

Asymptotics of partial sums of the Dirichlet series of the arithmetic derivative

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Received January 10, 2019; accepted November 25, 2019

Abstract. For $\emptyset \neq P \subseteq \mathbb{P}$, let D_P be the arithmetic subderivative function with respect to P on \mathbb{Z}_+ , let ζ_{D_P} be the function defined by the Dirichlet series of D_P , and let σ_{D_P} denote its abscissa of convergence. Under certain assumptions concerning s and P , we present asymptotic formulas for the partial sums of $\zeta_{D_P}(s)$ and show that $\sigma_{D_P} = 2$. We also express $\zeta_{D_P}(s)$, $s > 2$, using the Riemann zeta function.

AMS subject classifications: 11N37, 11N56

Key words: Abscissa of convergence, arithmetic derivative, Dirichlet series

1. Introduction

Let $n \in \mathbb{Z}_+$. There exists a unique sequence of nonnegative integers (with only finitely many positive terms)

$$(\nu_p(n))_{p \in \mathbb{P}},$$

where \mathbb{P} stands for the set of primes, such that

$$n = \prod_{p \in \mathbb{P}} p^{\nu_p(n)}.$$

We use the approach mostly from [1, 3, 5, 7]. Let $\emptyset \neq P \subseteq \mathbb{P}$. The *arithmetic subderivative* of n with respect to P is

$$D_P(n) = n'_P := \sum_{p \in P} n'_p,$$

where n'_p is the *arithmetic partial derivative* of n with respect to $p \in \mathbb{P}$, defined by

$$D_p(n) = n'_p = n'_{\{p\}} := \frac{\nu_p(n)}{p} n.$$

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The *arithmetic derivative* of n is

$$D(n) = n' := n'_\mathbb{P} = \sum_{p \in \mathbb{P}} n'_p.$$

We define the (*arithmetic*) *logarithmic subderivative*, *logarithmic partial derivative*, and *logarithmic derivative* of n , respectively, as follows:

$$\text{ld}_P(n) = \frac{n'_P}{n}, \quad \text{ld}_p(n) = \frac{n'_p}{n}, \quad \text{ld}(n) = \frac{n'}{n}.$$

Let f be an arithmetic function. There exists $\sigma_f \in \mathbb{R} \cup \{\pm\infty\}$ such that its *Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad s \in \mathbb{C},$$

converges if $\Re(s) > \sigma_f$ (\Re denotes the real part) and diverges if $\Re(s) < \sigma_f$ (see [6, p. 108, Theorem 3]). We call σ_f the *abscissa of convergence* of this series and define the function ζ_f by

$$\zeta_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad \Re(s) < \sigma_f.$$

For example, let the function u be identically one. The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta_u(s), \quad \text{and} \quad \sigma_u = 1.$$

Our paper originates from three results due to Barbeau [1]. The first one gives an upper bound for n' using n :

Lemma 1 (see [1, p. 118] or [7, Theorem 9]). *Let $n \in \mathbb{Z}_+$. Then*

$$n' \leq \frac{n \log n}{2 \log 2}.$$

In the next theorem, the first and second formula describe the asymptotic behavior of

$$\sum_{1 \leq n \leq x} \text{ld}(n) \quad \text{and} \quad \sum_{1 \leq n \leq x} n' :$$

Theorem 1 (see [1, pp. 119–121] or [7, Theorem 24]). *Asymptotically,*

$$\sum_{1 \leq n \leq x} \text{ld}(n) = Cx + O(\log x \log \log x)$$

and

$$\sum_{1 \leq n \leq x} n' = C \frac{x^2}{2} + O(x^{1+\delta}).$$

Here

$$C = \sum_{p \in \mathbb{P}} \frac{1}{p(p-1)} = 0.749\dots, \quad (1)$$

and $\delta > 0$ is arbitrary.

In the proofs cited above, actually $x \in \mathbb{Z}_+$, but they can easily be extended to hold for $x \in \mathbb{R}$, $x \geq 1$.

Our goal is to find asymptotic formulas for the partial sums of $\zeta_{D_P}(s)$, in other words, for the sums

$$\sum_{1 \leq n \leq x} \frac{n'}{n^s} \text{ and } \sum_{1 \leq n \leq x} \frac{n'_p}{n^s} \text{ and, more generally, for } \sum_{1 \leq n \leq x} \frac{n'_P}{n^s}, \quad (2)$$

where $s \in \mathbb{R}$. As a corollary, we will see that $\sigma_D = \sigma_{D_p} = \sigma_{D_P} = 2$. For $s = 1$ and $s = 0$, the formulas concerning the first sum are already given in Theorem 1. Lastly, we express $\zeta_{D_P}(s)$, $s > 2$, using the values of ζ .

Our main tool is the following Abel's summation formula:

Lemma 2 (see [6, p. 3, Theorem 1]). *Let (a_n) be a sequence of complex numbers, let $x > 1$, and let $g : [1, x] \rightarrow \mathbb{C}$ be a continuously differentiable function. Then*

$$\sum_{1 \leq n \leq x} a_n g(n) = \left(\sum_{1 \leq n \leq x} a_n \right) g(x) - \int_1^x \left(\sum_{1 \leq n \leq t} a_n \right) g'(t) dt.$$

2. Partial sums of $\zeta_D(2)$

In this section, we consider the first sum of (2) with $s = 2$. We obtain the following result:

Theorem 2. *Asymptotically,*

$$\sum_{1 \leq n \leq x} \frac{n'}{n^2} = C \log x + O(1).$$

Proof. Applying Lemma 2 to

$$a_n = \frac{n'}{n}, \quad g(x) = \frac{1}{x},$$

we obtain

$$\sum_{1 \leq n \leq x} \frac{n'}{n^2} = \sum_{1 \leq n \leq x} \frac{n'}{n} \frac{1}{n} = H(x) + K(x),$$

where

$$H(x) = \left(\sum_{1 \leq n \leq x} \frac{n'}{n} \right) \frac{1}{x}, \quad K(x) = \int_1^x \left(\sum_{1 \leq n \leq t} \frac{n'}{n} \right) \frac{1}{t^2} dt.$$

By Theorem 1,

$$H(x) = C + O(x^{-1} \log x \log \log x) = O(1) \quad (3)$$

and

$$\begin{aligned} K(x) &= \int_1^x (Ct + O(\log t \log \log t)) \frac{1}{t^2} dt \\ &= \int_1^x C \frac{1}{t} dt + \int_1^x O(t^{-2} \log t \log \log t) dt \\ &= C \log x + O\left(\int_1^x t^{-2} \log t \log \log t dt\right). \end{aligned}$$

Further, since

$$\log t \log \log t = O(t^\delta)$$

for any $\delta \in (0, 1)$, we have

$$\begin{aligned} K(x) &= C \log x + O\left(\int_1^x t^{\delta-2} dt\right) = C \log x + O(x^{\delta-1}) + O(1) \\ &= C \log x + O(1). \end{aligned} \tag{4}$$

Now, the claim follows from (3) and (4). \square

Corollary 1. *It holds that $\sigma_D = 2$.*

Proof. By Lemma 1,

$$0 \leq \frac{n'}{n^s} \leq \frac{n \log n}{2n^s \log 2} = \frac{\log n}{2n^{s-1} \log 2}. \tag{5}$$

If $s > 2$, then the series

$$\sum_{n=1}^{\infty} \frac{\log n}{n^{s-1}}$$

converges. By using (5), we conclude that the series

$$\sum_{n=1}^{\infty} \frac{n'}{n^s}$$

converges, too. Hence $\sigma_D \geq 2$. On the other hand, since by Theorem 2 the series

$$\sum_{n=1}^{\infty} \frac{n'}{n^2}$$

diverges, we have $\sigma_D \leq 2$. \square

3. Partial sums of $\zeta_D(s)$, $1 \neq s < 2$

Next, we study the first sum of (2) in the case of $1 \neq s < 2$.

Theorem 3. *Let $1 \neq s < 2$. Asymptotically,*

$$\sum_{1 \leq n \leq x} \frac{n'}{n^s} = \frac{C}{2-s} x^{2-s} + R(x),$$

where $R(x)$ is defined as follows: If $1 < s < 2$, then $R(x) = O(1)$. If $s < 1$, then $R(x) = O(x^{\delta-(s-1)})$ for any $\delta > 0$.

Proof. Assume first that $1 < s < 2$. We proceed as in the proof of Theorem 2 but take

$$g(x) = \frac{1}{x^{s-1}}.$$

Then

$$\sum_{1 \leq n \leq x} \frac{n'}{n^s} = H(x) + K(x),$$

where

$$H(x) = Cx^{2-s} + O(x^{1-s} \log x \log \log x) = Cx^{2-s} + O(1)$$

and

$$\begin{aligned} K(x) &= \int_1^x (Ct + O(\log t \log \log t)) \frac{s-1}{t^s} dt \\ &= C(s-1) \int_1^x \frac{dt}{t^{s-1}} + (s-1) \int_1^x O(t^{-s} \log t \log \log t) dt \\ &= C \frac{s-1}{2-s} x^{2-s} + O(1) + O\left(\int_1^x t^{-s} \log t \log \log t dt\right) \\ &= C \frac{s-1}{2-s} x^{2-s} + O(1) + O\left(\int_1^x t^{\delta-s} dt\right) \\ &= C \frac{s-1}{2-s} x^{2-s} + O(1) + O(x^{\delta-(s-1)}) + O(1). \end{aligned}$$

We can restrict ourselves to $0 < \delta \leq s-1$. Then $\delta - (s-1) \leq 0$, which implies that

$$K(x) = C \frac{s-1}{2-s} x^{2-s} + O(1)$$

and further,

$$H(x) + K(x) = C \left(1 + \frac{s-1}{2-s}\right) x^{2-s} + O(1) = \frac{C}{2-s} x^{2-s} + O(1),$$

completing the proof in this case.

If $s < 1$, then

$$K(x) = C \frac{s-1}{2-s} x^{2-s} + O(x^{\delta-(s-1)}),$$

and we can proceed as above. \square

Note that this theorem is a generalization of the latter part of Theorem 1; just set $s = 0$.

4. Partial sums of $\zeta_{D_p}(1)$

We show that the asymptotic formulas for the partial sums of $\zeta_D(s)$ given in Theorems 1–3 have variants for those of $\zeta_{D_p}(s)$. In these variants, the coefficient C given in (1) is replaced by C_p defined as

$$C_p = \frac{1}{p(p-1)}, \quad p \in \mathbb{P}.$$

Note that $C = \sum_{p \in \mathbb{P}} C_p$.

We begin the study of the partial sums of $\zeta_{D_p}(s)$ with $s = 1$.

Theorem 4. *Let $p \in \mathbb{P}$. Asymptotically,*

$$\sum_{1 \leq n \leq x} \text{ld}_p(n) = C_p x + O(\log x).$$

Proof. It is easy to see that it is enough to consider the sum

$$\sum_{k=1}^n \text{ld}_p(k) = \text{ld}_p \prod_{k=1}^n k = \text{ld}_p(n!).$$

We modify the proof of the first part of Theorem 1. By [2, Theorem 416],

$$n! = \prod_{q \in \mathbb{P}} q^{\mu_q(n)}, \quad (6)$$

where

$$\mu_q(n) = \sum_{m=1}^{\infty} \left\lfloor \frac{n}{q^m} \right\rfloor = \sum_{m=1}^{\alpha(n)} \left\lfloor \frac{n}{q^m} \right\rfloor, \quad \alpha(n) = \left\lfloor \frac{\log n}{\log q} \right\rfloor. \quad (7)$$

Now, denoting by $\stackrel{(i)}{=}$ that the equation follows from the formula (i), we obtain

$$\begin{aligned} \text{ld}_p(n!) &\stackrel{(6)}{=} \text{ld}_p \prod_{q \in \mathbb{P}} q^{\mu_q(n)} = \frac{\mu_p(n)}{p} \stackrel{(7)}{=} \frac{1}{p} \sum_{m=1}^{\alpha(n)} \left\lfloor \frac{n}{p^m} \right\rfloor \\ &= \frac{1}{p} \sum_{m=1}^{\alpha(n)} \frac{n}{p^m} + \frac{1}{p} \sum_{m=1}^{\alpha(n)} O(1) \stackrel{(7)}{=} n \sum_{m=2}^{\alpha(n)+1} \frac{1}{p^m} + O(\log n) \\ &= n \sum_{m=2}^{\infty} \frac{1}{p^m} - n \sum_{m=\alpha(n)+2}^{\infty} \frac{1}{p^m} + O(\log n) \\ &= C_p n - \frac{n}{p^{\alpha(n)+1}(p-1)} + O(\log n). \end{aligned}$$

It remains to study the complexity of

$$A(n) = \frac{n}{p^{\alpha(n)+1}(p-1)}.$$

Since

$$p^{\alpha(n)+1} \geq 2^{\alpha(n)+1} > n$$

by (7), it follows that $A(n) = O(1)$, and the proof is complete. \square

5. Partial sums of $\zeta_{D_p}(s)$ and $\zeta_{D_P}(s)$

In this section, we continue by studying the second sum of (2), where $1 \neq s \leq 2$. We first assume that $s = 2$.

Theorem 5. *Let $p \in \mathbb{P}$. Asymptotically,*

$$\sum_{1 \leq n \leq x} \frac{n'_p}{n^2} = C_p \log x + O(1).$$

Proof. The proof is analogous to that of Theorem 2. We apply Lemma 2 to

$$a_n = \frac{n'_p}{n}, \quad g(x) = \frac{1}{x},$$

and use Theorem 4. □

Corollary 2. *Let $p \in \mathbb{P}$. Then $\sigma_{D_p} = 2$.*

Proof. Clearly, $0 \leq n'_p \leq n'$ for all $n \in \mathbb{Z}_+$. Since $\sigma_D = 2$ by Corollary 1, we have $\sigma_{D_p} \geq 2$. On the other hand, since by Theorem 5 the series

$$\sum_{n=1}^{\infty} \frac{n'_p}{n^2}$$

diverges, it follows that $\sigma_{D_p} \leq 2$. □

Next, we consider the case of $1 \neq s < 2$.

Theorem 6. *Let $p \in \mathbb{P}$ and $1 \neq s < 2$. Asymptotically,*

$$\sum_{1 \leq n \leq x} \frac{n'_p}{n^s} = \frac{C_p}{2-s} x^{2-s} + R(x),$$

where $R(x)$ is as in Theorem 3.

Proof. The proof is a simple modification of that of Theorem 3. □

Corollary 3 (see Theorem 1). *Let $p \in \mathbb{P}$. Then*

$$\sum_{1 \leq n \leq x} n'_p = C_p \frac{x^2}{2} + O(x^{\delta+1})$$

for any $\delta > 0$.

Our results about $\zeta_{D_p}(s)$ can be extended to concern $\zeta_{D_P}(s)$ if $P \subset \mathbb{P}$ is nonempty and finite (or if $P = \mathbb{P}$, see Theorem 3). Then C_p is replaced by

$$C_P = \sum_{p \in P} \frac{1}{p(p-1)}.$$

For example, Theorem 4 and Theorem 6 ($s = 0$) extend to

$$\sum_{1 \leq n \leq x} \text{ld}_P(n) = C_P x + O(\log x), \quad \sum_{1 \leq n \leq x} n'_P = C_P \frac{x^2}{2} + O(x^{\delta+1}),$$

and Corollary 2 extends to $\sigma_{D_P} = 2$.

6. Reducing ζ_{D_P} to ζ

It is natural to expect that ζ_{D_P} has a close relation to the Riemann zeta function ζ . For ζ_{D_p} , this relation is already known in the following lemma (originally with different terminology and notation):

Lemma 3 (see [4, Lemma 6]). *Let $p \in \mathbb{P}$ and $s > 2$. Then*

$$\zeta_{D_p}(s) = \frac{\zeta(s-1)}{p^s - p}.$$

We extend this to ζ_{D_P} .

Theorem 7. *Let $\emptyset \neq P \subseteq \mathbb{P}$ and $s > 2$. Then*

$$\zeta_{D_P}(s) = \zeta(s-1) \sum_{p \in P} \frac{1}{p^s - p}.$$

Proof. We have

$$\zeta_{D_P}(s) = \sum_{n=1}^{\infty} \frac{n'_P}{n^s} = \sum_{n=1}^{\infty} \frac{n \sum_{p \in P} \frac{\nu_p(n)}{p}}{n^s} = \sum_{n=1}^{\infty} \sum_{p \in P} \frac{\nu_p(n)}{pn^{s-1}}. \quad (8)$$

Since the series (8) converges and all its terms are nonnegative, we can change the order of summation. Therefore, by the simple calculation and applying Lemma 3 we obtain

$$\begin{aligned} \zeta_{D_P}(s) &= \sum_{p \in P} \sum_{n=1}^{\infty} \frac{\nu_p(n)}{pn^{s-1}} = \sum_{p \in P} \sum_{n=1}^{\infty} \frac{n\nu_p(n)}{pn^s} = \sum_{p \in P} \sum_{n=1}^{\infty} \frac{n'_p}{n^s} \\ &= \sum_{p \in P} \zeta_{D_p}(s) = \sum_{p \in P} \frac{\zeta(s-1)}{p^s - p}, \end{aligned}$$

completing the proof. \square

In particular,

$$\zeta_D(s) = \zeta(s-1) \sum_{p \in \mathbb{P}} \frac{1}{p^s - p}.$$

7. Three further questions

In the case of $s \leq 2$, Theorems 1–3 give asymptotic formulas for the first sum of (2), and Theorems 4–6 give those for the second. What about the case of $s > 2$? Theorems 3 and 6 with $R(x) = O(1)$ hold also then, but since the main term has a smaller complexity than the error term, we get nothing reasonable out of them. The question about a nontrivial asymptotic formula for the second (and third) sum of (2) in the case of $s > 2$ therefore remains open.

As noted at the end of Section 5, our results about $\zeta_{D_p}(s)$ can be extended to $\zeta_{D_P}(s)$ if $P \subset \mathbb{P}$ is nonempty and finite or if $P = \mathbb{P}$. Can they be extended also if $P \subset \mathbb{P}$ is infinite? This question remains open, too.

Using advanced number-theoretic methods, the error terms of our asymptotic formulas can probably be improved, i.e., their complexity can be decreased. How could this be done? This is our third question.

8. Acknowledgment

We thank the referees. Their suggestions improved significantly the presentation of our paper.

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