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Existence of solutions for a system of fractional boundary value problems

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Abstract. We study the existence of solutions for a system of Riemann-Liouville fractional differential equations with nonlinearities dependent on fractional integrals, supplemented with uncoupled nonlocal boundary conditions which contain various fractional derivatives and Riemann-Stieltjes integrals. We use the fixed point theory in the proof of our main theorems.
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1. Introduction

We consider a nonlinear system of fractional differential equations

$$(S)\qquad \begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),v(t),I_{0+}^{\theta_1}u(t),I_{0+}^{\sigma_1}v(t))=0, & t\in(0,1),\\ D_{0+}^{\beta}v(t)+g(t,u(t),v(t),I_{0+}^{\theta_2}u(t),I_{0+}^{\sigma_2}v(t))=0, & t\in(0,1),\end{cases}$$

with uncoupled nonlocal boundary conditions

$$(BC)\qquad \begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, & D_{0+}^{\gamma_0}u(1)=\displaystyle\sum_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}u(t)\,dH_i(t),\\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, & D_{0+}^{\delta_0}v(1)=\displaystyle\sum_{i=1}^{q}\int_0^1 D_{0+}^{\delta_i}v(t)\,dK_i(t),\end{cases}$$

where $\alpha,\beta\in\mathbb{R}$, $\alpha\in(n-1,n]$, $\beta\in(m-1,m]$, $n,m\in\mathbb{N}$, $n\geq 2$, $m\geq 2$, $\theta_1,\theta_2,\sigma_1,\sigma_2>0$, $p,q\in\mathbb{N}$, $\gamma_i\in\mathbb{R}$ for all $i=0,\ldots,p$, $0\leq\gamma_1<\gamma_2<\cdots<\gamma_p<\alpha-1$, $\gamma_0\in[0,\alpha-1)$, $\delta_i\in\mathbb{R}$ for all $i=0,\ldots,q$, $0\leq\delta_1<\delta_2<\cdots<\delta_q<\beta-1$, $\delta_0\in[0,\beta-1)$, D_{0+}^{k} denotes the Riemann-Liouville derivative of order k (for $k=\alpha,\beta,\gamma_0,\gamma_i$, $i=1,\ldots,p$, δ_0,δ_i, $i=1,\ldots,q$), I_{0+}^{ζ} is the Riemann-Liouville integral of order ζ (for $\zeta=\theta_1,\sigma_1,\theta_2,\sigma_2$), the functions f and g are nonnegative, and the integrals from the boundary conditions (BC) are Riemann-Stieltjes integrals with H_i for $i=1,\ldots,p$ and K_i for $i=1,\ldots,q$ functions of bounded variation.

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Based on some theorems from the fixed point theory, in this paper we give conditions for the nonlinearities f and g such that problem $(S)-(BC)$ has at least one solution. The fractional equation

$$(E) \qquad D_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0,\ \ t\in(0,1),$$

with nonlocal boundary conditions

$$(BC_1) \qquad u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ \ D_{0+}^{p}u(1)=\sum_{i=1}^{m}a_iD_{0+}^{q}u(\xi_i),$$

where $\xi_i\in\mathbb{R}$, $i=1,\dots,m$, $0<\xi_1<\cdots<\xi_m<1$, p, $q\in\mathbb{R}$, $p\in[1,n-2]$, $q\in[0,p]$, was investigated in [8]. In [8], the nonlinearity f changes the sign and it is singular in the points $t=0,1$, and there the authors used the Guo-Krasnosel'skii fixed point theorem to prove the existence of positive solutions when the parameter λ belongs to various intervals. For some recent results on the existence, nonexistence and multiplicity of positive solutions or solutions for Riemann-Liouville, Caputo or Hadamard fractional differential equations and systems of fractional differential equations subject to various boundary conditions we refer the reader to the monographs [7, 21] and the papers [1, 2, 3, 4, 5, 6, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20].

2. Auxiliary results

We consider the fractional differential equation

$$D_{0+}^{\alpha}u(t)+h(t)=0,\ \ t\in(0,1), \qquad (1)$$

with boundary conditions

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ \ D_{0+}^{\gamma_0}u(1)=\sum_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}u(t)\,dH_i(t), \qquad (2)$$

where $h\in C(0,1)\cap L^1(0,1)$. We denote by

$$\Delta_1=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_0)}-\sum_{i=1}^{p}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_i)}\int_0^1 s^{\alpha-\gamma_i-1}\,dH_i(s).$$

By standard computations we obtain the following lemma.

Lemma 1. *If $\Delta_1\neq0$, then the function $u\in C[0,1]$ given by*

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\,ds+\frac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha-\gamma_0)}\int_0^1(1-s)^{\alpha-\gamma_0-1}h(s)\,ds$$
$$-\frac{t^{\alpha-1}}{\Delta_1}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i)}\int_0^1\left(\int_0^s(s-\tau)^{\alpha-\gamma_i-1}h(\tau)\,d\tau\right)dH_i(s),\ \ t\in[0,1],$$

is a solution of problem $(1)-(2)$.

We also consider the fractional differential equation

$$D_{0+}^{\beta}v(t) + k(t) = 0, \;\; t \in (0,1), \tag{3}$$

with boundary conditions

$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \;\; D_{0+}^{\delta_0}v(1) = \sum_{i=1}^{q} \int_0^1 D_{0+}^{\delta_i}v(t)\,dK_i(t), \tag{4}$$

where $k \in C(0,1) \cap L^1(0,1)$. We denote by

$$\Delta_2 = \frac{\Gamma(\beta)}{\Gamma(\beta-\delta_0)} - \sum_{i=1}^{q} \frac{\Gamma(\beta)}{\Gamma(\beta-\delta_i)} \int_0^1 s^{\beta-\delta_i-1}\,dK_i(s).$$

Lemma 2. *If $\Delta_2 \neq 0$, then the function $v \in C[0,1]$ given by*

$$\begin{aligned} v(t) = &-\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}k(s)\,ds + \frac{t^{\beta-1}}{\Delta_2\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1}k(s)\,ds \\ &- \frac{t^{\beta-1}}{\Delta_2} \sum_{i=1}^{q} \frac{1}{\Gamma(\beta-\delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\delta_i-1}k(\tau)\,d\tau \right) dK_i(s), \;\; t \in [0,1], \end{aligned}$$

is a solution of problem $(3)-(4)$.

Lemma 3 (see [2]). *If $z \in C[0,1]$ then for $\zeta > 0$ we have*

$$|I_{0+}^{\zeta}z(t)| \leq \frac{\|z\|}{\Gamma(\zeta+1)}, \;\; \forall\, t \in [0,1],$$

where $\|z\| = \sup_{t\in[0,1]} |z(t)|$.

We denote by $(I1)$ the following basic assumptions for problem $(S)-(BC)$ that will be used in the main theorems.

$(I1)$ $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n, m \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $\theta_1, \theta_2, \sigma_1, \sigma_2 > 0$, $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0,\ldots,p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p < \alpha - 1$, $\gamma_0 \in [0, \alpha-1)$, $\delta_i \in \mathbb{R}$ for all $i = 0,\ldots,q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q < \beta - 1$, $\delta_0 \in [0,\beta-1)$, $H_i : [0,1] \to \mathbb{R}$, $i = 1,\ldots,p$ and $K_j : [0,1] \to \mathbb{R}$, $j = 1,\ldots,q$ are functions of bounded variation, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$.

We introduce the following constants:

$$\begin{aligned} M_1 =& 1 + \frac{1}{\Gamma(\theta_1+1)}, \;\; M_2 = 1 + \frac{1}{\Gamma(\sigma_1+1)}, \;\; M_3 = 1 + \frac{1}{\Gamma(\theta_2+1)}, \\ M_4 =& 1 + \frac{1}{\Gamma(\sigma_2+1)},, \; M_5 = \max\{M_1, M_2\}, \;\; M_6 = \max\{M_3, M_4\}, \\ M_7 =& \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta_1|\Gamma(\alpha-\gamma_0+1)} + \frac{1}{|\Delta_1|} \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\gamma_i+1)} \left| \int_0^1 s^{\alpha-\gamma_i}\,dH_i(s) \right|, \\ M_9 =& \frac{1}{\Gamma(\beta+1)} + \frac{1}{|\Delta_2|\Gamma(\beta-\delta_0+1)} + \frac{1}{|\Delta_2|} \sum_{i=1}^{q} \frac{1}{\Gamma(\beta-\delta_i+1)} \left| \int_0^1 s^{\beta-\delta_i}\,dK_i(s) \right|, \end{aligned}$$

$$M_8 = M_7 - \frac{1}{\Gamma(\alpha+1)}, \ \ M_{10} = M_9 - \frac{1}{\Gamma(\beta+1)}. \qquad (5)$$

We consider the Banach space $X = C[0,1]$ with supremum norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We introduce the operator $A : Y \to Y$ defined by $A(u,v) = (A_1(u,v), A_2(u,v))$ for $(u,v) \in Y$, where the operators $A_1,\ A_2 : Y \to X$ are given by:

$$\begin{aligned}
A_1(u,v)(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),v(s),I_{0+}^{\theta_1}u(s),I_{0+}^{\sigma_1}v(s))\,ds \\
&+\frac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha-\gamma_0)}\int_0^1 (1-s)^{\alpha-\gamma_0-1} f(s,u(s),v(s),I_{0+}^{\theta_1}u(s),I_{0+}^{\sigma_1}v(s))\,ds - \frac{t^{\alpha-1}}{\Delta_1} \\
&\times\sum_{i=1}^p \frac{1}{\Gamma(\alpha-\gamma_i)}\int_0^1\left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} f(\tau,u(\tau),v(\tau),I_{0+}^{\theta_1}u(\tau),I_{0+}^{\sigma_1}v(\tau))\,d\tau\right)dH_i(s), \\
A_2(u,v)(t) &= -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,u(s),v(s),I_{0+}^{\theta_2}u(s),I_{0+}^{\sigma_2}v(s))\,ds \\
&+\frac{t^{\beta-1}}{\Delta_2\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\beta-\delta_0-1} g(s,u(s),v(s),I_{0+}^{\theta_2}u(s),I_{0+}^{\sigma_2}v(s))\,ds - \frac{t^{\beta-1}}{\Delta_2} \\
&\times\sum_{i=1}^q \frac{1}{\Gamma(\beta-\delta_i)}\int_0^1\left(\int_0^s (s-\tau)^{\beta-\delta_i-1} g(\tau,u(\tau),v(\tau),I_{0+}^{\theta_2}u(\tau),I_{0+}^{\sigma_2}v(\tau))\,d\tau\right)dK_i(s), \\
&\qquad \forall\, t\in[0,1],\ (u,v)\in Y.
\end{aligned} \qquad (6)$$

By using Lemmas 1 and 2, we note that if (u,v) is a fixed point of operator A, then (u,v) is a solution of problem $(S)-(BC)$.

3. Existence of solutions for $(S)-(BC)$

In this section, we will present some conditions for the nonlinearities f and g such that operator A has at least one fixed point, which is a solution of problem $(S)-(BC)$.

Theorem 1. *Assume that* $(I1)$ *and*

$(I2)$ *The functions* $f,\ g : [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ *are continuous and there exist* $L_1,\ L_2 > 0$ *such that*

$$|f(t,x_1,x_2,x_3,x_4) - f(t,\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3,\widetilde{x}_4)| \le L_1\sum_{i=1}^4 |x_i-\widetilde{x}_i|,$$
$$|g(t,y_1,y_2,y_3,y_4) - g(t,\widetilde{y}_1,\widetilde{y}_2,\widetilde{y}_3,\widetilde{y}_4)| \le L_2\sum_{i=1}^4 |y_i-\widetilde{y}_i|,$$

for all $t\in[0,1]$, $x_i, y_i, \widetilde{x}_i, \widetilde{y}_i \in \mathbb{R}$, $i = 1,\dots,4$,

hold. If $\Xi := L_1M_5M_7 + L_2M_6M_9 < 1$, *then problem* $(S)-(BC)$ *has at least one solution* $(u(t),v(t))$, $t\in[0,1]$, *where* M_5, M_6, M_7, M_9 *are given by (5).*

Proof. We consider the positive number r given by

$$r = (M_0M_7 + \widetilde{M}_0M_9)(1 - L_1M_5M_7 - L_2M_6M_9)^{-1},$$

where $M_0 = \sup_{t\in[0,1]} |f(t, 0, 0, 0, 0)|$, $\widetilde{M}_0 = \sup_{t\in[0,1]} |g(t, 0, 0, 0, 0)|$. We define the set $\overline{B}_r = \{(u, v) \in Y,\ \|(u, v)\|_Y \le r\}$ and first we show that $A(\overline{B}_r) \subset \overline{B}_r$. Let $(u, v) \in \overline{B}_r$. By using $(I2)$ and Lemma 3, for $f(t, u(t), v(t), I^{\theta_1}_{0+}u(t), I^{\sigma_1}_{0+}v(t))$ we deduce the following inequalities:

$$\begin{aligned}
&|f(t, u(t), v(t), I^{\theta_1}_{0+}u(t), I^{\sigma_1}_{0+}v(t))|\\
&\quad\le |f(t, u(t), v(t), I^{\theta_1}_{0+}u(t), I^{\sigma_1}_{0+}v(t)) - f(t, 0, 0, 0, 0)| + |f(t, 0, 0, 0, 0)|\\
&\quad\le L_1(|u(t)| + |v(t)| + |I^{\theta_1}_{0+}u(t)| + |I^{\sigma_1}_{0+}v(t)|) + M_0\\
&\quad\le L_1\left(\|u\| + \|v\| + \frac{\|u\|}{\Gamma(\theta_1 + 1)} + \frac{\|v\|}{\Gamma(\sigma_1 + 1)}\right) + M_0\\
&\quad= L_1\left(\left(1 + \frac{1}{\Gamma(\theta_1 + 1)}\right)\|u\| + \left(1 + \frac{1}{\Gamma(\sigma_1 + 1)}\right)\|v\|\right) + M_0\\
&\quad= L_1(M_1\|u\| + M_2\|v\|) + M_0\\
&\quad\le L_1M_5\|(u, v)\|_Y + M_0 \le L_1M_5r + M_0,\ \ \forall\, t \in [0, 1].
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
&|g(t, u(t), v(t), I^{\theta_2}_{0+}u(t), I^{\sigma_2}_{0+}v(t))|\\
&\quad\le |g(t, u(t), v(t), I^{\theta_2}_{0+}u(t), I^{\sigma_2}_{0+}v(t)) - g(t, 0, 0, 0, 0)| + |g(t, 0, 0, 0, 0)|\\
&\quad\le L_2(|u(t)| + |v(t)| + |I^{\theta_2}_{0+}u(t)| + |I^{\sigma_2}_{0+}v(t)|) + \widetilde{M}_0\\
&\quad\le L_2\left(\|u\| + \|v\| + \frac{\|u\|}{\Gamma(\theta_2 + 1)} + \frac{\|v\|}{\Gamma(\sigma_2 + 1)}\right) + \widetilde{M}_0\\
&\quad= L_2\left(\left(1 + \frac{1}{\Gamma(\theta_2 + 1)}\right)\|u\| + \left(1 + \frac{1}{\Gamma(\sigma_2 + 1)}\right)\|v\|\right) + \widetilde{M}_0\\
&\quad= L_2(M_3\|u\| + M_4\|v\|) + \widetilde{M}_0\\
&\quad\le L_2M_6\|(u, v)\|_Y + \widetilde{M}_0 \le L_2M_6r + \widetilde{M}_0,\ \ \forall\, t \in [0, 1].
\end{aligned}$$

Then by (6) (the definition of operators A_1 and A_2), we obtain

$$\begin{aligned}
|A_1(u, v)(t)| \le& \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}(L_1M_5r + M_0)\, ds\\
&+ \frac{t^{\alpha-1}}{|\Delta_1|\Gamma(\alpha - \gamma_0)}\int_0^1 (1 - s)^{\alpha-\gamma_0-1}(L_1M_5r + M_0)\, ds\\
&+ \frac{t^{\alpha-1}}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha - \gamma_i)}\left|\int_0^1\left(\int_0^s (s - \tau)^{\alpha-\gamma_i-1}(L_1M_5r + M_0)d\tau\right)dH_i(s)\right|\\
=&(L_1M_5r + M_0)\left[\frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{\alpha-1}}{|\Delta_1|\Gamma(\alpha - \gamma_0 + 1)}\right.\\
&\left.+ \frac{t^{\alpha-1}}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha - \gamma_i + 1)}\left|\int_0^1 s^{\alpha-\gamma_i}\, dH_i(s)\right|\right],\ \ \forall\, t \in [0, 1].
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}\|A_1(u,v)\| \leq & (L_1M_5r+M_0)\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta_1|\Gamma(\alpha-\gamma_0+1)}\right.\\ &\left.+\frac{1}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i+1)}\left|\int_0^1 s^{\alpha-\gamma_i}\,dH_i(s)\right|\right] \qquad (7)\\ =&(L_1M_5r+M_0)M_7.\end{aligned}$$

Arguing as before, we find

$$\begin{aligned}|A_2(u,v)(t)| \leq & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(L_2M_6r+\widetilde{M}_0)\,ds\\ &+\frac{t^{\beta-1}}{|\Delta_2|\Gamma(\beta-\delta_0)}\int_0^1(1-s)^{\beta-\delta_0-1}(L_2M_6r+\widetilde{M}_0)\,ds\\ &+\frac{t^{\beta-1}}{|\Delta_2|}\sum_{i=1}^{q}\frac{1}{\Gamma(\beta-\delta_i)}\left|\int_0^1\left(\int_0^s(s-\tau)^{\beta-\delta_i-1}(L_2M_6r+\widetilde{M}_0)\,d\tau\right)dK_i(s)\right|\\ =&(L_2M_6r+\widetilde{M}_0)\left[\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\beta-1}}{|\Delta_2|\Gamma(\beta-\delta_0+1)}\right.\\ &\left.+\frac{t^{\beta-1}}{|\Delta_2|}\sum_{i=1}^{q}\frac{1}{\Gamma(\beta-\delta_i+1)}\left|\int_0^1 s^{\beta-\delta_i}\,dK_i(s)\right|\right],\ \ \forall\, t\in[0,1].\end{aligned}$$

Then we have

$$\begin{aligned}\|A_2(u,v)\| \leq & (L_2M_6r+\widetilde{M}_0)\left[\frac{1}{\Gamma(\beta+1)}+\frac{1}{|\Delta_2|\Gamma(\beta-\delta_0+1)}\right.\\ &\left.+\frac{1}{|\Delta_2|}\sum_{i=1}^{q}\frac{1}{\Gamma(\beta-\delta_i+1)}\left|\int_0^1 s^{\beta-\delta_i}\,dK_i(s)\right|\right] \qquad (8)\\ =&(L_2M_6r+\widetilde{M}_0)M_9.\end{aligned}$$

By relations (7) and (8) we deduce

$$\|A(u,v)\|_Y=\|A_1(u,v)\|+\|A_2(u,v)\|\leq(L_1M_5r+M_0)M_7+(L_2M_6r+\widetilde{M}_0)M_9=r,$$

for all $(u,v)\in\overline{B}_r$, which implies that $A(\overline{B}_r)\subset\overline{B}_r$.

Next, we prove that operator A is a contraction. For $(u_i,v_i)\in\overline{B}_r$, $i=1,2$, and for each $t\in[0,1]$ we obtain

$$\begin{aligned}&|A_1(u_1,v_1)(t)-A_1(u_2,v_2)(t)|\\ &\leq\left|-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[f(s,u_1(s),v_1(s),I_{0+}^{\theta_1}u_1(s),I_{0+}^{\sigma_1}v_1(s))\right.\right.\\ &\quad\left.\left.-\ f(s,u_2(s),v_2(s),I_{0+}^{\theta_1}u_2(s),I_{0+}^{\sigma_1}v_2(s))\right]ds\right|\\ &\quad+\frac{t^{\alpha-1}}{|\Delta_1|\Gamma(\alpha-\gamma_0)}\int_0^1(1-s)^{\alpha-\gamma_0-1}|f(s,u_1(s),v_1(s),I_{0+}^{\theta_1}u_1(s),I_{0+}^{\sigma_1}v_1(s))\end{aligned}$$

$$
\begin{aligned}
&- f(s,u_2(s),v_2(s),I_{0+}^{\theta_1}u_2(s),I_{0+}^{\sigma_1}v_2(s))|\,ds\\
&+\frac{t^{\alpha-1}}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i)}\bigg|\int_0^1\bigg(\int_0^s(s-\tau)^{\alpha-\gamma_i-1}|f(\tau,u_1(\tau),v_1(\tau),I_{0+}^{\theta_1}u_1(\tau),I_{0+}^{\sigma_1}v_1(\tau))\\
&-\ f(\tau,u_2(\tau),v_2(\tau),I_{0+}^{\theta_1}u_2(\tau),I_{0+}^{\sigma_1}v_2(\tau))|\,d\tau\bigg)\ dH_i(s)\bigg|\\
\le&\frac{L_1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Big[|u_1(s)-u_2(s)|+|v_1(s)-v_2(s)|+|I_{0+}^{\theta_1}u_1(s)-I_{0+}^{\theta_1}u_2(s)|\\
&+|I_{0+}^{\sigma_1}v_1(s)-I_{0+}^{\sigma_1}v_2(s)|\Big]\ ds\\
&+\frac{t^{\alpha-1}L_1}{|\Delta_1|\Gamma(\alpha-\gamma_0)}\int_0^1(1-s)^{\alpha-\gamma_0-1}\,[|u_1(s)-u_2(s)|+|v_1(s)-v_2(s)|\\
&+|I_{0+}^{\theta_1}u_1(s)-I_{0+}^{\theta_1}u_2(s)|+|I_{0+}^{\sigma_1}v_1(s)-I_{0+}^{\sigma_1}v_2(s)|]\ ds\\
&+\frac{t^{\alpha-1}L_1}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i)}\left|\int_0^1\left(\int_0^s(s-\tau)^{\alpha-\gamma_i-1}\,[|u_1(\tau)-u_2(\tau)|+|v_1(\tau)-v_2(\tau)|\right.\right.\\
&\left.\left.+\ |I_{0+}^{\theta_1}u_1(\tau)-I_{0+}^{\theta_1}u_2(\tau)|+|I_{0+}^{\sigma_1}v_1(\tau)-I_{0+}^{\sigma_1}v_2(\tau)|\Big]\ d\tau\right)dH_i(s)\right|\\
\le&\frac{L_1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bigg[\|u_1-u_2\|+\|v_1-v_2\|+\frac{1}{\Gamma(\theta_1+1)}\|u_1-u_2\|\\
&+\frac{1}{\Gamma(\sigma_1+1)}\|v_1-v_2\|\bigg]\ ds\\
&+\frac{t^{\alpha-1}L_1}{|\Delta_1|\Gamma(\alpha-\gamma_0)}\int_0^1(1-s)^{\alpha-\gamma_0-1}\bigg[\|u_1-u_2\|+\|v_1-v_2\|+\frac{1}{\Gamma(\theta_1+1)}\|u_1-u_2\|\\
&+\frac{1}{\Gamma(\sigma_1+1)}\|v_1-v_2\|\bigg]\,ds\\
&+\frac{t^{\alpha-1}L_1}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i)}\left|\int_0^1\left(\int_0^s(s-\tau)^{\alpha-\gamma_i-1}\,[\|u_1-u_2\|+\|v_1-v_2\|\right.\right.\\
&\left.\left.+\ \frac{1}{\Gamma(\theta_1+1)}\|u_1-u_2\|+\frac{1}{\Gamma(\sigma_1+1)}\|v_1-v_2\|\bigg]\,d\tau\right)dH_i(s)\right|\\
\le&\frac{L_1M_5}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(\|u_1-u_2\|+\|v_1-v_2\|)\,ds\\
&+\frac{t^{\alpha-1}L_1M_5}{|\Delta_1|\Gamma(\alpha-\gamma_0)}\int_0^1(1-s)^{\alpha-\gamma_0-1}(\|u_1-u_2\|+\|v_1-v_2\|)\,ds\\
&+\frac{t^{\alpha-1}L_1M_5}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i)}\left|\int_0^1\left(\int_0^s(s-\tau)^{\alpha-\gamma_i-1}\,d\tau\right)dH_i(s)\right|\\
&\times(\|u_1-u_2\|+\|v_1-v_2\|)\\
=&L_1M_5\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha-1}}{|\Delta_1|\Gamma(\alpha-\gamma_0+1)}\right.\\
&\left.+\ \frac{t^{\alpha-1}}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i+1)}\left|\int_0^1 s^{\alpha-\gamma_i}\,dH_i(s)\right|\right)(\|u_1-u_2\|+\|v_1-v_2\|).
\end{aligned}
$$

Then we conclude

$$\|A_1(u_1,v_1) - A_1(u_2,v_2)\| \le L_1M_5M_7(\|u_1-u_2\| + \|v_1-v_2\|). \tag{9}$$

By similar computation, we also find

$$\|A_2(u_1,v_1) - A_2(u_2,v_2)\| \le L_2M_6M_9(\|u_1-u_2\| + \|v_1-v_2\|). \tag{10}$$

Therefore, by (9) and (10) we obtain

$$\begin{aligned}\|A(u_1,v_1) - A(u_2,v_2)\|_Y =&\|A_1(u_1,v_1) - A_1(u_2,v_2)\| + \|A_2(u_1,v_1) - A_2(u_2,v_2)\|\\ \le&(L_1M_5M_7 + L_2M_6M_9)(\|u_1-u_2\| + \|v_1-v_2\|)\\ =&\Xi\|(u_1,v_1)-(u_2,v_2)\|_Y.\end{aligned}$$

By using the condition $\Xi < 1$, we deduce that operator A is a contraction. By the Banach contraction mapping principle, we conclude that operator A has a unique fixed point $(u,v) \in \overline{B}_r$, which is a solution of problem $(S)-(BC)$ on $[0,1]$. □

Theorem 2. *Assume that* $(I1)$ *and*

$(I3)$ *The functions* f, $g : [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ *are continuous and there exist real constants* c_i, $d_i \ge 0$, $i = 0,\dots,4$, *and at least one of* c_0 *and* d_0 *is positive, such that*

$$|f(t,x_1,x_2,x_3,x_4)| \le c_0 + \sum_{i=1}^{4} c_i|x_i|, \ \ |g(t,y_1,y_2,y_3,y_4)| \le d_0 + \sum_{i=1}^{4} d_i|y_i|,$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1,\dots,4$,

hold. If $\Xi_1 := \max\{M_{11}, M_{12}\} < 1$, *where* $M_{11} = (c_1 + \frac{c_3}{\Gamma(\theta_1+1)})M_7 + (d_1 + \frac{d_3}{\Gamma(\theta_2+1)})M_9$ *and* $M_{12} = (c_2 + \frac{c_4}{\Gamma(\sigma_1+1)})M_7 + (d_2 + \frac{d_4}{\Gamma(\sigma_2+1)})M_9$, *then the boundary value problem* $(S)-(BC)$ *has at least one solution* $(u(t),v(t))$, $t \in [0,1]$.

Proof. We prove that operator A is completely continuous. By the continuity of functions f and g we obtain that operators A_1 and A_2 are continuous, and then A is a continuous operator. Next, we prove that A is a compact operator. Let $\Omega \subset Y$ be a bounded set. Then there exist positive constants L_3 and L_4 such that

$$|f(t,u(t),v(t),I_{0+}^{\theta_1}u(t),I_{0+}^{\sigma_1}v(t))| \le L_3, \ \ |g(t,u(t),v(t),I_{0+}^{\theta_2}u(t),I_{0+}^{\sigma_2}v(t))| \le L_4,$$

for all $(u,v) \in \Omega$ and $t \in [0,1]$. Therefore, as in the proof of Theorem 1 we deduce that

$$|A_1(u,v)(t)| \le L_3M_7, \ \ |A_2(u,v)(t)| \le L_4M_9, \ \ \forall\, t \in [0,1], \ \ (u,v) \in \Omega.$$

So we obtain

$$\|A_1(u,v)\| \le L_3M_7, \ \ \|A_2(u,v)\| \le L_4M_9, \ \ \|A(u,v)\|_Y \le L_3M_7 + L_4M_9, \ \ \forall (u,v) \in \Omega,$$

and then $A(\Omega)$ is uniformly bounded.

Next, we will prove that the functions from $A(\Omega)$ are equicontinuous. Let $(u,v) \in \Omega$ and $t_1,\, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|A_1(u,v)(t_2) - A_1(u,v)(t_1)| \\
&\quad \leq \Big| -\frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s,u(s),v(s),I_{0+}^{\theta_1}u(s),I_{0+}^{\sigma_1}v(s))\, ds \\
&\quad\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s,u(s),v(s),I_{0+}^{\theta_1}u(s),I_{0+}^{\sigma_1}v(s))\, ds \Big| \\
&\quad\quad + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta_1|\Gamma(\alpha-\gamma_0)} \Big| \int_0^1 (1-s)^{\alpha-\gamma_0-1} f(s,u(s),v(s),I_{0+}^{\theta_1}u(s),I_{0+}^{\sigma_1}v(s))\, ds \Big| \\
&\quad\quad + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\gamma_i)} \\
&\quad\quad \times \Big| \int_0^1 \Big(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} f(\tau,u(\tau),v(\tau),I_{0+}^{\theta_1}u(\tau),I_{0+}^{\sigma_1}v(\tau))\, d\tau \Big) dH_i(s) \Big| \\
&\quad \leq \frac{L_3}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] ds + \frac{L_3}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\, ds \\
&\quad\quad + \frac{L_3(t_2^{\alpha-1} - t_1^{\alpha-1})}{|\Delta_1|\Gamma(\alpha-\gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1}\, ds + \frac{L_3(t_2^{\alpha-1} - t_1^{\alpha-1})}{|\Delta_1|} \\
&\quad\quad \times \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\gamma_i)} \Big| \int_0^1 \Big(\int_0^s (s-\tau)^{\alpha-\gamma_i-1}\, d\tau \Big) dH_i(s) \Big| \\
&\quad = \frac{L_3}{\Gamma(\alpha+1)} (t_2^{\alpha} - t_1^{\alpha}) + L_3 M_8 (t_2^{\alpha-1} - t_1^{\alpha-1}).
\end{aligned}$$

Then

$$|A_1(u,v)(t_2) - A_1(u,v)(t_1)| \to 0, \text{ as } t_2 \to t_1, \text{ uniformly with respect to } (u,v) \in \Omega.$$

In a similar manner, we find

$$|A_2(u,v)(t_2) - A_2(u,v)(t_1)| \leq \frac{L_4}{\Gamma(\beta+1)} (t_2^{\beta} - t_1^{\beta}) + L_4 M_{10} (t_2^{\beta-1} - t_1^{\beta-1}),$$

and so

$$|A_2(u,v)(t_2) - A_2(u,v)(t_1)| \to 0, \text{ as } t_2 \to t_1, \text{ uniformly with respect to } (u,v) \in \Omega.$$

Thus $A_1(\Omega)$ and $A_2(\Omega)$ are equicontinuous, and then $A(\Omega)$ is also equicontinuous. Hence by the Arzela-Ascoli theorem, we conclude that $A(\Omega)$ is relatively compact, and then A is compact. Therefore, we deduce that A is completely continuous.

We will show next that the set $V = \{(u,v) \in Y,\ (u,v) = \nu A(u,v),\ 0 < \nu < 1\}$ is bounded. Let $(u,v) \in V$, that is $(u,v) = \nu A(u,v)$ for some $\nu \in (0,1)$. Then for any $t \in [0,1]$ we have $u(t) = \nu A_1(u,v)(t)$, $v(t) = \nu A_2(u,v)(t)$. Hence we find $|u(t)| \leq |A_1(u,v)(t)|$ and $|v(t)| \leq |A_2(u,v)(t)|$ for all $t \in [0,1]$.

By $(I3)$ we obtain

$$
\begin{aligned}
|u(t)| \leq & |A_1(u,v)(t)| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[c_0 + c_1|u(s)| + c_2|v(s)| + c_3|I_{0+}^{\theta_1} u(s)| + c_4|I_{0+}^{\sigma_1} v(s)|\right] ds \\
& + \frac{t^{\alpha-1}}{|\Delta_1|\Gamma(\alpha-\gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \left[c_0 + c_1|u(s)| + c_2|v(s)| + c_3|I_{0+}^{\theta_1} u(s)| \right. \\
& \left. + c_4|I_{0+}^{\sigma_1} v(s)|\right] ds \\
& + \frac{t^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha-\gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} \left[c_0 + c_1|u(\tau)| + c_2|v(\tau)| \right. \right. \right. \\
& \left. \left. \left. + c_3|I_{0+}^{\theta_1} u(\tau)| + c_4|I_{0+}^{\sigma_1} v(\tau)|\right] d\tau \right) dH_i(s) \right| \\
\leq & \left(c_0 + c_1\|u\| + c_2\|v\| + \frac{c_3}{\Gamma(\theta_1+1)}\|u\| + \frac{c_4}{\Gamma(\sigma_1+1)}\|v\| \right) \left[\frac{t^\alpha}{\Gamma(\alpha+1)} \right. \\
& \left. + \frac{t^{\alpha-1}}{|\Delta_1|\Gamma(\alpha-\gamma_0+1)} + \frac{t^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha-\gamma_i+1)} \left| \int_0^1 s^{\alpha-\gamma_i}\, dH_i(s) \right| \right].
\end{aligned}
$$

Then we deduce

$$
\|u\| \leq \left[c_0 + \left(c_1 + \frac{c_3}{\Gamma(\theta_1+1)} \right) \|u\| + \left(c_2 + \frac{c_4}{\Gamma(\sigma_1+1)} \right) \|v\| \right] M_7.
$$

In a similar manner, we have

$$
\|v\| \leq \left[d_0 + \left(d_1 + \frac{d_3}{\Gamma(\theta_2+1)} \right) \|u\| + \left(d_2 + \frac{d_4}{\Gamma(\sigma_2+1)} \right) \|v\| \right] M_9,
$$

and therefore

$$
\|(u,v)\|_Y \leq c_0 M_7 + d_0 M_9 + M_{11}\|u\| + M_{12}\|v\| \leq c_0 M_7 + d_0 M_9 + \Xi_1 \|(u,v)\|_Y.
$$

Because $\Xi_1 < 1$, we obtain

$$
\|(u,v)\|_Y \leq (c_0 M_7 + d_0 M_8)(1-\Xi_1)^{-1}, \ \ \forall\, (u,v) \in V.
$$

So, we conclude that the set V is bounded.

By using the Leray-Schauder alternative theorem, we deduce that operator A has at least one fixed point, which is a solution of our problem $(S)-(BC)$. □

Theorem 3. *Assume that* $(I1)$, $(I2)$ *and*

$(I4)$ *There exist the functions* $\phi_1, \phi_2 \in C([0,1],[0,\infty))$ *such that*

$$
|f(t,x_1,x_2,x_3,x_4)| \leq \phi_1(t), \ \ |g(t,x_1,x_2,x_3,x_4)| \leq \phi_2(t),
$$

for all $t \in [0,1], \ \ x_i \in \mathbb{R}, \ \ i = 1,\ldots,4,$

hold. If $\Xi_2 := L_1 M_5 \frac{1}{\Gamma(\alpha+1)} + L_2 M_6 \frac{1}{\Gamma(\beta+1)} < 1$*, then problem* $(S)-(BC)$ *has at least one solution on* $[0,1]$.

Proof. We fix $r_1 > 0$ such that $r_1 \geq M_7\|\phi_1\| + M_9\|\phi_2\|$. We consider the set $\overline{B}_{r_1} = \{(u,v) \in Y,\ \|(u,v)\|_Y \leq r_1\}$, and introduce the operators $D = (D_1, D_2) : \overline{B}_{r_1} \to Y$ and $E = (E_1, E_2) : \overline{B}_{r_1} \to Y$, where D_1, D_2, E_1, $E_2 : \overline{B}_{r_1} \to X$ are defined by

$$\begin{aligned}
D_1(u,v)(t) =& -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),v(s),I_{0+}^{\theta_1}u(s),I_{0+}^{\sigma_1}v(s))\,ds,\\
E_1(u,v)(t) =& \frac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha-\gamma_0)}\int_0^1 (1-s)^{\alpha-\gamma_0-1} f(s,u(s),v(s),I_{0+}^{\theta_1}u(s),I_{0+}^{\sigma_1}v(s))\,ds\\
&-\frac{t^{\alpha-1}}{\Delta_1}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i)}\\
&\times\int_0^1\left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} f(\tau,u(\tau),v(\tau),I_{0+}^{\theta_1}u(\tau),I_{0+}^{\sigma_1}v(\tau))\,d\tau\right)dH_i(s),\\
D_2(u,v)(t) =& -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,u(s),v(s),I_{0+}^{\theta_2}u(s),I_{0+}^{\sigma_2}v(s))\,ds,\\
E_2(u,v)(t) =& \frac{t^{\beta-1}}{\Delta_2\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\beta-\delta_0-1} g(s,u(s),v(s),I_{0+}^{\theta_2}u(s),I_{0+}^{\sigma_2}v(s))\,ds\\
&-\frac{t^{\beta-1}}{\Delta_2}\sum_{i=1}^{q}\frac{1}{\Gamma(\beta-\delta_i)}\\
&\times\int_0^1\left(\int_0^s (s-\tau)^{\beta-\delta_i-1} g(\tau,u(\tau),v(\tau),I_{0+}^{\theta_2}u(\tau),I_{0+}^{\sigma_2}v(\tau))\,d\tau\right)dK_i(s),
\end{aligned} \tag{11}$$

for $t \in [0,1]$ and $(u,v) \in \overline{B}_{r_1}$. So $A_1 = D_1 + E_1$, $A_2 = D_2 + E_2$ and $A = D + E$.

By using $(I4)$ for all (u_1,v_1), $(u_2,v_2) \in \overline{B}_{r_1}$ we obtain that

$$\begin{aligned}
\|D(u_1,v_1)+&E(u_2,v_2)\|_Y\\
\leq&\|D(u_1,v_1)\|_Y + \|E(u_2,v_2)\|_Y\\
=&\|D_1(u_1,v_1)\| + \|D_2(u_1,v_1)\| + \|E_1(u_2,v_2)\| + \|E_2(u_2,v_2)\|\\
\leq&\frac{1}{\Gamma(\alpha+1)}\|\phi_1\| + \frac{1}{\Gamma(\beta+1)}\|\phi_2\|\\
&+\left(\frac{1}{|\Delta_1|\Gamma(\alpha-\gamma_0+1)} + \frac{1}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i+1)}\left|\int_0^1 s^{\alpha-\gamma_i}\,dH_i(s)\right|\right)\|\phi_1\|\\
&+\left(\frac{1}{|\Delta_2|\Gamma(\beta-\delta_0+1)} + \frac{1}{|\Delta_2|}\sum_{i=1}^{q}\frac{1}{\Gamma(\beta-\delta_i+1)}\left|\int_0^1 s^{\beta-\delta_i}\,dK_i(s)\right|\right)\|\phi_2\|\\
=&M_7\|\phi_1\| + M_9\|\phi_2\| \leq r_1.
\end{aligned}$$

Hence $D(u_1,v_1) + E(u_2,v_2) \in \overline{B}_{r_1}$ for all (u_1,v_1), $(u_2,v_2) \in \overline{B}_{r_1}$.

The operator D is a contraction because

$$\begin{aligned}
\|D(u_1,v_1)-D(u_2,v_2)\|_Y &= \|D_1(u_1,v_1)-D_1(u_2,v_2)\|+\|D_2(u_1,v_1)-D_2(u_2,v_2)\| \\
&\leq \left(L_1M_5\frac{1}{\Gamma(\alpha+1)}+L_2M_6\frac{1}{\Gamma(\beta+1)}\right)(\|u_1-u_2\|+\|v_1-v_2\|) \\
&= \Xi_2\|(u_1,v_1)-(u_2,v_2)\|_Y,
\end{aligned}$$

for all (u_1,v_1), $(u_2,v_2)\in\overline{B}_{r_1}$, and $\Xi_2<1$.

The continuity of f and g implies that operator E is continuous on $\overline{B}_{r_1}$. We prove in what follows that E is compact. The functions from $E(\overline{B}_{r_1})$ are uniformly bounded because

$$\begin{aligned}
&\|E(u,v)\|_Y \\
&= \|E_1(u,v)\|+\|E_2(u,v)\| \\
&\leq \left(\frac{1}{|\Delta_1|\Gamma(\alpha-\gamma_0+1)}+\frac{1}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i+1)}\left|\int_0^1 s^{\alpha-\gamma_i}\,dH_i(s)\right|\right)\|\phi_1\| \\
&\quad+\left(\frac{1}{|\Delta_2|\Gamma(\beta-\delta_0+1)}+\frac{1}{|\Delta_2|}\sum_{i=1}^{q}\frac{1}{\Gamma(\beta-\delta_i+1)}\left|\int_0^1 s^{\beta-\delta_i}\,dK_i(s)\right|\right)\|\phi_2\| \\
&= M_8\|\phi_1\|+M_{10}\|\phi_2\|,\ \ \forall\,(u,v)\in\overline{B}_{r_1}.
\end{aligned}$$

We prove now that the functions from $E(\overline{B}_{r_1})$ are equicontinuous. We denote by

$$\begin{aligned}
\Psi_{r_1} &= \sup\Big\{|f(t,u,v,x,y)|,\ t\in[0,1],\ |u|\leq r_1,\ |v|\leq r_1, \\
&\qquad |x|\leq\frac{r_1}{\Gamma(\theta_1+1)},\ |y|\leq\frac{r_1}{\Gamma(\sigma_1+1)}\Big\}, \\
\Theta_{r_1} &= \sup\Big\{|g(t,u,v,x,y)|,\ t\in[0,1],\ |u|\leq r_1,\ |v|\leq r_1, \\
&\qquad |x|\leq\frac{r_1}{\Gamma(\theta_2+1)},\ |y|\leq\frac{r_1}{\Gamma(\sigma_2+1)}\Big\}.
\end{aligned} \tag{12}$$

Then for $(u,v)\in\overline{B}_{r_1}$, and t_1, $t_2\in[0,1]$ with $t_1<t_2$, we obtain

$$\begin{aligned}
&|E_1(u,v)(t_2)-E_1(u,v)(t_1)| \\
&\leq\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{|\Delta_1|\Gamma(\alpha-\gamma_0)}\int_0^1(1-s)^{\alpha-\gamma_0-1}\Psi_{r_1}\,ds \\
&\quad+\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i)}\left|\int_0^1\left(\int_0^s(s-\tau)^{\alpha-\gamma_i-1}\Psi_{r_1}\,d\tau\right)dH_i(s)\right| \\
&\leq\Psi_{r_1}(t_2^{\alpha-1}-t_1^{\alpha-1}) \\
&\quad\times\left[\frac{1}{|\Delta_1|\Gamma(\alpha-\gamma_0+1)}+\frac{1}{|\Delta_1|}\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\gamma_i+1)}\left|\int_0^1 s^{\alpha-\gamma_i}\,dH_i(s)\right|\right]
\end{aligned}$$

$$=\Psi_{r_1} M_8 (t_2^{\alpha-1} - t_1^{\alpha-1}),$$

$$|E_2(u,v)(t_2) - E_2(u,v)(t_1)| \le \frac{t_2^{\beta-1} - t_1^{\beta-1}}{|\Delta_2|\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} \Theta_{r_1}\, ds$$

$$+ \frac{t_2^{\beta-1} - t_1^{\beta-1}}{|\Delta_2|} \sum_{i=1}^{q} \frac{1}{\Gamma(\beta-\delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\delta_i-1} \Theta_{r_1}\, d\tau \right) dK_i(s) \right|$$

$$\le \Theta_{r_1} (t_2^{\beta-1} - t_1^{\beta-1})$$

$$\times \left[\frac{1}{|\Delta_2|\Gamma(\beta-\delta_0+1)} + \frac{1}{|\Delta_2|} \sum_{i=1}^{q} \frac{1}{\Gamma(\beta-\delta_i+1)} \left| \int_0^1 s^{\beta-\delta_i}\, dK_i(s) \right| \right]$$

$$=\Theta_{r_1} M_{10} (t_2^{\beta-1} - t_1^{\beta-1}).$$

Hence we find

$$|E_1(u,v)(t_2) - E_1(u,v)(t_1)| \to 0, \ \ |E_2(u,v)(t_2) - E_2(u,v)(t_1)| \to 0,$$

as $t_2 \to t_1$ uniformly with respect to $(u,v) \in \overline{B}_{r_1}$. So, $E_1(\overline{B}_{r_1})$ and $E_2(\overline{B}_{r_1})$ are equicontinuous, and then $E(\overline{B}_{r_1})$ is also equicontinuous. By using the Arzela-Ascoli theorem, we deduce that the set $E(\overline{B}_{r_1})$ is relatively compact. Therefore, E is a compact operator on $\overline{B}_{r_1}$. By the Krasnolsel'skii theorem for the sum of two operators (see [12]), we conclude that there exists a fixed point of operator $D+E(=A)$, which is a solution of problem $(S)-(BC)$. □

Theorem 4. *Assume that* $(I1)$, $(I2)$ *and* $(I4)$ *hold. If* $\Xi_3 := L_1 M_5 M_8 + L_2 M_6 M_{10} < 1$, *then problem* $(S)-(BC)$ *has at least one solution* (u,v) *on* $[0,1]$.

Proof. We consider again a positive number $r_1 \ge M_7\|\phi_1\| + M_9\|\phi_2\|$, and the operators D and E defined on $\overline{B}_{r_1}$ given by (11). As in the proof of Theorem 3, we obtain that $D(u_1,v_1) + E(u_2,v_2) \in \overline{B}_{r_1}$ for all (u_1,v_1), $(u_2,v_2) \in \overline{B}_{r_1}$.

The operator E is a contraction because

$$\|E(u_1,v_1) - E(u_2,v_2)\|_Y$$
$$= \|E_1(u_1,v_1) - E_1(u_2,v_2)\| + \|E_2(u_1,v_1) - E_2(u_2,v_2)\|$$
$$\le L_1 M_5 M_8 (\|u_1-u_2\| + \|v_1-v_2\|) + L_2 M_6 M_{10} (\|u_1-u_2\| + \|v_1-v_2\|)$$
$$= (L_1 M_5 M_8 + L_2 M_6 M_{10}) \|(u_1,v_1) - (u_2,v_2)\|_Y = \Xi_3 \|(u_1,v_1) - (u_2,v_2)\|_Y,$$

for all (u_1,v_1), $(u_2,v_2) \in \overline{B}_{r_1}$, with $\Xi_3 < 1$.

Next, the continuity of f and g implies that operator D is continuous on $\overline{B}_{r_1}$. We show now that D is compact. The functions from $D(\overline{B}_{r_1})$ are uniformly bounded because

$$\|D(u,v)\|_Y = \|D_1(u,v)\| + \|D_2(u,v)\|$$
$$\le \frac{1}{\Gamma(\alpha+1)} \|\phi_1\| + \frac{1}{\Gamma(\beta+1)} \|\phi_2\|, \ \ \forall\, (u,v) \in \overline{B}_{r_1}.$$

Now we prove that $D(\overline{B}_{r_1})$ is equicontinuous. By using Ψ_{r_1} and Θ_{r_1} defined in (12), for $(u,v) \in \overline{B}_{r_1}$ and $t_1,\, t_2 \in [0,1]$ with $t_1 < t_2$ we find that

$$\begin{aligned} |D_1(u,v)(t_2) - D_1(u,v)(t_1)| &\le \frac{\Psi_{r_1}}{\Gamma(\alpha+1)}(t_2^{\alpha} - t_1^{\alpha}), \\ |D_2(u,v)(t_2) - D_2(u,v)(t_1)| &\le \frac{\Theta_{r_1}}{\Gamma(\beta+1)}(t_2^{\beta} - t_1^{\beta}). \end{aligned}$$

Then we obtain

$$|D_1(u,v)(t_2) - D_1(u,v)(t_1)| \to 0, \;\; |D_2(u,v)(t_2) - D_2(u,v)(t_1)| \to 0,$$

as $t_2 \to t_1$ uniformly with respect to $(u,v) \in \overline{B}_{r_1}$. We deduce that $D_1(\overline{B}_{r_1})$ and $D_2(\overline{B}_{r_1})$ are equicontinuous, and so $D(\overline{B}_{r_1})$ is equicontinuous. By using the Arzela-Ascoli theorem, we conclude that the set $D(\overline{B}_{r_1})$ is relatively compact. Then D is a compact operator on $\overline{B}_{r_1}$. By the Krasnosel'skii theorem we deduce that there exists a fixed point of operator $D+E(=A)$, which is a solution of problem $(S)-(BC)$. $\square$

Theorem 5. *Assume that* $(I1)$ *and*

$(I5)$ *The functions* $f,\, g : [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ *are continuous and there exist the constants* $a_i \ge 0,\ i = 0,\dots,4$ *with at least one nonzero, the constants* $b_i,\ i = 0,\dots,4$ *with at least one nonzero, and* $l_i,\ m_i \in (0,1),\ i = 1,\dots,4$ *such that*

$$\begin{aligned} |f(t,x_1,x_2,x_3,x_4)| &\le a_0 + \sum_{i=1}^{4} a_i |x_i|^{l_i}, \\ |g(t,y_1,y_2,y_3,y_4)| &\le b_0 + \sum_{i=1}^{4} b_i |y_i|^{m_i}, \end{aligned}$$

for all $t \in [0,1],\ x_i,\ y_i \in \mathbb{R},\ i = 1,\dots,4,$

hold. Then problem $(S)-(BC)$ *has at least one solution.*

Proof. Let $\overline{B}_R = \{(u,v) \in Y,\ \ \|(u,v)\|_Y \le R\}$, where

$$\begin{aligned} R \ge \max\Bigg\{ & 10a_0M_7, (10a_1M_7)^{\frac{1}{1-l_1}}, (10a_2M_7)^{\frac{1}{1-l_2}}, \left(\frac{10a_3M_7}{(\Gamma(\theta_1+1))^{l_3}}\right)^{\frac{1}{1-l_3}}, \\ & \left(\frac{10a_4M_7}{(\Gamma(\sigma_1+1))^{l_4}}\right)^{\frac{1}{1-l_4}}, 10b_0M_9, (10b_1M_9)^{\frac{1}{1-m_1}}, (10b_2M_9)^{\frac{1}{1-m_2}}, \\ & \left(\frac{10b_3M_9}{(\Gamma(\theta_2+1))^{m_3}}\right)^{\frac{1}{1-m_3}}, \left(\frac{10b_4M_9}{(\Gamma(\sigma_2+1))^{m_4}}\right)^{\frac{1}{1-m_4}} \Bigg\}. \end{aligned}$$

We prove that $A : \overline{B}_R \to \overline{B}_R$. For $(u,v) \in \overline{B}_R$, we deduce

$$|A_1(u,v)(t)| \le \left(a_0 + a_1R^{l_1} + a_2R^{l_2} + a_3\frac{R^{l_3}}{(\Gamma(\theta_1+1))^{l_3}} + a_4\frac{R^{l_4}}{(\Gamma(\sigma_1+1))^{l_4}}\right)M_7 \le \frac{R}{2},$$

$$|A_2(u,v)(t)| \le \left(b_0 + b_1R^{m_1} + b_2R^{m_2} + b_3\frac{R^{m_3}}{(\Gamma(\theta_2+1))^{m_3}} + b_4\frac{R^{m_4}}{(\Gamma(\sigma_2+1))^{m_4}}\right)M_9 \le \frac{R}{2},$$

for all $t \in [0,1]$. Then we obtain

$$\|A(u,v)\|_Y = \|A_1(u,v)\| + \|A_2(u,v)\| \le R, \ \forall\, (u,v) \in \overline{B}_R,$$

which implies that $A(\overline{B}_R) \subset \overline{B}_R$.

Because the functions f and g are continuous, we conclude that operator A is continuous on $\overline{B}_R$. In addition, the functions from $A(\overline{B}_R)$ are uniformly bounded and equicontinuous. Indeed, by using the notations (12) with r_1 replaced by R, for any $(u,v) \in \overline{B}_R$ and $t_1,\ t_2 \in [0,1]$, $t_1 < t_2$ we find that

$$\begin{aligned} |A_1(u,v)(t_2) - A_1(u,v)(t_1)| \le& \frac{\Psi_R}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha) + \Psi_R M_8 (t_2^{\alpha-1} - t_1^{\alpha-1}),\\ |A_2(u,v)(t_2) - A_2(u,v)(t_1)| \le& \frac{\Theta_R}{\Gamma(\beta+1)}(t_2^\beta - t_1^\beta) + \Theta_R M_{10} (t_2^{\beta-1} - t_1^{\beta-1}). \end{aligned}$$

Therefore,

$$|A_1(u,v)(t_2) - A_1(u,v)(t_1)| \to 0, \ \ |A_2(u,v)(t_2) - A_2(u,v)(t_1)| \to 0, \ \text{as} \ \ t_2 \to t_1,$$

uniformly with respect to $(u,v) \in \overline{B}_R$. By the Arzela-Ascoli theorem, we deduce that $A(\overline{B}_R)$ is relatively compact, and then A is a compact operator. By the Schauder fixed point theorem, we conclude that operator A has at least one fixed point (u,v) in $\overline{B}_R$, which is a solution of our problem $(S)-(BC)$. □

Theorem 6. *Assume that* $(I1)$ *and*

$(I6)$ *The functions* $f,\ g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *are continuous and there exist* $p_i \ge 0$, $i = 0,\dots,4$ *with at least one nonzero,* $q_i \ge 0$, $i = 0,\dots,4$ *with at least one nonzero, and nondecreasing functions* $h_i,\ k_i \in C([0,\infty),[0,\infty))$ $i = 1,\dots,4$ *such that*

$$\begin{aligned} |f(t,x_1,x_2,x_3,x_4)| \le& p_0 + \sum_{i=1}^4 p_i h_i(|x_i|),\\ |g(t,y_1,y_2,y_3,y_4)| \le& q_0 + \sum_{i=1}^4 q_i k_i(|y_i|), \end{aligned}$$

for all $t \in [0,1],\ x_i,\ y_i \in \mathbb{R},\ i = 1,\dots,4,$

hold. If there exists $\Xi_0 > 0$ *such that*

$$\begin{aligned} &\left(p_0 + p_1 h_1(\Xi_0) + p_2 h_2(\Xi_0) + p_3 h_3\left(\frac{\Xi_0}{\Gamma(\theta_1+1)}\right) + p_4 h_4\left(\frac{\Xi_0}{\Gamma(\sigma_1+1)}\right)\right) M_7\\ &+ \left(q_0 + q_1 k_1(\Xi_0) + q_2 k_2(\Xi_0) + q_3 k_3\left(\frac{\Xi_0}{\Gamma(\theta_2+1)}\right) + q_4 k_4\left(\frac{\Xi_0}{\Gamma(\sigma_2+1)}\right)\right) M_9 < \Xi_0, \end{aligned} \tag{13}$$

then problem $(S)-(BC)$ *has at least one solution on* $[0,1]$.

Proof. We consider the set $\overline{B}_{\Xi_0} = \{(u,v) \in Y, \ \|(u,v)\|_Y \le \Xi_0\}$, where Ξ_0 is given in the theorem. We will prove that $A : \overline{B}_{\Xi_0} \to \overline{B}_{\Xi_0}$. For $(u,v) \in \overline{B}_{\Xi_0}$ and $t \in [0,1]$ we obtain

$$\begin{aligned}|A_1(u,v)(t)| \le & \Big(p_0 + p_1 h_1(\Xi_0) + p_2 h_2(\Xi_0) \\ & + p_3 h_3\left(\frac{\Xi_0}{\Gamma(\theta_1+1)}\right) + p_4 h_4\left(\frac{\Xi_0}{\Gamma(\sigma_1+1)}\right)\Big) M_7, \\ |A_2(u,v)(t)| \le & \Big(q_0 + q_1 k_1(\Xi_0) + q_2 k_2(\Xi_0) \\ & + q_3 k_3\left(\frac{\Xi_0}{\Gamma(\theta_2+1)}\right) + q_4 k_4\left(\frac{\Xi_0}{\Gamma(\sigma_2+1)}\right)\Big) M_9,\end{aligned}$$

and then for all $(u,v) \in \overline{B}_{\Xi_0}$ we have

$$\begin{aligned}\|A(u,v)\|_Y & \\ \le & \left(p_0 + p_1 h_1(\Xi_0) + p_2 h_2(\Xi_0) + p_3 h_3\left(\frac{\Xi_0}{\Gamma(\theta_1+1)}\right) + p_4 h_4\left(\frac{\Xi_0}{\Gamma(\sigma_1+1)}\right)\right) M_7 \\ & + \left(q_0 + q_1 k_1(\Xi_0) + q_2 k_2(\Xi_0) + q_3 k_3\left(\frac{\Xi_0}{\Gamma(\theta_2+1)}\right) + q_4 k_4\left(\frac{\Xi_0}{\Gamma(\sigma_2+1)}\right)\right) M_9 \\ & < \Xi_0.\end{aligned}$$

Then $A(\overline{B}_{\Xi_0}) \subset \overline{B}_{\Xi_0}$. In a similar manner used in the proof of Theorem 5, we can prove that operator A is completely continuous.

We suppose now that there exists $(u,v) \in \partial B_{\Xi_0}$ such that $(u,v) = \nu A(u,v)$ for some $\nu \in (0,1)$. We obtain as above that $\|(u,v)\|_Y \le \|A(u,v)\|_Y < \Xi_0$, which is a contradiction because $(u,v) \in \partial B_{\Xi_0}$. Then by the nonlinear alternative of Leray-Schauder type, we conclude that operator A has a fixed point $(u,v) \in \overline{B}_{\Xi_0}$, and so problem $(S)-(BC)$ has at least one solution. □

4. Examples

Let $\alpha = \frac{5}{2}$ $(n=3)$, $\beta = \frac{10}{3}$ $(m=4)$, $\theta_1 = \frac{1}{3}$, $\sigma_1 = \frac{9}{4}$, $\theta_2 = \frac{16}{5}$, $\sigma_2 = \frac{25}{6}$, $\gamma_0 = \frac{4}{3}$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{3}{4}$, $\delta_0 = \frac{11}{5}$, $\delta_1 = \frac{1}{6}$, $\delta_2 = \frac{15}{7}$, $H_1(t) = t^3$, $t \in [0,1]$, $H_2(t) = \{0,\ t \in [0,\frac{1}{3});\ 2,\ t \in [\frac{1}{3},1]\}$, $K_1(t) = \{0,\ t \in [0,\frac{1}{2});\ 4,\ t \in [\frac{1}{2},1],\}$, $K_2(t) = t^2$, $t \in [0,1]$.

We consider the system of fractional differential equations

$$(S_0) \qquad \begin{cases} D_{0+}^{5/2} u(t) + f(t, u(t), v(t), I_{0+}^{1/3} u(t), I_{0+}^{9/4} v(t)) = 0, \ \ t \in (0,1), \\ D_{0+}^{10/3} v(t) + g(t, u(t), v(t), I_{0+}^{16/5} u(t), I_{0+}^{25/6} v(t)) = 0, \ \ t \in (0,1), \end{cases}$$

with the boundary conditions

$$(BC_0) \quad \begin{cases} u(0) = u'(0) = 0, \ \ D_{0+}^{4/3} u(1) = 3\int_0^1 t^2 D_{0+}^{1/2} u(t)\,dt + 2D_{0+}^{3/4} u\left(\frac{1}{3}\right), \\ v(0) = v'(0) = v''(0) = 0, \ \ D_{0+}^{11/5} v(1) = 4D_{0+}^{1/6} v\left(\frac{1}{2}\right) + 2\int_0^1 t D_{0+}^{15/7} v(t)\,dt. \end{cases}$$

We obtain $\Delta_1 \approx -0.83314732 \neq 0$ and $\Delta_2 \approx -0.85088584 \neq 0$. So assumption $(I1)$ is satisfied. In addition, we have $M_1 \approx 2.11984652$, $M_2 \approx 1.39227116$, $M_3 \approx$

1.12892098, $M_4 \approx 1.03231866$, $M_5 = M_1$, $M_6 = M_3$, $M_7 \approx 1.98819306$, $M_8 \approx 1.68729195$, $M_9 \approx 1.95523852$, $M_{10} \approx 1.84725332$.

Example 1. *We consider the functions*

$$\begin{aligned} f(t,x_1,x_2,x_3,x_4) =& \frac{1}{\sqrt{4+t^2}} + \frac{|x_1|}{7(t+1)^3(1+|x_1|)} - \frac{t}{8}\arctan x_2 \\ &+ \frac{t^2}{t+9}\cos x_3 - \frac{1}{2(t+10)}\sin^2 x_4, \\ g(t,y_1,y_2,y_3,y_4) =& \frac{2t}{t^2+9} - \frac{1}{10}\sin y_1 + \frac{|y_2|}{4(2+|y_2|)} + \frac{1}{12}\arctan y_3 - \frac{t}{t+20}\cos^2 y_4, \end{aligned}$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1,\ldots,4$. *We find the inequalities*

$$|f(t,x_1,x_2,x_3,x_4) - f(t,\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3,\widetilde{x}_4)| \le \frac{1}{7}\sum_{i=1}^{4}|x_i - \widetilde{x}_i|,$$
$$|g(t,y_1,y_2,y_3,y_4) - g(t,\widetilde{y}_1,\widetilde{y}_2,\widetilde{y}_3,\widetilde{y}_4)| \le \frac{1}{8}\sum_{i=1}^{4}|y_i - \widetilde{y}_i|,$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1,\ldots,4$. *So we have* $L_1 = \frac{1}{7}$, $L_2 = \frac{1}{8}$, *and* $\Xi \approx 0.878 < 1$. *Therefore, assumption* $(I2)$ *is satisfied, and by Theorem 1 we deduce that problem* $(S_0)-(BC_0)$ *has at least one solution* $(u(t),v(t))$, $t \in [0,1]$.

Example 2. *We consider the functions*

$$\begin{aligned} f(t,x_1,x_2,x_3,x_4) =& \frac{t+1}{t^2+3}(3\sin t + \frac{1}{4}\sin x_1) - \frac{1}{(t+3)^2}x_2 + \frac{t}{4}\arctan x_3 - \cos x_4, \\ g(t,y_1,y_2,y_3,y_4) =& \frac{e^{-t}}{1+t^2} - \frac{1}{3}\sin y_2 + \cos^2 y_3 + \frac{1}{5}\arctan y_4, \end{aligned}$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1,\ldots,4$. *Because we have*

$$\begin{aligned} |f(t,x_1,x_2,x_3,x_4)| \le& \frac{5}{2} + \frac{1}{8}|x_1| + \frac{1}{9}|x_2| + \frac{1}{4}|x_3|, \\ |g(t,y_1,y_2,y_3,y_4)| \le& 2 + \frac{1}{3}|y_2| + \frac{1}{5}|y_4|, \end{aligned}$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1,\ldots,4$, *then assumption* $(I3)$ *is satisfied with* $c_0 = \frac{5}{2}$, $c_1 = \frac{1}{8}$, $c_2 = \frac{1}{9}$, $c_3 = \frac{1}{4}$, $c_4 = 0$, $d_0 = 2$, $d_1 = 0$, $d_2 = \frac{1}{3}$, $d_3 = 0$, $d_4 = \frac{1}{5}$. *Besides, we obtain* $M_{11} \approx 0.805142$, $M_{12} \approx 0.885295$ *and* $\Xi_1 = M_{12} < 1$. *Then by Theorem 2 we conclude that problem* $(S_0)-(BC_0)$ *has at least one solution* $(u(t),v(t))$, $t \in [0,1]$.

Example 3. *We consider the functions*

$$\begin{aligned} f(t,x_1,x_2,x_3,x_4) =& -\frac{1}{4}x_1^{3/5} + \frac{1}{2(1+t)}\arctan x_4^{2/3}, \\ g(t,y_1,y_2,y_3,y_4) =& \frac{e^{-t}}{1+t^4} - \frac{1}{3}|y_2|^{1/2} + \sin|y_3|^{3/4}, \end{aligned}$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1, \ldots, 4$. *Because we obtain*

$$|f(t,x_1,x_2,x_3,x_4)| \leq \frac{1}{4}|x_1|^{3/5}+\frac{1}{2}|x_4|^{2/3}, \quad |g(t,y_1,y_2,y_3,y_4)| \leq 1+\frac{1}{3}|y_2|^{1/2}+|y_3|^{3/4},$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1, \ldots, 4$, *then assumption* $(I5)$ *is satisfied with* $a_0 = 0$, $a_1 = \frac{1}{4}$, $a_2 = 0$, $a_3 = 0$, $a_4 = \frac{1}{2}$, $b_0 = 1$, $b_1 = 0$, $b_2 = \frac{1}{3}$, $b_3 = 1$, $b_4 = 0$, $l_1 = \frac{3}{5}$, $l_4 = \frac{2}{3}$, $m_2 = \frac{1}{2}$, $m_3 = \frac{3}{4}$. *Therefore, by Theorem 5 we deduce that problem* $(S_0) - (BC_0)$ *has at least one solution* $(u(t), v(t))$, $t \in [0,1]$.

Example 4. *We consider the functions*

$$\begin{aligned} f(t,x_1,x_2,x_3,x_4) &= \frac{(1-t)^3}{10} + \frac{e^{-t}x_2^3}{20(1+x_1^2)} - \frac{t^2 x_3^{1/3}}{5}, \\ g(t,y_1,y_2,y_3,y_4) &= \frac{t^2}{20} + \frac{1-t^2}{25}y_1^2 - \frac{1}{30}y_4^{1/5}, \end{aligned}$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1, \ldots, 4$. *Because we have*

$$\begin{aligned} |f(t,x_1,x_2,x_3,x_4)| &\leq \frac{1}{10} + \frac{1}{20}|x_2|^3 + \frac{1}{5}|x_3|^{1/3}, \\ |g(t,y_1,y_2,y_3,y_4)| &\leq \frac{1}{20} + \frac{1}{25}|y_1|^2 + \frac{1}{30}|y_4|^{1/5}, \end{aligned}$$

for all $t \in [0,1]$, x_i, $y_i \in \mathbb{R}$, $i = 1, \ldots, 4$, *then assumption* $(I6)$ *is satisfied with* $p_0 = \frac{1}{10}$, $p_1 = 0$, $p_2 = \frac{1}{20}$, $p_3 = \frac{1}{5}$, $p_4 = 0$, $h_1(x) = 0$, $h_2(x) = x^3$, $h_3(x) = x^{1/3}$, $h_4(x) = 0$, $q_0 = \frac{1}{20}$, $q_1 = \frac{1}{25}$, $q_2 = 0$, $q_3 = 0$, $q_4 = \frac{1}{30}$, $k_1(x) = x^2$, $k_2(x) = 0$, $k_3(x) = 0$, $k_4(x) = x^{1/5}$. *For* $\Xi_0 = 2$, *condition (13) is satisfied because*

$$\begin{aligned} &\left(p_0 + p_1 h_1(2) + p_2 h_2(2) + p_3 h_3\left(\frac{2}{\Gamma(\theta_1+1)}\right) + p_4 h_4\left(\frac{2}{\Gamma(\sigma_1+1)}\right)\right) M_7 \\ &+ \left(q_0 + q_1 k_1(2) + q_2 k_2(2) + q_3 k_3\left(\frac{2}{\Gamma(\theta_2+1)}\right) + q_4 k_4\left(\frac{2}{\Gamma(\sigma_2+1)}\right)\right) M_9 \\ &\approx 1.96264 < 2. \end{aligned}$$

Therefore, by Theorem 6 we conclude that problem $(S_0) - (BC_0)$ *has at least one solution* $(u(t), v(t))$, $t \in [0,1]$.

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