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Spectra and fine spectra of the upper triangular band matrix $U(a; 0; b)$ on the Hahn sequence space

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Abstract. The aim of this paper is to obtain the spectra and fine spectra of the matrix $U(a; 0; b)$ on the Hahn space. Also, we explore some ideas of how to study the problem for a general form of the matrix, namely, the matrix $U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1})$, where the non-zero diagonals are the entries of an oscillatory sequence.

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Key words: upper triangular band matrix, spectrum, fine spectrum, approximate point spectrum

1. Introduction

In numerical analysis, matrices from the finite element or finite difference problems are often banded. Such matrices can be viewed as descriptions of the coupling between problem variables; the bandedness corresponds to the fact that variables are not coupled over arbitrarily large distances. Such matrices can be further divided - for instance, banded matrices exist where every element in the band is nonzero. These often arise when discretizing one-dimensional problems. Problems in higher dimensions also lead to banded matrices, in which case the band itself also tends to be sparse. For instance, a partial differential equation on a square domain (using central differences) will yield a matrix with a bandwidth equal to the square root of the matrix dimension, but inside the band only 5 diagonals are nonzero. Unfortunately, applying Gaussian elimination (or equivalently an LU decomposition) to such a matrix results in the band being filled in by many non-zero elements (see [13]). And so, the resolvent set of the band operators is important for solving such problems.

Spectral theory is one of the most useful tools in science. There are many applications of mathematics and physics which contain matrix theory, control theory, function theory, differential and integral equations, complex analysis, and quantum physics. For example, atomic energy levels are determined and therefore the frequency of a laser or the spectral signature of a star is obtained in quantum mechanics. The resolvent set of band operators is important for solving the above explanation problems. Band matrices emerge in many areas of mathematics and its applications. Tridiagonal, or more general, banded matrices are used in telecommunication system analysis, finite difference methods for solving partial differential

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equations, linear recurrence systems with non-constant coefficients, etc. (see [16]); so, it is natural to ask the question of whether one can obtain some results about the spectral decomposition of a $U(a;0;b)$ matrix.

Let X and Y be Banach spaces, and $L : X \to Y$ a bounded linear operator. By

$$R(L) = \{y \in Y : y = Lx,\ x \in X\}$$

we denote the range of L and by $B(X)$, we show the set of all bounded linear operators on X into itself.

Let $L : D(L) \to X$ be a linear operator, defined on $D(L) \subset X$, where $D(L)$ denotes the domain of L and X is a complex normed linear space. Let $L_\lambda := \lambda I - L$ for $L \in B(X)$ and $\lambda \in \mathbb{C}$, where I is the identity operator. L_λ^{-1} is known as the resolvent operator of L.

The resolvent set of L is the set of complex numbers λ of L such that L_λ^{-1} exists, is bounded and is defined on a set which is dense in X and denoted by $\rho(L, X)$. Its complement is given by $\mathbb{C}\backslash\rho(L; X)$ which is called the spectrum of L denoted by $\sigma(L, X)$.

The spectrum $\sigma(L, X)$ is a union of three disjoint sets as follows: The point spectrum $\sigma_p(L, X)$ is the set such that L_λ^{-1} does not exist. If the operator L_λ^{-1} is defined on a dense subspace of X and is unbounded, then $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L, X)$ of L. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L, X)$ of L if the operator L_λ^{-1} exists, but its domain of definition, i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X, then in this case L_λ^{-1} may be bounded or unbounded. From the above definitions we have

$$\sigma(L, X) = \sigma_p(L, X) \cup \sigma_c(L, X) \cup \sigma_r(L, X) \tag{1}$$

and

$$\sigma_p(L, X) \cap \sigma_c(L, X) = \emptyset,\ \sigma_p(L, X) \cap \sigma_r(L, X) = \emptyset,\ \sigma_r(L, X) \cap \sigma_c(L, X) = \emptyset.$$

By w we denote the space of all sequences. Well-known examples of Banach sequence spaces are the spaces ℓ_∞, c, c_0 and bv of bounded, convergent, null and bounded variation sequences, respectively. Also, by ℓ_p, bv_p we denote the spaces of all $p-$absolutely summable sequences and $p-$bounded variation sequences, respectively.

Hahn [10] introduced the space h of all sequence $x = (x_k) \in c_0$ such that

$$\sum_{k=0}^{\infty} k\,|x_{k+1} - x_k|$$

is finite. The norm

$$\|x\|_h = \sum_{k=1}^{\infty} k\,|x_{k+1} - x_k| + \sup_k |x_k|$$

was defined on the space h by Hahn [10]. Rao ([12] Proposition 2.1) defined a new norm of h given by

$$\|x\|_h = \sum_{k=1}^{\infty} k\,|x_{k+1} - x_k|\,.$$

The dual space of h is norm isomorphic to the Banach space

$$\sigma_\infty = \left\{ x = (x_k) \in w : \sup_n \frac{1}{n} \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

Many investigators studied the spectrum and fine spectrum of linear operators on some sequence spaces. In 2013, Tripathy and Saikia [14] calculated the spectrum of the Cesàro operator C_1 on $\overline{bv_0} \cap \ell_\infty$. In 2014, Paul and Tripathy [11] studied the spectrum of the operator $D(r,0,0,s)$ over the sequence spaces ℓ_p and bv_p. In 2016, Yeşilkayagil and Kirişci [17] calculated the fine spectrum of the forward difference operator on the Hahn space.

2. Fine spectrum

The upper triangular matrix $U(a;0;b)$ is an infinite matrix with non-zero diagonals that are the entries of an oscillatory sequence of the form

$$U(a;0;b) = \begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & 0 & b_2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_0 & 0 & b_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_1 & 0 & b_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & b_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix} \quad (b_0, b_1, b_2 \neq 0). \tag{2}$$

In this paper, we will calculate the spectral decomposition of the above matrix.

Lemma 1 (see [12, Proposition 10]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(h)$ from h to itself if and only if*

(*i*) $\sum_{n=1}^{\infty} n\,|a_{nk} - a_{n+1,k}|$ *converges, for each k;*

(*ii*) $\sup_k \frac{1}{k} \sum_{n=1}^{\infty} n \left| \sum_{v=1}^{k} (a_{nv} - a_{n+1,v}) \right| < \infty$;

(*iii*) $\lim_{n\to\infty} a_{nk} = 0$, *for each k.*

Theorem 1. *$U(a;0;b) : h \to h$ is a bounded linear operator if and only if $a_n + b_n = a_{n+1} + b_{n+1}$, $n = 0, 1, 2$.*

Proof. Let us use Lemma 1 for the proof.

(i) $\sum_{n=1}^{\infty} n\,|a_{nk} - a_{n+1,k}| = \begin{cases} |a_0|, & k = 1 \\ 3\,|a_1|, & k = 2 \\ (2k-1)\,|a_{k-1}| + (2k-5)\,|b_k|, & k \geq 3 \end{cases}$ is convergent.

Herein $a_x = a_y$, $b_x = b_y$ for $x \equiv y \pmod 3$.

(ii)

$$\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(a_{nv}-a_{n+1,v})\right|$$
$$=\begin{cases} \frac{\left(3\left\lfloor\frac{k-1}{3}\right\rfloor-1\right)\left\lfloor\frac{k-1}{3}\right\rfloor}{2}\left|a_0-a_1+b_0-b_1\right|+\frac{\left(3\left\lfloor\frac{k-2}{3}\right\rfloor-1\right)\left\lfloor\frac{k-2}{3}\right\rfloor}{2}\left|a_1-a_2+b_1-b_2\right| \\ +\frac{\left(3\left\lfloor\frac{k-3}{3}\right\rfloor-1\right)\left\lfloor\frac{k-3}{3}\right\rfloor}{2}\left|a_2-a_0+b_2-b_0\right|+(k-2)\left|a_{k-3}-a_{k-2}+b_{k-3}\right| \\ +(k-1)\left|a_{k-2}-a_{k-1}\right|+k\left|a_{k-1}\right|, k\geq 2 \\ \left|a_0\right|, k=1 \end{cases}$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Herein $a_x = a_y$, $b_x = b_y$ for $x \equiv y \pmod 3$. Therefore

$$\frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(a_{nv}-a_{n+1,v})\right|$$

is convergent if and only if $a_n + b_n = a_{n+1} + b_{n+1}$, $n = 0, 1, 2$. Hence

$$\sup_k \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(a_{nv}-a_{n+1,v})\right| < \infty$$

is convergent if and only if $a_n + b_n = a_{n+1} + b_{n+1}$, $n = 0, 1, 2$.

(iii) For each k, it is clear that $\lim_{n\to\infty} a_{nk} = 0$.

Thus the assertion of Lemma 1 holds. □

Lemma 2 (see [9, p.59]). *T has a dense range if and only if T^* is 1-1.*

Lemma 3 (see [9, p.60]). *T has a bounded inverse if and only if T^* is onto.*

Theorem 2. $\sigma_p(U(a;0;b),h) = \{\lambda \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| < |b_0|\,|b_1|\,|b_2|\}$.

Proof. Let λ be an eigenvalue of the operator $U(a;0;b)$. Then there exists $x \neq \theta = (0,0,0,...)$ in h such that $U(a;0;b)x = \lambda x$. Then we have

$$\begin{cases} x_{6n} &= q^n x_0, \\ x_{6n+1} &= q^n x_1, \\ x_{6n+2} &= \frac{\lambda-a_0}{b_0}q^n x_0, \\ x_{6n+3} &= \frac{\lambda-a_1}{b_1}q^n x_1, \qquad\qquad n\geq 0 \\ x_{6n+4} &= \frac{(\lambda-a_0)(\lambda-a_2)}{b_0 b_2}q^n x_0, \\ x_{6n+5} &= \frac{(\lambda-a_0)(\lambda-a_1)}{b_0 b_1}q^n x_1. \end{cases}$$

where $q = \frac{(\lambda-a_0)(\lambda-a_1)(\lambda-a_2)}{b_0 b_1 b_2}$. Thus we get

$$|x_{6k+r} - x_{6k+r+1}| = |K_r|\,|q|^k\,,\ r = \overline{0,5},$$

where

$$K_r := \begin{cases} x_0 - x_1, & r = 0 \\ x_1 - \dfrac{\lambda - a_0}{b_0} x_0, & r = 1 \\ \dfrac{\lambda - a_0}{b_0} x_0 - \dfrac{\lambda - a_1}{b_1} x_1, & r = 2 \\ \dfrac{\lambda - a_1}{b_1} x_1 - \dfrac{(\lambda - a_0)(\lambda - a_2)}{b_0 b_2} x_0, & r = 3 \\ \dfrac{(\lambda - a_0)(\lambda - a_2)}{b_0 b_2} x_0 - \dfrac{(\lambda - a_0)(\lambda - a_1)}{b_0 b_1} x_1, & r = 4 \\ \dfrac{(\lambda - a_0)(\lambda - a_1)}{b_0 b_1} x_1 - q x_0, & r = 5 \end{cases}$$

and so

$$(6k + r) \left| x_{6k+r} - x_{6k+r+1} \right| = |K_r| (6k + r) |q|^k \ , \ r = \overline{0,5}.$$

Therefore we have

$$\begin{aligned} \sum_{n=1}^{\infty} n \left| x_n - x_{n+1} \right| &= \sum_{k=1}^{\infty} (6k + r) \left| x_{6k+r} - x_{6k+r+1} \right| \\ &= |K_r| \sum_{k=1}^{\infty} (6k + r) |q|^k . \end{aligned}$$

Since

$$\lim_{k \to \infty} \frac{(6k + r + 6) |q|^{k+1}}{(6k + r) |q|^k} = |q|$$

from D'Alembert's ratio test, the series

$$\sum_{k=1}^{\infty} (6k + r) |q|^k$$

is convergent if and only if $|q| < 1$ and hence, $x = (x_n) \in h$ if and only if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|$. Therefore,

$$\sigma_p(U(a;0;b), h) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\} .$$

□

If $T : h \longmapsto h$ is a bounded linear operator represented by a matrix A, then it is known that the adjoint operator $T^* : h^* \longmapsto h^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space h^* of h is isometrically isomorphic to the Banach space $\sigma_\infty = \left\{ x = (x_k) \in w : \sup_n \frac{1}{n} \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}$.

Theorem 3. $\sigma_p(U(a;0;b)^*, h^* \cong \sigma_\infty) = \emptyset$.

Proof. Let λ be an eigenvalue of the operator $U(a;0;b)^*$. Then there exists $x \neq \theta = (0,0,0,...)$ in σ_∞ such that $U(a;0;b)^* x = \lambda x$.
Then, we have

$$
\begin{aligned}
a_0 x_0 &= \lambda x_0 && (3)\\
a_1 x_1 &= \lambda x_1 && (4)\\
b_0 x_0 + a_2 x_2 &= \lambda x_2 && (5)\\
b_1 x_1 + a_0 x_3 &= \lambda x_3 && (6)\\
b_2 x_2 + a_1 x_4 &= \lambda x_4 && (7)\\
b_0 x_3 + a_2 x_5 &= \lambda x_5 && (8)\\
b_1 x_4 + a_0 x_6 &= \lambda x_6 && (9)\\
b_2 x_5 + a_1 x_7 &= \lambda x_7 && (10)\\
b_0 x_6 + a_2 x_8 &= \lambda x_8 && (11)\\
&\vdots
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&n = 3k, \quad b_0 x_n + a_2 x_{n+2} = \lambda x_{n+2} && (12)\\
&n = 3k+1, \quad b_1 x_n + a_0 x_{n+2} = \lambda x_{n+2} && (13)\\
&n = 3k+2, \quad b_2 x_n + a_1 x_{n+2} = \lambda x_{n+2}
\end{aligned}
$$

Let $x_0 \neq 0$; then we obtain that $\lambda = a_0$ from (3), $x_1 = 0$ from (6), $x_4 = 0$ from (9), $x_2 = 0$ from (7) and $x_0 = 0$ from (5). But this contradicts our assumption.

Now let $x_0 = 0$ and $x_1 \neq 0$; then we obtain that $\lambda = a_1$ from (4), $x_2 = 0$ from (7), $x_5 = 0$ from (10), $x_3 = 0$ from (8), $x_1 = 0$ from (6). But this contradicts with our assumption.

Similarly let $x_0 = 0$, $x_1 = 0$ and $x_2 \neq 0$; then we obtain that $\lambda = a_1$ from (5), $x_6 = 0$ from (11), $x_4 = 0$ from (9), $x_2 = 0$ from (7). But this contradicts with our assumption.

Finally, let x_{3k+1} be the first non-zero of the sequence (x_n). If $n = 3k$, then from (12) we have $\lambda = a_2$. Again from (12) for $n = 3k+3$ we have $b_0 x_{3k+3} + a_2 x_{3k+5} = a_2 x_{3k+5}$; then we get $x_{3k+3} = 0$. But from (13) for $n = 3k+1$ we have $b_1 x_{3k+1} + a_0 x_{3k+3} = a_2 x_{3k+3}$, we have $x_{3k+1} = 0$, a contradiction.

Similarly, if x_{3k} or $x_{3k} + 2$ is the first non-zero of the sequence (x_n), we get a contradiction.
Hence, $\sigma_p(U(a;0;b)^*, c_0^* \cong \ell_1) = \emptyset$. □

Theorem 4. $\sigma_r(U(a;0;b), h) = \emptyset$.

Proof. Since $\sigma_r(A,h) = \sigma_p(A^*, \sigma_\infty) \backslash \sigma_p(A,h)$, Theorems 2 and 3 give us the required result. □

Lemma 4.

$$\sum_{k=1}^{n} \left(\sum_{i=0}^{k-1} a_i b_{ki} \right) = \sum_{i=0}^{n-1} a_i \left(\sum_{k=i+1}^{n} b_{ki} \right),$$

where (a_k) *and* (b_{nk}) *are real numbers.*

Proof. It is clear. □

Theorem 5. $\sigma_c(U(a;0;b),h) = \{\lambda \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| = |b_0|\,|b_1|\,|b_2|\}$ *and* $\sigma(U(a;0;b),h) = \{\lambda \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| \le |b_0|\,|b_1|\,|b_2|\}$.

Proof. Let us take $y = (y_n) \in \sigma_\infty$ such that $(U(a;0;b) - \lambda I)^* x = y$ for some $x = (x_n)$. Then we get a system of linear equations:

$$\begin{aligned}
(a_0 - \lambda)x_0 &= y_0\\
(a_1 - \lambda)x_1 &= y_1\\
&\vdots\\
b_0 x_{3n} + (a_2 - \lambda)x_{3n+2} &= y_{3n+2}, \quad n \ge 0\\
b_1 x_{3n+1} + (a_0 - \lambda)x_{3n+3} &= y_{3n+3}\\
b_2 x_{3n+2} + (a_1 - \lambda)x_{3n+4} &= y_{3n+4}\\
&\vdots
\end{aligned}$$

Solving these equations, we have

$$x_{2n+t} = \frac{1}{a_{2n+t} - \lambda}\left[y_{2n+t} + \sum_{k=0}^{n-1}(-1)^{n-k} y_{2k+t} \prod_{\nu=1}^{n-k} \frac{b_{2n-2\nu+t}}{a_{2n-2\nu+t} - \lambda}\right],$$
$$t = 0,1; n = 1,2,\ldots$$

Herein $a_x = a_y$, $b_x = b_y$ for $x \equiv y \pmod 3$.
Therefore we get

$$\begin{aligned}
\frac{1}{2n+t}\left|\sum_{k=0}^{2n+t} x_k\right| &= \frac{1}{2n+t}\left|x_0 + x_1 + x_2 + x_3 + \cdots + x_{2n+t}\right|\\
&= \frac{1}{2n+t}\left|x_0 + x_1 + \sum_{k=1}^{n} x_{2k+t}\right|\\
&\le \frac{1}{2n+t}\left|\frac{y_0}{a_0 - \lambda} + \frac{y_1}{a_1 - \lambda}\right|\\
&\quad + \frac{1}{2n+t}\left|\sum_{k=1}^{n}\frac{1}{a_{2k+t} - \lambda}\left[y_{2k+t} + \sum_{i=0}^{k-1}(-1)^{k-i} y_{2i+t} \prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t} - \lambda}\right]\right|
\end{aligned}$$

$$\leq \frac{1}{2n+t}\left|\frac{y_0}{a_0-\lambda}+\frac{y_1}{a_1-\lambda}\right|$$
$$+\frac{1}{2n+t}\left|\sum_{k=1}^{n}\frac{y_{2k+t}}{a_{2k+t}-\lambda}\right|$$
$$+\frac{1}{2n+t}\left|\sum_{k=1}^{n}\frac{1}{a_{2k+t}-\lambda}\sum_{i=0}^{k-1}(-1)^{k-i}y_{2i+t}\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda}\right|$$
$$\leq \frac{1}{2n+t}\left|\frac{y_0}{a_0-\lambda}+\frac{y_1}{a_1-\lambda}\right|$$
$$+\max_{m=0}^{2}\left|\frac{1}{a_m-\lambda}\right|\left[\frac{1}{2n+t}\left|\sum_{k=1}^{n}y_{2k+t}\right|+\frac{1}{2n+t}\left|\sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^{k-i}y_{2i+t}\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda}\right|\right].$$

Thus

$$\frac{1}{2n+t}\left|\sum_{k=0}^{2n+t}x_k\right|$$
$$\leq \max_{m=0}^{2}\left|\frac{1}{a_m-\lambda}\right|\left[2\left\|y\right\|_{\sigma\infty}+\frac{1}{2n+t}\left|\sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^{k-i}y_{2i+t}\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda}\right|\right].$$

Now, we consider the sum

$$\frac{1}{2n+t}\left|\sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^{k-i}y_{2i+t}\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda}\right|.$$

In Lemma 4, if we take $a_i = y_{2i+t}$ and $b_{ki} = (-1)^{k-i}\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda}$, then we have

$$\frac{1}{2n+t}\left|\sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^{k-i}y_{2i+t}\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda}\right|$$
$$=\frac{1}{2n+t}\left|\sum_{i=0}^{n-1}y_{2i+t}\sum_{k=i+1}^{n}(-1)^{k-i}\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda}\right|.$$

Also, since $\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda} = Mp^{k-i}$, where M is constant and

$$p=\left(\frac{b_2b_1b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)}\right)^{1/3},$$

the last equation turns into the sum $\frac{|M|}{2n+t}\left|\sum_{i=0}^{n-1}y_{2i+t}\sum_{k=i+1}^{n}p^{k-i}\right|$. Then

$$\frac{|M|}{2n+t}\left|\sum_{i=0}^{n-1}y_{2i+t}\sum_{k=i+1}^{n}(-1)^{k-i}p^{k-i}\right| = \left|\frac{Mp}{1-p}\right|\frac{1}{2n+t}\left|\sum_{i=0}^{n-1}y_{2i+t}\left(1-(-p)^{n-i}\right)\right|$$

$$= \left|\tfrac{Mp}{1-p}\right| \tfrac{1}{2n+t} \left|\sum_{i=0}^{n-1} y_{2i+t} - \sum_{i=0}^{n-1} y_{2i+t} (-p)^{n-i}\right|$$
$$\leq \left|\tfrac{Mp}{1-p}\right| \tfrac{1}{2n+t} \left|\sum_{i=0}^{n-1} y_{2i+t}\right| + \left|\tfrac{Mp}{1-p}\right| \tfrac{1}{2n+1} \left|\sum_{i=0}^{n-1} y_{2i+t} (-p)^{n-i}\right|.$$

Hence

$$\begin{aligned} &\frac{|M|}{2n+t} \left|\sum_{i=0}^{n-1} y_{2i+t} \sum_{k=i+1}^{n} (-1)^{k-i} p^{k-i}\right| \\ &\qquad \leq \left|\frac{Mp}{1-p}\right| \|y\|_{\sigma\infty} + \left|\frac{Mp^{n+1}}{1-p}\right| \frac{1}{2n+t} \left|\sum_{i=0}^{n-1} y_{2i+t} (-p)^{-i}\right|. \end{aligned} \tag{14}$$

If we take $a_i = y_{2i+t}$, $b_i = (-p)^{-i}$ and apply Abel's partial summation formula to the sum $\sum_{i=0}^{n-1} \frac{y_{2i+t}}{(-p)^i}$, we obtain

$$\sum_{i=0}^{n-1} \frac{y_{2i+t}}{(-p)^i} = \frac{1}{(-p)^n} \sum_{i=0}^{n} y_{2i+t} + \sum_{i=0}^{n-2} \frac{p+1}{(-p)^{i+1}} \sum_{k=0}^{i} y_{2k+t}$$

since $s_n = \sum_{i=0}^{n} y_{2i+t}$, $\Delta b_i = \frac{p+1}{(-p)^{i+1}}$. Thus

$$\begin{aligned} &\left|\frac{Mp^{n+1}}{1-p}\right| \frac{1}{2n+t} \left|\sum_{i=0}^{n-1} y_{2i+t} (-p)^{-i}\right| \\ &= \left|\frac{Mp^{n+1}}{1-p}\right| \frac{1}{2n+t} \left|\frac{1}{(-p)^n} \sum_{i=0}^{n} y_{2i+t} + \sum_{i=0}^{n-2} \frac{p+1}{(-p)^{i+1}} \sum_{k=0}^{i} y_{2k+t}\right| \\ &\leq \left|\frac{Mp}{1-p}\right| \frac{1}{2n+t} \left|\sum_{i=0}^{n} y_{2i+t}\right| + \left|\frac{M(p+1)p^{n+1}}{1-p}\right| \frac{1}{2n+t} \left|\sum_{i=0}^{n-2} \frac{1}{(-p)^{i+1}} \sum_{k=0}^{i} y_{2k+t}\right| \\ &\leq \left|\frac{Mp}{1-p}\right| \frac{1}{2n+t} \left|\sum_{i=0}^{n} y_{2i+t}\right| + \left|\frac{M(p+1)p^{n}}{1-p}\right| \sum_{i=0}^{n-2} \frac{1}{|p|^i} \frac{1}{2n+t} \left|\sum_{k=0}^{i} y_{2k+t}\right| \\ &\leq \left|\frac{Mp}{1-p}\right| \|y\|_{\sigma\infty} + \left|\frac{M(p+1)p^{n}}{1-p}\right| \|y\|_{\sigma\infty} \sum_{i=0}^{n-2} \frac{1}{|p|^i} \end{aligned}$$

and we get

$$\left|\frac{Mp^{n+1}}{1-p}\right| \frac{1}{2n+t} \left|\sum_{i=0}^{n-1} y_{2i+t} (-p)^{-i}\right| \leq \left[1 + (p+1)\, p \frac{|p|^{n-1} - 1}{|p| - 1}\right] \left|\frac{Mp}{1-p}\right| \|y\|_{\sigma\infty}. \tag{15}$$

Replacing (15) in (14), we have

$$\begin{aligned} &\frac{|M|}{2n+t} \left|\sum_{i=0}^{n-1} y_{2i+t} \sum_{k=i+1}^{n} (-1)^{k-i} p^{k-i}\right| \\ &\qquad \leq 2 \left|\frac{Mp}{1-p}\right| \|y\|_{\sigma\infty} + \left|\frac{M(p+1)p^2}{1-p}\right| \|y\|_{\sigma\infty} \frac{|p|^{n-1} - 1}{|p| - 1}. \end{aligned} \tag{16}$$

Replacing (16) in (14), we have

$$\begin{aligned}\frac{1}{2n+t}&\left|\sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^{k-i}y_{2i+t}\prod_{\nu=1}^{k-i}\frac{b_{2k-2\nu+t}}{a_{2k-2\nu+t}-\lambda}\right|\\&\leq\left[2+(p+1)\,p\frac{|p|^{n-1}-1}{|p|-1}\right]\left|\frac{Mp}{1-p}\right|\|y\|_{\sigma_\infty}.\end{aligned}\tag{17}$$

Finally, replacing (17) in (14), we get

$$\frac{1}{2n+t}\left|\sum_{k=0}^{2n+t}x_k\right|\leq\|y\|_{\sigma_\infty}\max_{m=0}^{2}\left|\frac{1}{a_m-\lambda}\right|\left\{2+2\left|\frac{Mp}{1-p}\right|+\left|\frac{M\,(p+1)\,p^2}{1-p}\right|\frac{|p|^{n-1}-1}{|p|-1}\right\}.$$

Since

$$y=(y_n)\in\sigma_\infty, x=(x_n)\in\sigma_\infty \text{ iff } |p|=\left|\frac{b_2b_1b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)}\right|^{1/3}<1.$$

Consequently, if for $\lambda\in\mathbb{C}$, $|a_2-\lambda|\,|a_1-\lambda|\,|a_0-\lambda|>|b_2|\,|b_1|\,|b_0|$, then $(x_n)\in\sigma_\infty$. Therefore, the operator $(U(a;0;b)-\lambda I)^*$ is onto if $|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|>|b_0|\,|b_1|\,|b_2|$. Then by Lemma 3, $U(a;0;b)-\lambda I$ has a bounded inverse if $|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|>|b_0|\,|b_1|\,|b_2|$. So,

$$\sigma_c(U(a;0;b),h)\subseteq\{\lambda\in\mathbb{C}:|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|\leq|b_0|\,|b_1|\,|b_2|\}.$$

Since $\sigma(L,h)$ is the disjoint union of $\sigma_p(L,h)$, $\sigma_r(L,h)$ and $\sigma_c(L,h)$, therefore

$$\sigma(U(a;0;b),h)\subseteq\{\lambda\in\mathbb{C}:|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|\leq|b_0|\,|b_1|\,|b_2|\}.$$

By Theorem 2, we get

$$\{\lambda\in\mathbb{C}:|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|<|b_0|\,|b_1|\,|b_2|\}=\sigma_p(U(a;0;b),h)\subset\sigma(U(a;0;b),h).$$

Since $\sigma(L,h)$ is closed and thus,

$$\begin{aligned}\overline{\{\lambda\in\mathbb{C}:|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|<|b_0|\,|b_1|\,|b_2|\}}&\subset\overline{\sigma(U(a;0;b),h)}\\&=\sigma(U(a;0;b),h)\end{aligned}$$

and

$$\{\lambda\in\mathbb{C}:|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|\leq|b_0|\,|b_1|\,|b_2|\}\subset\sigma(U(a;0;b),h).$$

Hence,

$$\sigma(U(a;0;b),h)=\{\lambda\in\mathbb{C}:|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|\leq|b_0|\,|b_1|\,|b_2|\}$$

and so

$$\sigma_c(U(a;0;b),h)=\{\lambda\in\mathbb{C}:|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2|=|b_0|\,|b_1|\,|b_2|\}.$$

□

3. Subdivision of the spectrum

The spectrum $\sigma(L,X)$ is partitioned into three sets which are not necessarily disjoint as follows:

If there exists a sequence (x_n) in X such that $\|x_n\| = 1$ and $\|Lx_n\| \to 0$ as $n \to \infty$, then (x_n) is called a Weyl sequence for L.

We call the set

$$\sigma_{ap}(L,X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - L\}$$

the approximate point spectrum of L. Moreover, the set

$$\sigma_{\delta}(L,X) := \{\lambda \in \sigma(L,X) : \lambda I - L \text{ is not surjective}\}$$

is called a defect spectrum of L. Finally, the set

$$\sigma_{co}(L,X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - L)} \neq X\}$$

is called compression spectrum in the literature.

The following proposition is quite useful for calculating the separation of the spectrum of the linear operator in Banach spaces.

Proposition 1 (see [1, Proposition 1.3]). *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*

(*a*) $\sigma(L^*,X^*) = \sigma(L,X)$, (*b*) $\sigma_c(L^*,X^*) \subseteq \sigma_{ap}(L,X)$,

(*c*) $\sigma_{ap}(L^*,X^*) = \sigma_{\delta}(L,X)$, (*d*) $\sigma_{\delta}(L^*,X^*) = \sigma_{ap}(L,X)$,

(*e*) $\sigma_p(L^*,X^*) = \sigma_{co}(L,X)$, (*f*) $\sigma_{co}(L^*,X^*) \supseteq \sigma_p(L,X)$,

(*g*) $\sigma(L,X) = \sigma_{ap}(L,X) \cup \sigma_p(L^*,X^*) = \sigma_p(L,X) \cup \sigma_{ap}(L^*,X^*)$.

Goldberg's Classification of Spectrum

If $T \in B(X)$, then there are three cases for $R(T)$:

(I) $R(T) = X$, (II) $\overline{R(T)} = X$, but $R(T) \neq X$, (III) $\overline{R(T)} \neq X$
and three cases for T^{-1}:

(1) T^{-1} exists and is continuous, (2) T^{-1} exists but is discontinuous, (3) T^{-1} does not exist.

If these cases are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3 (see [9]).

$\sigma(L,X)$ can be divided into subdivisions $I_2\sigma(L,X) = \emptyset$, $I_3\sigma(L,X)$, $II_2\sigma(L,X)$, $II_3\sigma(L,X)$, $III_1\sigma(L,X)$, $III_2\sigma(L,X)$, $III_3\sigma(L,X)$. For example, if $T = \lambda I - L$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(L,X)$.

By the definitions given above and the introduction, we can write following table:

		1	2	3
		L_λ^{-1} exists and is bounded	L_λ^{-1} exists and is unbounded	L_λ^{-1} does not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L, X)$	–	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L, X)$	$\lambda \in \sigma_c(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$

Table 1:

Section 2 mentioned articles concerned with the decomposition of the spectrum defined by Goldberg. However, in [4], Durna and Yildirim investigated a subdivision of the spectra for factorable matrices on c_0, and in [2], Başar, Durna and Yildirim investigated subdivisions of the spectra for a generalized difference operator on the sequence spaces c_0 and c and in [5] Durna, have studied subdivision of the spectra for the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c. Moreover, in [3], Das calculated the spectrum and fine spectrum of the upper triangular matrix $U(r_1, r_2; s_1, s_2)$ over the sequence space c_0. In [8], El-Shabrawy and Abu-Janah determined spectra and fine spectra of a generalized difference operator $B(r, s)$ on the sequence spaces bv_0 and h, in [18], Yildirim and Durna examined the spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on ℓ_p, $(1 < p < \infty)$. In [15], the fine spectrum of the upper triangular matrix $U(r, 0, 0, s)$ over the squence spaces c_0 and c was studied by Tripathy and Das. In 2018, Durna et al. [6] studied a partition of the spectra for the generalized difference operator $B(r, s)$ on the sequence space cs, in [7], Durna studied a subdivision of spectra for some lower triangular doule-band matrices as operators on c_0 and in [19], Yildirim et al. studied the spectrum and fine spectrum of generalized Rhaly-Cesàro matrices on c_0 and c.

In this section, we will take $a_0 = a_1 = a_2 = a$ and $b_0 = b_1 = b_2 = b$.

Lemma 5.

$$\sum_{n=2}^{\infty} \left(\sum_{k=0}^{n-2} a_n b_{nk} \right) = \sum_{k=0}^{\infty} \left(\sum_{n=2+k}^{\infty} a_n b_{nk} \right),$$

where (a_k) *and* (b_{nk}) *are positive real numbers.*

Proof. It is clear. □

Theorem 6. *If* $|\lambda - a| < |b|$*, then* $\lambda \in I_3\sigma(U(a; 0; b), h)$.

Proof. Suppose that $|\lambda - a| < |b|$ and so from Theorem 2, $\lambda \in \sigma_p(U(a; 0; b), h)$. Hence, λ satisfies Golberg's condition 3. We shall show that $U(a; 0; b) - \lambda I$ is onto

when $|\lambda - a| < |b|$.
Let us take $y = (y_n) \in h$ such that $(U(a;0;b) - \lambda I)x = y$ for $x = (x_n)$. Then

$$(a - \lambda)\, x_k + b x_{k+2} = y_k \ , \ \ k \geq 0$$

Calculating x_k, we get

$$x_{2n+t} = \frac{1}{b}\left[y_{2n-2+t} + \sum_{k=0}^{n-2} y_{2k+t} \left(\frac{\lambda - a}{b} \right)^{n-k-1} \right] + x_t \left(\frac{\lambda - a}{b} \right)^{n}, \qquad (18)$$
$$t = 0, 1; \ \ n = 2, 3, \ldots$$

We must show that $x = (x_k) \in h$. Since

$$\sum_{n=1}^{\infty} n\, |x_n - x_{n+1}| = \sum_{n=1}^{\infty} (2n-1)\, |x_{2n-1} - x_{2n}| + \sum_{n=1}^{\infty} 2n\, |x_{2n} - x_{2n+1}|,$$

let us investigate whether the series $\sum_{n=1}^{\infty} 2n\, |x_{2n} - x_{2n+1}|$ is convergent. Similarly, we can show that the series $\sum_{n=1}^{\infty} (2n-1)\, |x_{2n-1} - x_{2n}|$ is convergent. Since

$$x_{2n} - x_{2n+1} = \frac{1}{b}\left[y_{2n-2} - y_{2n-1} + \sum_{k=0}^{n-2} (y_{2k} - y_{2k+1}) \left(\frac{\lambda - a}{b} \right)^{n-k-1} \right] + (x_0 - x_1) \left(\frac{\lambda - a}{b} \right)^{n},$$

we have

$$2n\, |x_{2n} - x_{2n+1}| \leq \frac{1}{|b|}\left[2n\, |y_{2n-2} - y_{2n-1}| + 2n \sum_{k=0}^{n-2} |y_{2k} - y_{2k+1}| \left| \frac{\lambda - a}{b} \right|^{n-k-1} \right] + |x_0 - x_1| \left| \frac{\lambda - a}{b} \right|^{n}.$$

Therefore

$$\sum_{n=1}^{\infty} 2n\, |x_{2n} - x_{2n+1}| \leq \frac{|x_2 - x_3|}{|b|} + \frac{1}{|b|} \sum_{n=2}^{\infty} 2n\, |y_{2n-2} - y_{2n-1}| + \frac{1}{|b|} \sum_{n=1}^{\infty} 2n \sum_{k=0}^{n-2} |y_{2k} - y_{2k+1}| \left| \frac{\lambda - a}{b} \right|^{n-k-1} + |x_0 - x_1| \sum_{n=2}^{\infty} \left| \frac{\lambda - a}{b} \right|^{n}. \qquad (19)$$

Since $|\lambda - a| < |b|$, the series $\sum_{n=2}^{\infty} \left| \frac{\lambda - a}{b} \right|^{n}$ is convergent. And also, since $y = (y_n) \in h$, the series $\sum_{n=2}^{\infty} 2n\, |y_{2n-2} - y_{2n-1}|$ is convergent. Hence, it is enough to show that the series

$$\sum_{n=1}^{\infty} 2n \sum_{k=0}^{n-2} |y_{2k} - y_{2k+1}| \left| \frac{\lambda - a}{b} \right|^{n-k-1}$$

is convergent, for the series $\sum_{n=1}^{\infty} 2n |x_{2n} - x_{2n+1}|$ is to be convergent. From Lemma 5, we get

$$\begin{aligned} \sum_{n=1}^{\infty} 2n \sum_{k=0}^{n-2} |y_{2k} - y_{2k+1}| \left| \frac{\lambda - a}{b} \right|^{n-k-1} &= \sum_{k=0}^{\infty} \sum_{n=k+2}^{\infty} 2n |y_{2k} - y_{2k+1}| \left| \frac{\lambda - a}{b} \right|^{n-k-1} \\ &= \sum_{k=0}^{\infty} 2 |y_{2k} - y_{2k+1}| \sum_{n=k+2}^{\infty} n \left| \frac{\lambda - a}{b} \right|^{n-k-1}. \end{aligned} \tag{20}$$

Setting $r := \left| \frac{\lambda - a}{b} \right|$. Since $\lambda \in \sigma_p(U(a_0, a_1, a_2;), h)$, $r < 1$. Thus

$$\begin{aligned} \sum_{n=k+2}^{\infty} nr^{n-k-1} &= \sum_{n=1}^{\infty} (n + k + 1) r^n = \sum_{n=1}^{\infty} nr^n + (k+1) \sum_{n=1}^{\infty} r^n \\ &= \frac{r}{(1-r)^2} + (k+1) \frac{r}{1-r} \end{aligned}$$

is valid. Replacing this in (20), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} 2n \sum_{k=0}^{n-2} |y_{2k} - y_{2k+1}| r^{n-k-1} \\ &\quad = \sum_{k=0}^{\infty} 2 |y_{2k} - y_{2k+1}| \left(\frac{r}{(1-r)^2} + (k+1) \frac{r}{1-r} \right) \\ &\quad = \frac{2r}{(1-r)^2} \sum_{k=0}^{\infty} |y_{2k} - y_{2k+1}| + \frac{r}{1-r} \sum_{k=0}^{\infty} 2(k+1) |y_{2k} - y_{2k+1}|. \end{aligned}$$

Thus since $y = (y_n) \in h$, the series

$$\sum_{k=0}^{\infty} 2(k+1) |y_{2k} - y_{2k+1}|$$

is convergent. Also since $|y_{2k} - y_{2k+1}| \leq (k+1) |y_{2k} - y_{2k+1}|$ for $k \in \mathbb{N}$, $y = (y_n) \in h$ implies that the series

$$\sum_{k=0}^{\infty} |y_{2k} - y_{2k+1}|$$

is convergent from the comparison test. Therefore from (19), $y = (y_n) \in h$ and $\lambda \in \sigma_p(U(a_0, a_1, a_2;), h)$ imply that the series

$$\sum_{n=1}^{\infty} 2n |x_{2n} - x_{2n+1}|$$

is convergent.

Finally, if we show that $\lim_{n\to\infty} x_n = 0$, there exists $x = (x_n) \in h$ for all $y = (y_n) \in h$. Since $y = (y_n) \in h$, $\lim_{n\to\infty} y_n = 0$ and since every convergent sequence is bounded, there exists $N > 0$ such that $|y_n| \leq N$ for all $n \in \mathbb{N}$. Also, since $\lambda \in \sigma_p(U(a_0, a_1, a_2;), h)$ implies $r < 1$, from (18), we have

$$\begin{aligned}\lim_{n\to\infty} x_{2n+t} &= \lim_{n\to\infty} \frac{1}{b} y_{2n-2+t} + \lim_{n\to\infty} \frac{1}{b} \sum_{k=0}^{n-2} y_{2k+t} \left(\frac{\lambda - a}{b}\right)^{n-k-1} + x_t \lim_{n\to\infty} \left(\frac{\lambda - a}{b}\right)^{n} \\ &= \lim_{n\to\infty} \frac{1}{b} \sum_{k=0}^{n-2} y_{2k+t} \left(\frac{\lambda - a}{b}\right)^{n-k-1}. \end{aligned} \tag{21}$$

Hence we have

$$\begin{aligned} 0 &\leq \left| \lim_{n\to\infty} \frac{1}{b} \sum_{k=0}^{n-2} y_{2k+t} \left(\frac{\lambda - a}{b}\right)^{n-k-1} \right| \leq \lim_{n\to\infty} \frac{1}{|b|} \sum_{k=0}^{n-2} |y_{2k+t}| \left|\frac{\lambda - a}{b}\right|^{n-k-1} \\ &= \frac{1}{|b|} \lim_{n\to\infty} \sum_{k=0}^{n-2} |y_{2k+t}| \left|\frac{\lambda - a}{b}\right|^{n-k-1} = \frac{1}{|b|} \lim_{n\to\infty} \frac{\sum_{k=0}^{n-2} |y_{2k+t}| \left|\frac{b}{\lambda - a}\right|^{k}}{\left|\frac{b}{\lambda - a}\right|^{n-1}}. \end{aligned} \tag{22}$$

If we take

$$a_n = \sum_{k=0}^{n-2} |y_{2k+t}| \left|\frac{b}{\lambda - a}\right|^{k} \text{ and } b_n = \left|\frac{b}{\lambda - a}\right|^{n-1},$$

then (b_n) holds conditions of the Stolz theorem since $\left|\frac{b}{\lambda - a}\right| > 1$. Therefore, from (22), we obtain

$$\begin{aligned} 0 &\leq \frac{1}{|b|} \lim_{n\to\infty} \frac{\sum_{k=0}^{n-2} |y_{2k+t}| \left|\frac{b}{\lambda - a}\right|^{k}}{\left|\frac{b}{\lambda - a}\right|^{n-1}} = \frac{1}{|b|} \lim_{n\to\infty} \frac{|y_{2n-2+t}| \left|\frac{b}{\lambda - a}\right|^{n-1}}{\left|\frac{b}{\lambda - a}\right|^{n-1} \left(\left|\frac{b}{\lambda - a}\right| - 1\right)} \\ &= \frac{1}{|b| \left(\left|\frac{b}{\lambda - a}\right| - 1\right)} \lim_{n\to\infty} |y_{2n-2+t}| = 0. \end{aligned}$$

Thus from the sandwich theorem and (21), $\lim_{n\to\infty} x_{2n+t} = 0$, $t = 0, 1$ and so $\lim_{n\to\infty} x_n = 0$. Thus, $(x_n) \in h$ iff $|\lambda - a| < |b|$. Therefore, $U(a;0;b) - \lambda I$ is onto. So, $\lambda \in I$. Hence we get the required result. $\square$

Corollary 1. $III_1\sigma(U(a;0;b), h) = III_2\sigma(U(a;0;b), h) = \emptyset$.

Proof. Since $\sigma_r(L, h) = III_1\sigma(L, h) \cup III_2\sigma(L, h)$ from Table 1, the required result is obtained from Theorem 4 with $a_0 = a_1 = a_2 = a$ and $b_0 = b_1 = b_2 = b$. $\square$

Corollary 2. $II_3\sigma(U(a;0;b),h) = III_3\sigma(U(a;0;b),h) = \emptyset$.

Proof. Since $\sigma_p(L,h) = I_3\sigma(L,h) \cup II_3\sigma(L,h) \cup III_3\sigma(L,h)$ from Table 1, the required result is obtained from Theorem 2 and Theorem 6 with $a_0 = a_1 = a_2 = a$ and $b_0 = b_1 = b_2 = b$. □

Theorem 7. *It holds:*

(*a*) $\sigma_{ap}(U(a;0;b),h) = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}$;

(*b*) $\sigma_\delta(U(a;0;b),h) = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\}$;

(*c*) $\sigma_{co}(U(a;0;b),h) = \emptyset$.

Proof. (a): From Table 1, we get

$$\sigma_{ap}(L,h) = \sigma(L,h) \backslash III_1\sigma(L,h).$$

And so $\sigma_{ap}(U(a;0;b),h) = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}$ from Corollary 1.

(b): From Table 1, we have

$$\sigma_\delta(L,h) = \sigma(L,h) \backslash I_3\sigma(L,h).$$

So using Theorem 5 and 6 with $a_0 = a_1 = a_2 = a$ and $b_0 = b_1 = b_2 = b$, we get the required result.

(c): By Proposition 1 (e), we have

$$\sigma_p(L^*,h^*) = \sigma_{co}(L,h).$$

Using Theorem 3 with $a_0 = a_1 = a_2 = a$ and $b_0 = b_1 = b_2 = b$, we get the required result. □

Corollary 3. *It holds:*

(*a*) $\sigma_{ap}(U(a;0;b)^*, h^* \cong \sigma_\infty) = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\}$;

(*b*) $\sigma_\delta(U(a;0;b)^*, h^* \cong \sigma_\infty) = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}$.

Proof. Using Proposition 1 (c) and (d), we have

$$\sigma_{ap}(U(a;0;b)^*, h^* \cong \sigma_\infty) = \sigma_\delta(U(a;0;b),h)$$

and

$$\sigma_\delta(U(a;0;b)^*, h^* \cong \sigma_\infty) = \sigma_{ap}(U(a;0;b),h).$$

Using Theorem 7 (a) and (b) with $a_0 = a_1 = a_2 = a$ and $b_0 = b_1 = b_2 = b$, we get the required results. □

4. Results

We can generalize our operator

$$U(a_0,a_1,\dots,a_{n-1};0;b_0,b_1,\dots,b_{n-1}) = \begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & 0 & \ddots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_{n-1} & 0 & b_{n-1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & 0 & b_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 & b_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

where $b_0, b_1, \dots, b_{n-1} \neq 0$.

One can get all our results obtained in the previous section as follows.

Theorem 8. *If*

$$S = \left\{ \lambda \in \mathbb{C} : \prod_{k=0}^{n-1} \left| \frac{\lambda - a_k}{b_k} \right| \le 1 \right\},$$

$\mathring{S}$ is the interior of the set S and ∂S is the boundary of the set S. Then the following holds:

1. $\sigma_p(U(a_0,a_1,\dots,a_{n-1};0;b_0,b_1,\dots,b_{n-1}),h) = \mathring{S}$;
2. $\sigma_p(U(a_0,a_1,\dots,a_{n-1};0;b_0,b_1,\dots,b_{n-1})^*,h^* \cong \sigma_\infty) = \emptyset$;
3. $\sigma_r(U(a_0,a_1,\dots,a_{n-1};0;b_0,b_1,\dots,b_{n-1}),h) = \emptyset$;
4. $\sigma_c(U(a_0,a_1,\dots,a_{n-1};0;b_0,b_1,\dots,b_{n-1}),h) = \partial S$;
5. $\sigma(U(a_0,a_1,\dots,a_{n-1};0;b_0,b_1,\dots,b_{n-1}),h) = S$.

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