : \ d_i = 0 \ or \ 1 \right\}, \tag{7}$$

with the following arrows on the right-hand side:

$$\overline{d_1 \ldots 01 \ldots d_n} \to \overline{d_1 \ldots 10 \ldots d_n} \quad and \quad \overline{d_1 \ldots d_{n-1}0} \to \overline{d_1 \ldots d_{n-1}1}.$$

Proof. Bijection (7) follows from the definition of the LS notation. Obviously, the "adding one box" operation has the effect of switching a pair of consecutive digits $\overline{01}$ to $\overline{10}$ (if the added box is not the last possible in a row), or changing the last digit $\overline{0}$ to $\overline{1}$ (if the added box is the last possible in a row). □
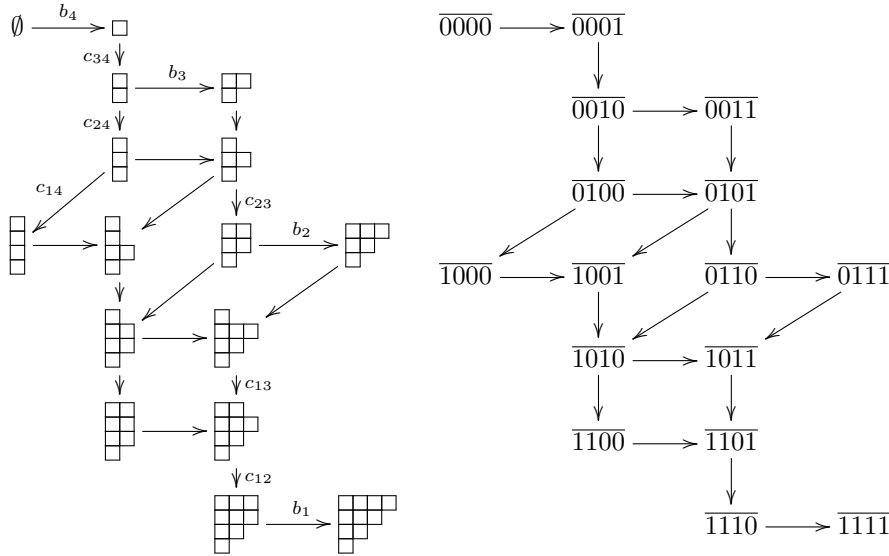

Figure 2: $W^{\mathfrak{p}}$ for $\mathfrak{p} = \circ\!\!-\!\!\circ\!\!-\!\!\circ\!\!\Leftarrow\!\!\times$ *both in terms of generalized Young diagrams (on the left), and using the LS notation (on the right)*

Proposition 3. *Let* $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \in \mathfrak{h}^*$ *and* $w \in W^{\mathfrak{p}}$. *Let* $i_1 < i_2 < \ldots < i_k$ *denote the positions of digit* 1 *in the LS notation for* w. *Then*

$$w\lambda = [\lambda_1, \lambda_2, \ldots, \widehat{\lambda_{i_1}}, \ldots, \widehat{\lambda_{i_2}}, \ldots, \widehat{\lambda_{i_k}}, \ldots, \lambda_n, -\lambda_{i_k}, -\lambda_{i_{k-1}}, \ldots, -\lambda_{i_1} \,|\,]. \tag{8}$$

The coordinates with $\widehat{hat}$ are omitted. In other words, the positions of digit 1 *are precisely the positions of the coordinates of* ρ *that become negative in* $w\rho$.

Proof. Suppose first that $k = 1$. Then $w = \overline{0\ldots010\ldots0}$, and the corresponding generalized Young diagram has just one column of $n + 1 - i_1$ boxes. It follows that

$$w = \sigma_{c_{i_1,n}} \circ \ldots \circ \sigma_{c_{n-1,n}} \circ \sigma_{b_n}.$$

Applying this composition to λ gives $[\lambda_1, \ldots, \widehat{\lambda_{i_1}}, \ldots, \lambda_n, -\lambda_{i_1}]$.

In general, the same principle is applicable. One can decompose w into columns, and calculate the action of each column from the left to the right. More precisely,

$$w = C_k \circ C_{k-1} \circ \ldots \circ C_1, \ \text{where} \ \ C_j = \sigma c_{i_j+1-j,n+1-j} \circ \ldots \circ \sigma c_{n-j,n+1-j} \circ \sigma b_{n+1-j}.$$

Applying this to λ gives (8).   □

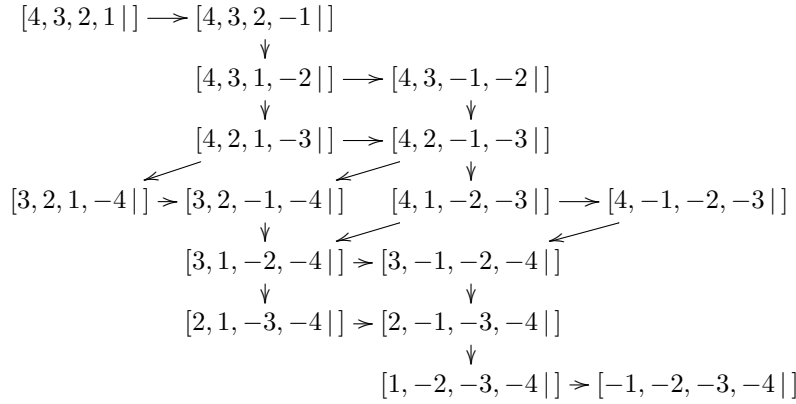One can use Proposition 3 to translate between the right-hand side of Figure 2 and Figure 3.

$$
\begin{array}{l}
[4,3,2,1\,|\,] \longrightarrow [4,3,2,-1\,|\,] \\
\qquad\qquad\qquad\quad \downarrow \\
\qquad [4,3,1,-2\,|\,] \longrightarrow [4,3,-1,-2\,|\,] \\
\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
\qquad [4,2,1,-3\,|\,] \longrightarrow [4,2,-1,-3\,|\,] \\
\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
[3,2,1,-4\,|\,] \succ [3,2,-1,-4\,|\,] \quad [4,1,-2,-3\,|\,] \longrightarrow [4,-1,-2,-3\,|\,] \\
\qquad\qquad \downarrow \\
\qquad [3,1,-2,-4\,|\,] \succ [3,-1,-2,-4\,|\,] \\
\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
\qquad [2,1,-3,-4\,|\,] \succ [2,-1,-3,-4\,|\,] \\
\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad [1,-2,-3,-4\,|\,] \succ [-1,-2,-3,-4\,|\,]
\end{array}
$$

Figure 3: $W^{\mathfrak{p}}\rho$ *for* $\mathfrak{p} = \circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\circ\!\!=\!\!\blacktriangleleft\!\!\times$, *compare with Figure 2*

2.5. Description of the singular Hasse diagrams

Take an integral weight λ such that $\lambda + \rho$ is dominant and denote by Σ its set of simple singular roots. Consider the $W^{\mathfrak{p}}$-orbit of $\lambda + \rho$ (which is the same as the affine $W^{\mathfrak{p}}$-orbit of λ, up to the shift of coordinates by ρ), and look for the elements that are strictly $\mathfrak{p}$-dominant. The results that are not strictly $\mathfrak{p}$-dominant do not correspond to a homogeneous vector bundle over G/P. The remaining part is what is called the *singular orbit* attached to the pair $(\mathfrak{p}, \Sigma)$ or $(\mathfrak{p}, \lambda)$. In the identification (7), we can recognize those LS words that belong to the singular Hasse diagram (2):

Proposition 4. *Suppose* $\Sigma = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$. *If the long simple root* $\alpha_n \notin \Sigma$, *then* $W^{\mathfrak{p},\Sigma} = \{\overline{d_1 d_2 \ldots d_n} : \overline{d_{i_k} d_{i_k+1}} = \overline{01} \ \text{for} \ k = 1, \ldots, s\}$. *Otherwise* $\alpha_n = \alpha_{i_s}$, *and* $W^{\mathfrak{p},\Sigma} = \{\overline{d_1 d_2 \ldots d_{n-1} 0} : \overline{d_{i_k} d_{i_k+1}} = \overline{01} \ \text{for} \ k = 1, \ldots, s-1\}$.

Proof. Easily follows from definitions and (8).   □

Two different cases, depending on whether $\alpha_n \notin \Sigma$ or $\alpha_n \in \Sigma$, will be referred to as the *singularity of the first kind*, and the *singularity of the second kind*, respectively. In Figure 4, we give an example of a singular orbit of each kind in rank 4. In the
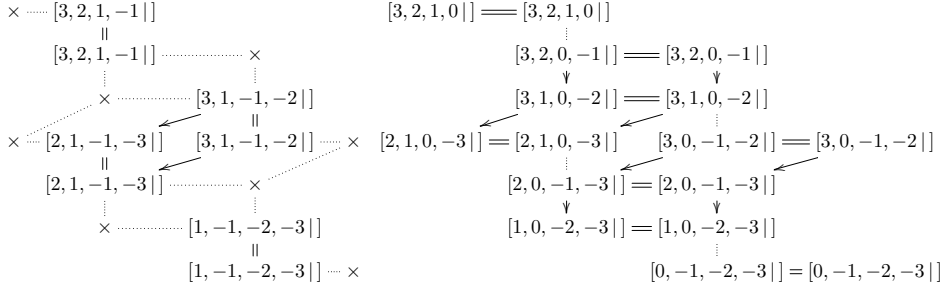

Figure 4: *Singular orbit for* $[3, 2, 1, 1]$ *and* $[3, 2, 1, 0]$

right-hand side orbit, there are some dotted lines between non-zero objects. These represent the standard morphisms that turn out to be zero (for a direct proof, see [17, p. 69], but it also follows from a much stronger result, namely Lemma 1).

In both of these orbits there are some non-standard operators, which are not visible in the figures. These missing operators will be constructed using the Penrose transform from an appropriately chosen twistor space. They will be (together with the standard operators) the differentials in the singular BGG complex. The construction of non-standard operators will be more complicated for the singularity of the second kind.

3. Construction of non-standard operators

From now on, we will work with a fixed weight λ such that $\lambda + \rho$ is orthogonal to only one simple root. In this case, we say that $\lambda + \rho$ is *semi-regular*. So, $\Sigma = \{\alpha_k\}$ for some $k \leq n$. In case of a singularity of the first kind ($k < n$), the minimal choice for $\lambda + \rho$ is

$$\lambda + \rho = [n-1, n-2, \ldots, n-k+1, \underbrace{\mathbf{n-k}}_{k}, \underbrace{\mathbf{n-k}}_{k+1}, n-k-1, \ldots, 2, 1]. \qquad (9)$$

In case of the singularity of the second kind ($k = n$), the minimal choice for $\lambda + \rho$ is

$$\lambda + \rho = [n-1, n-2, \ldots, 2, 1, \mathbf{0}].$$

We will work with this minimal choice for $\lambda + \rho$, but we want to note that in the construction of non-standard operators that follows, the minimality is not important, only the order among the coordinates of $\lambda + \rho$ plays a role. Equivalently, one can apply the Jantzen-Zuckerman translation functors to obtain non-minimal cases from the minimal ones. Of course, for a non-minimal $\lambda + \rho$, the orders of the constructed operators will increase (see Remark 3).

3.1. Double fibration

Consider the following double fibration:

$$
G/Q = \times\!\!-\!\!\circ\cdots\cdots\circ\!\!-\!\!\circ\!\!\Longleftarrow\!\!\times \tag{10}
$$

$$
G/R = \times\!\!-\!\!\circ\cdots\cdots\circ\!\!-\!\!\circ\!\!\Longleftarrow\!\!\circ \qquad\qquad G/P = \circ\!\!-\!\!\circ\cdots\cdots\circ\!\!-\!\!\circ\!\!\Longleftarrow\!\!\times.
$$

Recall that G/P can be realized as the Lagrangian Grassmannian $\mathrm{iGr}(n,2n)$, G/R as the isotropic Grassmannian $\mathrm{iGr}(1,2n)$ (biholomorphic to $\mathbb{P}^{2n-1}$), and G/Q as the space of isotropic flags of the type $(1,n)$. More precisely, double fibration (10) becomes:

$$
\{(L,W)\colon \dim L = 1,\ \dim W = n,\ L \leq W \text{ isotropic}\}
$$

$$
\{L\colon \dim L = 1\} \qquad\qquad \{W\colon \dim W = n,\ W \text{ isotropic}\},
$$

where η and τ are the projections to the components. Take $X \subseteq G/P$ to be the big affine cell. It consists of subspaces spanned by the columns of the matrix $\begin{pmatrix} I \\ C \end{pmatrix}$ (w.r.t a fixed symplectic basis), where I is an identity $n \times n$ matrix, and C a symmetric $n \times n$ matrix. The symmetric matrices $\mathrm{Sym}_n(\mathbb{C}) \cong \mathfrak{u}^-$ give the cannonical affine coordinates on X. Put $Y := \tau^{-1}(X)$ and $Z := \eta(Y)$. A general element in the fiber $\tau^{-1}(W)$, $W \in X$, is a pair (L,W), where any non-zero vector in L is a linear combination of the columns of $\begin{pmatrix} I \\ C \end{pmatrix}$. The coefficients in this linear combination are uniquely determined by L up to a non-zero scalar, so they define a point in the projective space $\mathbb{P}^{n-1}$. From this, it follows that we have a biholomorphic bijection

$\mathrm{Sym}_n(\mathbb{C}) \times \mathbb{P}^{n-1} \cong Y$ given by $(C,y) \mapsto \left(\begin{pmatrix} y \\ C \cdot y \end{pmatrix}, \begin{pmatrix} I \\ C \end{pmatrix} \right)$, where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ are

the projective coordinates. Now, the restricted double fibration is:

$$
\begin{array}{ccc}
& Y & \\
\eta \swarrow & & \searrow \tau \\
Z & & X,
\end{array}
\qquad\qquad
\begin{array}{ccc}
& (C,y) & \\
\eta \swarrow & & \searrow \tau \\
\begin{pmatrix} y \\ C \cdot y \end{pmatrix} & & C.
\end{array}
\tag{11}
$$

Proposition 5. *We have* $Z = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{P}^{2n-1} \ :\ \text{at least one } y_i \neq 0 \right\}.$

Proof. The condition in the curly brackets is necessary because y are projective coordinates. For the converse, assume $y_1 = 1$, and observe:

$$\begin{pmatrix} z_1 - \sum_{i=2}^{n} z_i \cdot y_i & z_2 & \ldots & z_n \\ z_2 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ z_n & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

The proof is analogous if some other $y_i = 1$. $\qquad\square$

Proposition 6. *The fibers of $\eta \colon Y \to Z$ are smoothly contractible.*

Proof. Given $\begin{pmatrix} y \\ z \end{pmatrix} \in Z$, the condition $C \cdot y = z$ is given by linear equations in the entries of the matrix C. So, the fiber $\eta^{-1}\begin{pmatrix} y \\ z \end{pmatrix} \subseteq \mathrm{Sym}_n(\mathbb{C}) \times y$ is a certain affine subspace of Y, and therefore smoothly contractible. $\qquad\square$

Suppose $X' \subseteq X$ is a convex open subset, and put $Y' := \tau^{-1}(X')$, $Z' := \eta(Y')$. In this new restricted double fibration, the fibers of $\eta \colon Y' \to Z'$ are equal to the intersection of an affine set (the fibers of η in Y) and a convex set (a copy of X' in Y), and are therefore also smoothly contractible. So, we have a valid setup for the Penrose transform locally, around any point in the Lagrangian Grassmannian.

3.2. Relative Hasse diagrams

Fix $k \in \{1, \ldots, n\}$, and consider the homogeneous sheaf $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/R, where

$$\tilde{\lambda} + \rho = [n - k \,|\, n - 1, n - 2, \ldots, 2, 1]. \tag{12}$$

The weight $\tilde{\lambda} = [-k \,|\, 0, 0, \ldots, 0, 0]$ is obviously $\mathfrak{r}$-dominant, so the sheaf $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ is indeed well defined. To calculate the relative BGG resolution (3) of the inverse image $\eta^{-1}\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/Q, we need the relative Hasse diagram $W_{\mathfrak{r}}^{\mathfrak{q}}$. The fiber of η is $R/Q \cong \underset{\alpha_2}{\circ}\!\!-\!\!\circ\cdots\circ\!\!-\!\!\overset{\Longleftarrow}{\underset{\alpha_n}{\times}}$. So, $W_{\mathfrak{r}}^{\mathfrak{q}}$ (and so the relative BGG resolution) has the same shape as the regular Hasse diagram in rank $n-1$. As a subset of $W_{\mathfrak{g}}$, $W_{\mathfrak{r}}^{\mathfrak{q}}$ operates on the last $n-1$ coordinates and ignores the first coordinate of a weight. Therefore, $W_{\mathfrak{r}}^{\mathfrak{q}}$ can be identified with the following subgraph of $W^{\mathfrak{p}}$:

$$W_{\mathfrak{r}}^{\mathfrak{q}} = \left\{ w \in W^{\mathfrak{p}} \ : \ w = \overline{0\, d_1 d_2 \ldots d_{n-1}} \right\}. \tag{13}$$

To apply the Bott-Borel-Weil theorem for calculating the higher direct images along τ, it is convenient to understand the relative Hasse diagram $W_{\mathfrak{p}}^{\mathfrak{q}}$. The fiber of τ is $P/Q \cong \underset{\alpha_1}{\times}\!\!-\!\!\circ\cdots\circ\!\!-\!\!\underset{\alpha_{n-1}}{\circ} \cong \mathbb{P}^{n-1}$, also $|1|$-graded, so by an easy calculation:

$$W_{\mathfrak{p}}^{\mathfrak{q}} = \{\mathrm{Id} \xrightarrow{a_{12}} \bullet \xrightarrow{a_{13}} \bullet \xrightarrow{a_{14}} \ldots \xrightarrow{a_{1n}} \bullet\} \subseteq W_{\mathfrak{g}}. \tag{14}$$

Example 1. *Take $\lambda + \rho = [3,2,1,1]$, so $\tilde{\lambda} + \rho = [1\,|\,3,2,1]$. The relative BGG resolution of the sheaf $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/Q is obtained by applying (13) to $\tilde{\lambda} + \rho$:*

$$\eta^{-1}\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda}) \twoheadrightarrow [1\,|\,3,2,1\,|\,] \twoheadrightarrow [1\,|\,3,2,-1\,|\,]$$
$$[1\,|\,3,1,-2\,|\,] \twoheadrightarrow [1\,|\,3,-1,-2\,|\,]$$
$$[1\,|\,2,1,-3\,|\,] \twoheadrightarrow [1\,|\,2,-1,-3\,|\,]$$
$$[1\,|\,1,-2,-3\,|\,] \twoheadrightarrow [1\,|\,-1,-2,-3\,|\,] \twoheadrightarrow 0.$$

To calculate higher direct images, remove the first bar in each weight in the resolution. If a weight has two coordinates equal, it is $\mathfrak{p}$-singular, and so all its higher direct images are 0. Otherwise, the surviving higher direct image is obtained by arranging the coordinates in the strictly decreasing order, and the degree is equal to the number of the transpositions of adjacent coordinates needed to move the first coordinate to its correct position (follows from (14)). We organize this data on the first page of the spectral sequence (5), which is in this example

$$\begin{array}{|llllll}
0 & [3,2,1,-1\,|\,] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & [3,1,-1,-2\,|\,] \twoheadrightarrow [2,1,-1,-3\,|\,] & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & [1,-1,-2,-3\,|\,].
\end{array}$$

Compare this to Figure 4. Two non-standard operators to be constructed in this case are $[3,2,1,-1\,|\,] \to [3,1,-1,-2\,|\,]$ and $[2,1,-1,-3\,|\,] \to [1,-1,-2,-3\,|\,]$. Note that the objects in the spectral sequence are not really homogeneous sheaves, but rather their sections over X. We will omit $\Gamma(X,-)$ from the notation, and write only sheaves, or the defining (ρ-shifted) weights, or their LS codes. Also note that a standard operator between two adjacent objects in the relative BGG resolution survives the higher direct image and appears in the spectral sequence as a standard operator, only if both these adjacent objects survive in the same degree. This follows from the functoriality of the direct images.

Example 2. *Take $\lambda + \rho = [3,2,1,0]$, so $\tilde{\lambda} + \rho = [0\,|\,3,2,1]$. This is the singularity of the second kind. The relative BGG resolution is the same as in Example 1, except that instead of 1 there is 0 before the first bar. Note that now every weight survives a higher direct image. The first page of spectral sequence (5) is:*

$$\begin{array}{|llllllll}
[3,2,1,0\,|\,] & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & [3,2,0,-1\,|\,] \twoheadrightarrow [3,1,0,-2\,|\,] \twoheadrightarrow [2,1,0,-3\,|\,] & 0 & 0 & 0 \\
0 & 0 & 0 & [3,0,-1,-2\,|\,] \twoheadrightarrow [2,0,-1,-3\,|\,] \twoheadrightarrow [1,0,-2,-3\,|\,] & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & [0,-1,-2,-3\,|\,].
\end{array}$$

Compare this to Figure 4. Two non-standard operators will be construted here: $[3,2,1,0\,|\,] \to [3,0,-1,-2\,|\,]$ and $[2,1,0,-3\,|\,] \to [0,-1,-2,-3\,|\,]$. Namely, the Enright-Shelton equivalence says that this orbit should decompose into two disjoint blocks with respect to the parity, each of the shape $\bullet \to \bullet \to \bullet \to \bullet$.

Example 3. *In Figure 5, we give the degrees of the surviving higher direct images in rank 5 in all semi-regular cases, from $\Sigma = \{\alpha_1\}$ to $\Sigma = \{\alpha_5\}$, respectively. The non-standard operators to be constructed are indicated by dashed arrows.*
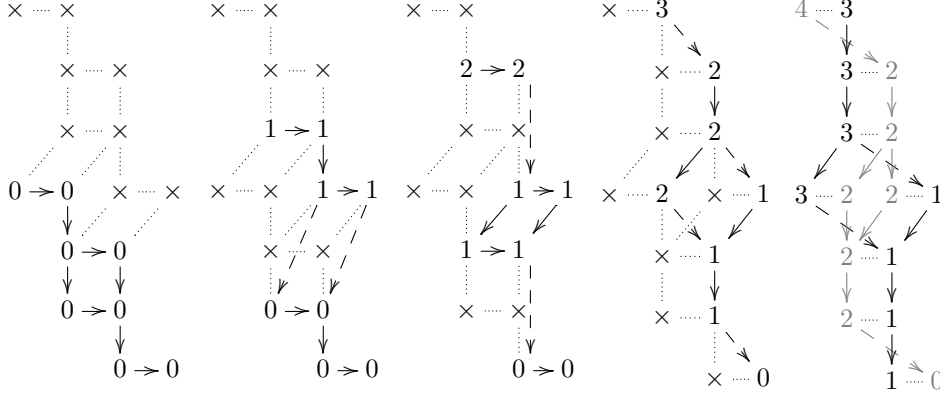

Figure 5: *Degrees of higher direct images in rank 5*

The main technical difference between the two kinds of singularities is the following: In the first kind, all non-standard operators to be constructed go across zero columns in the first page of the spectral sequence; in the second kind, the wanted non-standard operators go across columns with non-zero entries. We will deal with the two kinds separately.

3.3. Construction for the first kind

Suppose $\lambda + \rho$ is orthogonal to only one short simple root. We work with the minimal choice for $\lambda + \rho$, which is given in (9), so $\tilde{\lambda}$ is as in (12) for some fixed $k < n$.

Proposition 7. *In case of singularity of the first kind, the objects in the relative BGG resolution that survive a higher direct image are parametrized by the LS words of the form $\overline{0\, d_1 \dots d_{k-1}\, \mathbf{1}\, d_{k+1} \dots d_{n-1}}$. The surviving degree is equal to the number of the digits 0 among $d_1, \dots, d_{k-1}$. The result in this degree corresponds to $\overline{d_1 \dots d_{k-1}\, \mathbf{01}\, d_{k+1} \dots d_{n-1}} \in W^{\mathfrak{p},\Sigma}$.*

Moreover, the first page of spectral sequence (5) agrees with the singular orbit, including both the objects and the standard operators.

Proof. This is just a translation of the Bott-Borel-Weil theorem in our notation (see Remark 1). An element $w = \overline{0\, d_1 \dots d_{n-1}} \in W_{\mathfrak{r}}^{\mathfrak{q}}$ will make $\tilde{\lambda} + \rho$ $\mathfrak{p}$-regular if and only if it makes the coordinate entry $n - k$ (after the bar) negative. This will happen if and only if $d_k = 1$. The number of the transpositions of adjacent coordinates needed to make $w(\tilde{\lambda} + \rho)$ $\mathfrak{p}$-dominant is equal to the number of the coordinates in $w(\tilde{\lambda} + \rho)$ greater than $n - k$ (which can occur only on positions 2 to k); this equals the number of digits 0 among $d_1, \dots, d_{k-1}$. The last two statements are obvious. $\square$

We have an obvious bijection from the singular Hasse diagram (and the surviving part of the relative BGG) to the regular Hasse diagram of rank $n-2$, given by:

$$\overline{d_1 \ldots d_{k-1} \, \mathbf{01} \, d_{k+1} \ldots d_{n-1}} \mapsto \overline{d_1 \ldots d_{k-1} d_{k+1} \ldots d_{n-1}}. \tag{15}$$

However, this is not a directed-graph isomorphism. We need to "add" more arrows to the left-hand side. Those arrows are the missing non-standard operators, constructed in the following theorem.

Theorem 3. *There are non-standard invariant differential operators*

$$D \colon \mathcal{O}_{\mathfrak{p}}(\nu) \to \mathcal{O}_{\mathfrak{p}}(\nu'') \tag{16}$$

for all pairs ν, ν'' in the singular orbit of the first kind, given by

$$\nu = \overline{d_1 \ldots d_{k-2} \, \mathbf{0011} \, d_{k+3} \ldots d_n} \cdot \lambda, \quad \nu'' = \overline{d_1 \ldots d_{k-2} \, \mathbf{1010} \, d_{k+3} \ldots d_n} \cdot \lambda$$

for $k = 2, \ldots, n-2$, or by $\nu = \overline{d_1 \ldots d_{n-3} \, \mathbf{001}} \cdot \lambda$ and $\nu'' = \overline{d_1 \ldots d_{n-3} \, \mathbf{101}} \cdot \lambda$ for $k = n-1$.

If λ is minimal as in (9), operator (16) is of order 2.

Proof. Take X' to be an open ball inside the big affine cell in G/P, and consider the Penrose transform over the corresponding restricted double fibration. In the relative BGG resolution, we find and fix the following sequence:

$$\mu = \overline{0 \, d_1 \ldots d_{k-2} \, 011 \, d_{k+2} \ldots d_{n-1}} \cdot \tilde{\lambda}$$
$$\downarrow$$
$$\mu' = \overline{0 \, d_1 \ldots d_{k-2} \, 101 \, d_{k+2} \ldots d_{n-1}} \cdot \tilde{\lambda} \twoheadrightarrow \mu'' = \overline{0 \, d_1 \ldots d_{k-2} \, 110 \, d_{k+2} \ldots d_{n-1}} \cdot \tilde{\lambda}.$$

Denote $q = 1+$the number of the digits 0 among $d_1, \ldots, d_{k-2}$. Consider the (part of the) Čech bi-complex that calculates the higher direct images, described in Figure 6. Here the horizontal morphisms d_h are induced from the differentials of the relative BGG. The vertical morphisms d_v are the usual differentials in the Čech resolution. We have $d_v^2 = 0$, $d_h^2 = 0$, and for each square, $d_h d_v = -d_v d_h$. By definition, the vertical cohomologies are equal to the higher direct images of the corresponding sheaves. By Proposition 7,

$$H^q(\check{C}_\mu^\bullet, d_v) = \tau_*^q \mathcal{O}_{\mathfrak{q}}(\mu) = \mathcal{O}_{\mathfrak{p}}(\nu), \quad H^{q-1}(\check{C}_{\mu''}^\bullet, d_v) = \tau_*^{q-1} \mathcal{O}_{\mathfrak{q}}(\mu'') = \mathcal{O}_{\mathfrak{p}}(\nu'').$$

The cochain spaces with nontrivial cohomology will be denoted in bold font below. All other vertical cohomologies are trivial, including the complete middle column.

We will define operator (16) on the representatives of the cohomology classes in $H^q(\check{C}_\mu^\bullet, d_v)$. We want this operator to agree with a certain higher differential in the spectral sequence (5). This will be used in the proof of Theorem 7.

Take a cocycle $x \in \check{\mathbf{C}}_\mu^{\mathbf{q}}$. From $d_v d_h(x) = -d_h d_v(x) = 0$ it follows that $d_h(x) \in \check{C}_{\mu'}^q$ is a cocycle. Since $H^q(\check{C}_{\mu'}^\bullet, d_v) = 0$, it follows that $d_h(x) \in \operatorname{Im} d_v$. So, there is $y \in \check{C}_{\mu'}^{q-1}$ such that $d_v(y) = d_h(x)$. Then, $d_h(y) \in \check{\mathbf{C}}_{\mu''}^{\mathbf{q-1}}$, in the correct cochain space. The element $d_h(y)$ is a cocycle: $d_v d_h(y) = -d_h d_v(y) = -d_h^2(x) = 0$.

$$\begin{array}{ccc} \check{\mathbf{C}}^{\mathbf{q}}_{\mu} \longrightarrow \check{C}^{q}_{\mu'} \longrightarrow \check{C}^{q}_{\mu''} \\ \uparrow \qquad \uparrow \qquad \uparrow \\ \check{C}^{q-1}_{\mu} \longrightarrow \check{C}^{q-1}_{\mu'} \longrightarrow \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''} \\ \vdots \qquad \vdots \qquad \vdots \\ \mathcal{O}_{\mathfrak{q}}(\mu) \rightarrowtail \mathcal{O}_{\mathfrak{q}}(\mu') \rightarrowtail \mathcal{O}_{\mathfrak{q}}(\mu'') \end{array} \qquad \begin{array}{c} x \in \check{\mathbf{C}}^{\mathbf{q}}_{\mu} \vdash\!\!\!\rightarrow d_h(x) \in \check{C}^{q}_{\mu'} \\ \qquad \qquad \qquad \uparrow \\ y \in \check{C}^{q-1}_{\mu'} \longmapsto d_h(y) \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''}. \end{array}$$

Figure 6: *Diagram chasing over the Čech bi-complex (1)*

Next, we check that we have a well-defined map $[x] \mapsto [d_h(y)]$ on the cohomology classes. Take another cocycle x' in the same cohomology class $[x]$, and find y' so that $d_v(y') = d_h(x')$. Since $x - x' = d_v(t)$ for some $t \in \check{C}^{q-1}_{\mu}$, observe that

$$d_v(y - y' + d_h(t)) = d_h(x - x') - d_h d_v(t) = 0, \tag{17}$$

so we conclude that $y - y' + d_h(t) = d_v(t')$ for some $t' \in \check{C}^{q-2}_{\mu'}$. Finally,

$$d_v(-d_h(t')) = d_h(d_v(t')) = d_h(y) - d_h(y') + d_h^2(t) = d_h(y) - d_h(y') \in \operatorname{Im} d_v.$$

Therefore, we have a well-defined map (16), given by $D([x]) = [d_h(y)]$, which is by construction local and G-invariant. By Remark 2, it is a differential operator. By Remark 3, its order is given by the difference of the generalized conformal weights, which is easily seen to be 2 in the minimal case. □

Definition 1. *In case of singularity of the first kind, the singular orbit with all non-standard operators constructed in Theorem 3 included therein, is called the singular BGG complex of infinitesimal character $\lambda + \rho$.*

Theorem 4. *In case of singularity of the first kind:*

1. *The singular BGG complexes of rank n are directed-graph isomorphic to the regular one of rank $n - 2$.*

2. *Every square in the singular BGG complex anticommutes.*

3. *If we add up all objects of the singular BGG complex of the same degree, we get a cochain complex.*

Proof. It is easy to check 1: now (15) is a directed-graph isomorphism. Statement 3 follows from 2: $\left(\sum d_i\right)^2 = \sum_{i \neq j} d_i d_j = \sum_{i < j} (d_i d_j + d_j d_i) = 0$.

To prove 2, observe that the standard operators anticommute since this was already true in the relative BGG resolution. There are no squares with all operators non-standard. Therefore, we only need to check a combination of a standard and a non-standard operator. A typical situation in the relative BGG resolution that induces such a square is (the k-th coordinate is denoted in the bold font):

$$\begin{array}{ccc} \overline{0 \ldots 0 1 \ldots 0 \mathbf{1} 1 \ldots} & \rightarrowtail \overline{0 \ldots 0 1 \ldots \mathbf{1} 0 1 \ldots} & \rightarrowtail \overline{0 \ldots 0 1 \ldots \mathbf{1} 1 0 \ldots} \\ \downarrow & \downarrow & \downarrow \\ \overline{0 \ldots 1 0 \ldots 0 \mathbf{1} 1 \ldots} & \rightarrowtail \overline{0 \ldots 1 0 \ldots \mathbf{1} 0 1 \ldots} & \rightarrowtail \overline{0 \ldots 1 0 \ldots \mathbf{1} 1 0 \ldots.} \end{array}$$

(Other possible situations start with $\overline{0\ldots0\mathbf{1}\mathbf{1}\ldots01\ldots}$, or $\overline{0\ldots0\mathbf{1}\mathbf{1}\ldots0}$). Denote by μ, μ', μ'' the objects in the first row, and by θ, θ', θ'' the objects in the second row, and consider the Čech bi-complex above it, with the same notation as in the proof of Theorem 3. Denote by d all standard operators $\mu \to \theta$, $\mu' \to \theta'$ and $\mu'' \to \theta''$ in the relative BGG resolution. These are just horizontal differentials, but they go in a different direction than those we denoted by d_h in the definition of D. The maps d anticommute with both d_h and d_v. This is the part of the Čech bi-complex that is mapped to a square in the singular orbit with two parallel standard and two parallel non-standard operators:

$$
\begin{array}{ccc}
d(x) \in \check{\mathbf{C}}^{\mathbf{q}}_{\theta} & \longmapsto & d_h(d(x)) \\
\nearrow & & \uparrow \\
x \in \check{\mathbf{C}}^{\mathbf{q}}_{\mu} \longmapsto d_h(x) & & y' \longmapsto d_h(y') \in \check{\mathbf{C}}^{\mathbf{q}-\mathbf{1}}_{\theta''} \\
\uparrow & & \nearrow \; -d \\
y \longmapsto d_h(y) \in \check{\mathbf{C}}^{\mathbf{q}-\mathbf{1}}_{\mu''}.
\end{array}
$$

We need to show that $[d(d_h(y))] = -[d_h(y')]$. First, for y' we can take $d(y)$ without changing the class $D([d(x)])$ since $d_v(d(y)) = -d(d_v(y)) = -d(d_h(x)) = d_h(d(x))$. Therefore, $d_h(y') = d_h(d(y)) = -d(d_h(y))$. $\qquad\square$

3.4. Construction for the second kind

Suppose $\lambda + \rho$ is orthogonal only to the long simple root. We work with the minimal choice for $\lambda + \rho$, which is $\lambda + \rho = [n-1, \ldots, 1, 0\,|\,]$, so $\tilde{\lambda} + \rho = [0\,|\,n-1, \ldots, 1]$.

Proposition 8. *In case of singularity of the second kind, all objects in the relative BGG resolution survive a higher direct image. The surviving degree of an object parametrized by the LS word $w = \overline{0\,d_1\ldots d_{n-1}}$ is equal to the number of digits 0 among $d_1, \ldots, d_{n-1}$. The result of the direct image in this degree corresponds to $\overline{d_1\ldots d_{n-1}\,0} \in W^{\mathfrak{p},\Sigma}$. The first page of spectral sequence (5) agrees with the singular orbit, including both the objects and the standard operators.*

Proof. The same as for Proposition 7. $\qquad\square$

We split the singular orbit into the *even* and the *odd part*, parametrized by the following subsets of the singular Hasse diagram, respectively:

$$
W^{\mathfrak{p},\Sigma}_{\epsilon} = \left\{\overline{d_1\ldots d_{n-1}\,0} \text{ with the number of digits } 1 \text{ of parity } \epsilon\right\}, \quad \epsilon \in \{\text{even}, \text{odd}\}.
$$

Recall again that the number of digits 1 in $w \in W^{\mathfrak{p},\Sigma}$ is equal to the number of negative coordinates in $w\lambda$. Both $W^{\mathfrak{p},\Sigma}_{\epsilon}$ are in bijection with the regular Hasse diagram of rank $n-2$; in each case, the bijection is:

$$
\overline{d_1\ldots d_{n-2}d_{n-1}\,0} \mapsto \overline{d_1\ldots d_{n-2}}. \tag{18}
$$

Again, the idea is to add enough arrows on the left-hand side to make (18) a directed-graph isomorphism. By inspection, the missing arrows should occur in these situations: $\overline{\ldots000} \to \overline{\ldots110}$. For constructing them, we need a crucial fact about the singular orbit of the second kind (see [9, p. 63]):

Lemma 1 (Enright-Shelton)**.** *There are no non-zero morphisms between subquotients of objects from the blocks with different parities.*

Theorem 5. *There are non-standard invariant differential operators*

$$D \colon \mathcal{O}_\mathfrak{p}(\nu) \to \mathcal{O}_\mathfrak{p}(\nu''') \tag{19}$$

for all pairs ν, ν''' in the singular orbit of the second kind given by

$$\nu = \overline{d_1 \ldots d_{n-3} \, 00\mathbf{0}} \cdot \lambda, \quad \nu''' = \overline{d_1 \ldots d_{n-3} \, 11\mathbf{0}} \cdot \lambda.$$

If λ is minimal, operator (19) is of order 3.

Proof. Take X' to be an open ball inside the big affine cell in G/P, and consider the Penrose transform over the corresponding restricted double fibration. In the relative BGG resolution, we find and fix the following sequence:

$$\mu = \overline{0 \, d_1 \ldots d_{n-3} \, 00} \cdot \tilde{\lambda} \twoheadrightarrow \mu' = \overline{0 \, d_1 \ldots d_{n-3} \, 01} \cdot \tilde{\lambda}$$
$$\downarrow$$
$$\mu'' = \overline{0 \, d_1 \ldots d_{n-3} \, 10} \cdot \tilde{\lambda} \twoheadrightarrow \mu''' = \overline{0 \, d_1 \ldots d_{n-3} \, 11} \cdot \tilde{\lambda}.$$

Let $q = 2 +$ the number of digits 0 in $d_1, \ldots, d_{n-3}$. Let also $\nu' = \overline{d_1 \ldots d_{n-3} \, 010} \cdot \lambda$ and $\nu'' = \overline{d_1 \ldots d_{n-3} \, 100} \cdot \lambda$. Consider again the Čech bi-complex described in Figure 7. By Proposition 8, we have the following:

$$H^q(\check{C}^\bullet_\mu, d_v) = \tau_*^q \mathcal{O}_\mathfrak{q}(\mu) = \mathcal{O}_\mathfrak{p}(\nu),$$
$$H^{q-1}(\check{C}^\bullet_{\mu'}, d_v) = \tau_*^{q-1} \mathcal{O}_\mathfrak{q}(\mu') = \mathcal{O}_\mathfrak{p}(\nu'),$$
$$H^{q-1}(\check{C}^\bullet_{\mu''}, d_v) = \tau_*^{q-1} \mathcal{O}_\mathfrak{q}(\mu'') = \mathcal{O}_\mathfrak{p}(\nu''),$$
$$H^{q-2}(\check{C}^\bullet_{\mu'''}, d_v) = \tau_*^{q-2} \mathcal{O}_\mathfrak{q}(\mu''') = \mathcal{O}_\mathfrak{p}(\nu'''),$$

and all other vertical cohomologies are trivial. Note that we also have a standard operator $\mathcal{O}_\mathfrak{q}(\mu') \to \mathcal{O}_\mathfrak{q}(\mu'')$ that survives on the $(q-1)$-th cohomology,

$$d \colon H^{q-1}(\check{C}^\bullet_{\mu'}, d_v) \to H^{q-1}(\check{C}^\bullet_{\mu''}, d_v), \qquad d([y]) = [d_h(y)].$$

Again, we want to define operator (19) in such a way that it agrees with a certain higher differential in the spectral sequence (5). This will be used in the proof of Theorem 7.

As before, take a cocycle $x \in \check{\mathbf{C}}^\mathbf{q}_\mu$, and find $y \in \check{C}^{q-1}_{\mu'}$ such that $d_v(y) = d_h(x)$. Then, $d_h(y) \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''}$ is also a cocycle. But since $H^{q-1}(\check{C}^\bullet_{\mu''}, d_v) \neq 0$, we cannot conclude that $d_h(y) \in \operatorname{Im} d_v$ and proceed in the same way. To overcome this, we claim that the map

$$H^q(\check{C}^\bullet_\mu, d_v) \to H^{q-1}(\check{C}^\bullet_{\mu''}, d_v) \big/ \operatorname{Im} d, \qquad [x] \mapsto [d_h(y)] + \operatorname{Im} d \tag{20}$$

is well-defined. Take $x' = x + d_v(t)$ and choose y' so that $d_h(x') = d_v(y')$. The equation (17) shows that $[y - y' + d_h(t)] \in H^{q-1}(\check{C}^\bullet_{\mu'}, d_v)$. Moreover, observe that

$d([y-y'+d_h(t)]) = [d_h(y)]-[d_h(y')] \in \operatorname{Im} d$, which proves our claim. Since obviously ν and ν'' are of different parity, Lemma 1 implies that the map in (20) is trivial. Unwinding, this means that we can find a cocycle $y'' \in \check{\mathbf{C}}_{\mu'}^{\mathbf{q}-\mathbf{1}}$ so that $d_h(y)-d_h(y'') \in \operatorname{Im} d_v$. Consequently, we can replace y by $y-y''$ and continue our diagram chase downwards since now:

$$d_v(y-y'') = d_h(x),$$
$$d_h(y-y'') = d_v(z), \text{ for some } z \in \check{C}_{\mu''}^{q-2}.$$

Finally, $d_h(z) \in \check{\mathbf{C}}_{\mu'''}^{\mathbf{q}-\mathbf{2}}$ is a cocycle. We want to define $D([x]) = [d_h(z)]$. It still

Figure 7: *Diagram chasing over the Čech bi-complex (2)*

needs to be checked that D does not depend on various choices we made. For this, we introduce another auxiliary map

$$\tilde{d}\colon \operatorname{Ker} d \to H^{q-2}(\check{C}_{\mu'''}^{\bullet}, d_v) \tag{21}$$

defined as follows. For $[y] \in \operatorname{Ker} d$ choose $z \in \check{C}_{\mu''}^{q-2}$ such that $d_v(z) = d_h(y)$. We put $\tilde{d}([y]) := [d_h(z)]$. It is easy to check that $\tilde{d}$ is well defined (in the same way as for D in the proof of Theorem 3). Since ν' and ν''' are of different parity, Lemma 1 implies that the map $\tilde{d}$ is trivial. Suppose we have $x, x' \in \check{\mathbf{C}}_{\mu}^{\mathbf{q}}$ such that $x - x' = d_v(t)$ for some $t \in \check{C}_{\mu}^{q-1}$, and consider different choices for defining D:

Equation (17) again shows that $[y - y' + d_h(t)] \in H^{q-1}(\check{C}_{\mu'}^{\bullet}, d_v)$. Observe that $d([y - y' + d_h(t)]) = [d_h(y) - d_h(y')] = [d_v(z - z')] = 0$. So, $[y - y' + d_h(t)] \in \operatorname{Ker} d$, and therefore $0 = \tilde{d}([y - y' + d_h(t)]) = [d_h(z - z')] = [d_h(z)] - [d_h(z')]$. The last conclusions are analogous to the ones in case of singularity of the first kind. $\square$

Definition 2. *In case of singularity of the second kind, the even (resp. the odd) part of the singular orbit, with all non-standard operators constructed in Theorem 5 included therein, is called an even (resp. the odd) singular BGG complex of infinitesimal character $\lambda + \rho$.*

Theorem 6. *In case of singularity of the second kind:*

1. *The singular orbit of rank n consists of two singular BGG complexes, each of which is directed-graph isomorphic to the regular one of rank $n - 2$.*

2. *Every square in the singular BGG complex anticommutes.*

3. *If we add up all objects of the singular BGG complex of the same degree, we get a cochain complex.*

Proof. Analogous to the proof of Theorem 4. $\qquad\square$

4. Exactness of the singular BGG complex

Lemma 2. *Let Z be the twistor space of the restricted double fibration (11). For any coherent sheaf $\mathcal{F}$ on Z we have:*

$$H^k(Z, \mathcal{F}) = 0, \quad \text{for all } k \geq n. \tag{22}$$

Proof. From Proposition 5 it is obvious that Z is a union of n open subsets given by the equations $y_i \neq 0$, for $i = 1, \ldots, n$. Each of those is isomorphic to $\mathbb{C}^{2n-1}$, hence affine. Cartan's theorem B and the Leray theorem imply (22). $\qquad\square$

Theorem 7. *Each singular BGG complex is exact (in positive degrees) over the big affine cell X.*

Proof. Observe that spectral sequence (5) has on the abutment E_∞ the sections over X of the cohomologies of our singular BGG complex. This follows from the construction: non-standard operators were defined exactly as the induced differentials in the hypercohomology spectral sequence, so they appear on the last page of the spectral sequence before it stabilizes. Moreover, by Cartan's theorem B, the functor $\Gamma(X, -)$ is exact, so it commutes with taking cohomology of a cochain complex.

From Propositions 7 and 8, it follows that the non-trivial elements on the first page of the spectral sequence with the smallest $p + q$ are $\overline{0 \ldots 0 1 0 \ldots 0}$ for the first kind, and $\overline{0 \ldots 0}$ and $\overline{0 \ldots 0 1}$ for the second kind respectively, and each of them has $p + q = n - 1$. So $H^i(Z, \mathcal{O}_{\mathfrak{r}}(\tilde{\lambda}))$ measure the non-exactness of the singular BGG complex, up to the shift in degree by $n - 1$. Because of Lemma 2, singular BGG complexes are exact from the degree $n - (n - 1) = 1$ on. $\qquad\square$

Example 4. *The even singular BGG complex for $\lambda + \rho = [2, 1, 0]$ consists of one non-standard operator $D \colon \mathcal{O}_{\mathfrak{p}}(\,\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}\!\!\overset{-1}{\Longleftarrow\!\!\times}\,) \to \mathcal{O}_{\mathfrak{p}}(\,\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}\!\!\overset{-3}{\Longleftarrow\!\!\times}\,)$, surjective over the big cell. By finding the maximal vector of the corresponding homomorphism, we can find the formula for D in the local coordinates on the big cell given by $\mathfrak{u}^-$:*

$$D = 4\partial_{b_1}\partial_{b_2}\partial_{b_3} - \partial_{b_1}\partial_{c_{23}}^2 - \partial_{b_2}\partial_{c_{13}}^2 - \partial_{b_3}\partial_{c_{12}}^2 + \partial_{c_{12}}\partial_{c_{13}}\partial_{c_{23}}.$$

4.1. Conjectures

The following conjecture would imply local exactness of the singular BGG complex, that is, exactness in the category of sheaves:

Conjecture 1. *The conclusion of Lemma 2 is true for the twistor space Z', where X' is a suitably chosen, but arbitrarily small open subset.*

If λ is of a higher singularity, say $|\Sigma| = s > 1$, a reasonable thing to try would be the Penrose transform over the same double fibration (10), but instead of the first, the s-th nodes should be crossed. The needed vanishing result is:

Conjecture 2. *In the above setting we have $H^k(Z, \mathcal{F}) = 0$, for any coherent sheaf $\mathcal{F}$ on Z, and $k > s(n - s) - \frac{s(s-1)}{2}$.*

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