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On Diophantine, pronic and triangular triples of balancing numbers

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Abstract. In this paper, we search for some Diophantine triples of balancing numbers. We prove that, if $(6\pm2)B_nB_k+1$ and $(6\pm2)B_{n+2}B_k+1$ are both squares, then $k=n+1$, for any positive integer n. In addition, we define pronic m-tuples and triangular m-tuples, and prove some results related to pronic and triangular triples of balancing numbers.

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1. Introduction

Balancing numbers n and balancers r are solutions of the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r).$$

The sequence of balancing numbers $\{B_n\}_{n\geq1}$ satisfy the linear homogeneous binary recurrence $B_{n+1}=6B_n-B_{n-1}, n\geq1$, with initial terms $B_0=0, B_1=1$. Moreover, if a positive integer x is a balancing number, then x^2 is a triangular number, and consequently, $8x^2+1$ is a square and the positive square root of $8x^2+1$ is called a Lucas-balancing number. The sequence of Lucas-balancing numbers $\{C_n\}_{n\geq1}$ also satisfy the recurrence relation $C_{n+1}=6C_n-C_{n-1}, n\geq1$, with initial terms $C_0=1, C_1=3$ (see [1]). Further, cobalancing numbers n and cobalancers r are solutions of the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r).$$

The sequence of cobalancing numbers $\{b_n\}_{n\geq1}$ satisfy the non-homogeneous binary recurrence $b_{n+1}=6b_n-b_{n-1}+2, n\geq1$, with initial terms $b_0=0, b_1=0$. Moreover, a positive integer x is a cobalancing number if and only if $x(x+1)$ is a triangular number (see [13]).

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A set of m positive integers $\{a_1, a_2, \cdots, a_m\}$ is called a Diophantine m-tuple if for all $i, j \in \{1, 2, \cdots, m\}$ and $i \neq j$, $a_i a_j + 1$ is a perfect square. Diophantus was the first to discover the rational quadruples $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ with the above property. Subsequently, Fermat obtained the first Diophantine quadruple $\{1, 3, 8, 120\}$. Euler tried to extend this set to a Diophantine quintuple but did not succeed. However, he found a fifth term $\frac{777480}{8288641}$, which is a rational number. Moreover, he managed to find an infinite family of the Diophantine quadruple $\{a, b, a+b+2r, 4r(r+a)(r+b)\}$, starting with two numbers a and b such that $ab+1 = r^2$. Later on, Dujella [5] proved that no Diophantine sextuples exist. In this sequel He, Togbé and Ziegler [7] proved the nonexistence of Diophantine quintuples.

Many Diophantine triples and quadruples involving terms of binary recurrence sequences have been studied since 1977, some of which are available in [2, 3, 4, 8]. In [6], He, Luca and Togbé proved that if $\{F_{2n}, F_{2n+2}, F_k\}$ is a Diophantine triple, then $k \in \{2n+4, 2n-2\}$, except when $n = 2$, one additional solution $k = 1$ exists. Subsequently, Rihane, Hernane and Togbé [17] proved that if $\{P_{2n}, P_{2n+2}, 2P_k\}$ is a Diophantine triple, then $k \in \{2n, 2n+2\}$.

Let n be a positive integer. The balancing numbers B_n satisfy the identity $B_n B_{n+2} + 1 = B_{n+1}^2$ (see [14]). Thus, it is natural to search for a positive integer X which would make $\{B_n, B_{n+2}, X\}$ a Diophantine triple. To find such an X, one needs to solve a system of Diophantine equations

$$B_n X + 1 = Y^2, \quad B_{n+2} X + 1 = Z^2. \tag{1}$$

It is well known that a positive integer x is a pronic or triangular number as $4x+1$ or $8x+1$ is a square. Using these properties, we define a pronic and triangular m-tuple as follows:

Definition 1. *A set of m positive integers $\{a_1, a_2, \cdots, a_m\}$ is called a pronic (triangular) m-tuple if for all $i, j \in \{1, 2, \cdots, m\}$ and $i \neq j$, $a_i a_j$ is a pronic (triangular).*

Observe that a set of m positive integers $\{a_1, a_2, \cdots, a_m\}$ is a pronic m-tuple if for all $i, j \in \{1, 2, \cdots, m\}$ and $i \neq j$, $4a_i a_j + 1$ is a square. Similarly, $\{a_1, a_2, \cdots, a_m\}$ is a triangular m-tuple if for all $i, j \in \{1, 2, \cdots, m\}$ and $i \neq j$, $8a_i a_j + 1$ is a square.

In [15], Panda proved that the product of two consecutive balancing numbers is a pronic number as well as a triangular number. In particular, for every positive integer n, it is easy to see that

$$B_n B_{n+1} = b_{n+1}(b_{n+1} + 1) \quad \text{and} \quad B_n B_{n+1} = \frac{(B_n + b_{n+1})(B_n + b_{n+1} + 1)}{2}.$$

Thus, it is also natural to see if $\{B_n, B_{n+1}, X\}$ is a pronic or triangular triple for some positive integer X.

The aim of this paper is to continue in the spirit developed by He-Luca-Togbé [6] and prove the following theorems:

Theorem 1. *For a fixed positive integer n, if $\{B_n, B_{n+2}, 4B_k\}$ or $\{B_n, B_{n+2}, 8B_k\}$ is a Diophantine triple, then $k = n + 1$.*

Theorem 2. *For a fixed positive integer n, if $\{B_n, B_{n+1}, 2B_k\}$ is a pronic triple, then $k \in \{n, n+1\}$.*

Theorem 3. *There do not exist positive integers n, k such that $\{B_n, B_{n+1}, B_k\}$ is a triangular triple.*

We organize this paper as follows. In Section 2, we recall and prove some elementary results that will be useful for the proofs of our main results stated above. Section 3 helps us to see that we must consider only the two Diophantine triples $\{B_n, B_{n+2}, (6 \pm 2)B_k\}$ that will be studied in Section 4. The last section is devoted to exploring pronic and triangular triples of balancing numbers.

2. Preliminaries

In this section, we state some definitions and results on some properties of balancing numbers, algebraic numbers, logarithmic heights, continued fractions and convergents which are needed in the forthcoming sections. We recall or prove the following results.

At first, we need the definition of the height of an algebraic number.

Definition 2. *Let γ be any non-zero algebraic number of degree d over $\mathbb{Q}$ whose minimal polynomial over $\mathbb{Z}$ is a $\prod_{j=1}^{d}(X - \gamma^{(j)})$. We denote the absolute logarithmic height of γ by*

$$h(\gamma) = \frac{1}{d}\Big(\log a + \sum_{j=1}^{d} \log \max\left(1, |\gamma^{(j)}|\right)\Big).$$

Lemma 1 (see [12])**.** *Let Λ be a linear form in logarithms of multiplicatively independent totally real algebraic numbers $\alpha_1, \cdots, \alpha_N$ with rational integer coefficients $b_1, \cdots, b_N, (b_N \neq 0)$.*

Define $D := [\mathbb{Q}(\alpha_1, \cdots, \alpha_N) : \mathbb{Q}], A_j = \max\{Dh(\alpha_j), |\log\alpha_j|\}$ $(1 \leq j \leq N)$ and $E = \max\{1, \max\{|b_j|A_j/A_N; 1 \leq j \leq N\}\}$. Then

$$\log|\Lambda| > -C(N)C_0W_0D^2\Omega,$$

where

$$\begin{aligned} C(N) &:= \frac{8}{(N-1)!}(N+2)(2N+3)(4e(N+1))^{N+1} \\ C_0 &:= \log(e^{4.4N+7}N^{5.5}D^2\log(eD)), \\ W_0 &:= \log(1.5eED\log(eD)), \quad \Omega := A_1 \cdots A_N. \end{aligned}$$

Lemma 2 (see [11])**.** *Let $\gamma_1, \gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $b_1, b_2 \in \mathbb{Z}$ not both 0 and*

$$\Lambda = b_2\log\gamma_2 - b_1\log\gamma_1.$$

Define $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$. Let

$$h_i \geq \max\Big\{h(\gamma_i), \frac{|\log\gamma_i|}{D}, \frac{1}{D}\Big\}, \quad \text{for } i = 1, 2, \quad b' \geq \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

Then

$$\log|\Lambda| \geq -17.9 \cdot D^4\Big(\max\{\log b' + 0.38, \frac{30}{D}, 1\}\Big)^2 h_1h_2.$$

Lemma 3 (see [4, Lemma 5(a)]). *Assume that κ and μ are real numbers and M is a positive integer. Let P/Q be a convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = ||\mu Q|| - M \cdot ||\kappa Q||;$$

$||\cdot||$ denotes the distance to the nearest integer. If $\eta > 0$, then there is no solution to the inequality

$$0 < j\kappa - k + \mu < AB^{-j}$$

in integers j and k with

$$\frac{\log(AQ/\eta)}{\log B} \leq j \leq M.$$

The following lemma deals with some properties of balancing numbers and can be found in [14, 15].

Lemma 4. *If n is any positive integer, $\alpha = 3 + \sqrt{8}$ and $\overline{\alpha} = 3 - \sqrt{8}$, then:*

(*i*) $B_{2n} = 2B_nC_n$,

(*ii*) $\gcd(B_n, C_n) = 1$,

(*iii*) $2 \mid B_n$ *if and only if* $2 \mid n$,

(*iv*) $B_{n+1}B_{n-1} = B_n^2 - 1$,

(*v*) $B_n = (\alpha^n - \overline{\alpha}^n)/(4\sqrt{2})$,

(*vi*) $\alpha^{n-1} < B_n < \alpha^n$.

Lemma 5. *The Diophantine equation $B_n = x^2$ has the only solution $n = 1$. Further, $B_n = 2x^2$ has no solution in positive integer n.*

Proof. In [16], Panda proved that 1 is the only perfect square in the balancing sequence, and hence, $B_n = x^2$ has the only solution $n = 1$. Further, if $B_n = 2x^2$, then B_n is even which implies that n is even. Letting $n = 2n_1$, we have $B_{2n_1} = 2B_{n_1}C_{n_1} = 2x^2$ and so $B_{n_1}C_{n_1} = x^2$. Thus, $B_{n_1} = x_1^2, C_{n_1} = x_2^2$ for some integers x_1 and x_2. Using the solution of $B_n = x^2$ and the values $B_1 = 1, C_1 = 3$, we get $3 = x_2^2$, which is a contradiction. Hence, the result follows. $\square$

3. Diophantine triples of balancing numbers

Consider Pell's equation $V^2 - B_nB_{n+2}U^2 = 1$, where n is a fixed positive integer. Using the fundamental solution $(U, V) = (1, B_{n+1})$, the totality of the solution is given by

$$U_l^{(n)} = \frac{(B_{n+1} + \sqrt{B_nB_{n+2}})^l - (B_{n+1} - \sqrt{B_nB_{n+2}})^l}{2\sqrt{B_nB_{n+2}}}$$

and

$$V_l^{(n)} = \frac{(B_{n+1} + \sqrt{B_nB_{n+2}})^l + (B_{n+1} - \sqrt{B_nB_{n+2}})^l}{2}.$$

Moreover, for all $l \geq 0$, $U_l^{(n)}$ and $V_l^{(n)}$ satisfy the recurrence

$$U_{l+2}^{(n)} = 2B_{n+1}U_{l+1}^{(n)} - U_l^{(n)}, \quad U_0^{(n)} = 0, U_1^{(n)} = 1$$

and

$$V_{l+2}^{(n)} = 2B_{n+1}V_{l+1}^{(n)} - V_l^{(n)}, \quad V_0^{(n)} = 1, V_1^{(n)} = B_{n+1},$$

respectively.

Eliminating X from (1), we get $B_{n+2}Y^2 - B_nZ^2 = B_{n+2} - B_n$ or, equivalently,

$$(B_{n+2}Y)^2 - B_nB_{n+2}Z^2 = B_{n+2}(B_{n+2} - B_n),$$

which is a generalized Pell's equation and we will find out two classes of solutions corresponding to $Y \equiv Z \pmod{B_{n+2}(B_{n+2} - B_n)}$.

The congruence

$$(B_{n+2}Z)^2 \equiv B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n) \pmod{B_{n+2}(B_{n+2} - B_n)} \tag{2}$$

holds and is implied by

$$B_{n+2}Z \equiv \pm\sqrt{B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n)} \pmod{B_{n+2}(B_{n+2} - B_n)} \tag{3}$$

and any solution to (3) is a solution to (2). In view of (3),

$$\frac{B_{n+2}Z + \sqrt{B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n)}}{B_{n+2}(B_{n+2} - B_n)}$$

or

$$\frac{B_{n+2}Z - \sqrt{B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n)}}{B_{n+2}(B_{n+2} - B_n)}$$

is an integer. Since

$$B_nB_{n+2}\left[\frac{B_{n+2}Z \pm \sqrt{B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n)}}{B_{n+2}(B_{n+2} - B_n)}\right]^2 + 1$$

$$= \left[\frac{B_nZ \pm \sqrt{B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n)}}{B_{n+2} - B_n}\right]^2,$$

it follows that either

$$\frac{B_{n+2}Z + \sqrt{B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n)}}{B_{n+2}(B_{n+2} - B_n)} = U$$

or

$$\frac{B_{n+2}Z - \sqrt{B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n)}}{B_{n+2}(B_{n+2} - B_n)} = U,$$

where $B_nB_{n+2}U^2 + 1 = V^2$. Letting

$$U = \frac{B_{n+2}Z \pm \sqrt{B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2} - B_n)}}{B_{n+2}(B_{n+2} - B_n)},$$

we get

$$[B_{n+2}(B_{n+2}-B_n)U - B_{n+2}Z]^2 = B_nB_{n+2}Z^2 + B_{n+2}(B_{n+2}-B_n),$$

which, on rearranging results in the quadratic equation

$$B_{n+2}(B_{n+2}-B_n)Z^2 - 2B_{n+2}^2(B_{n+2}-B_n)UZ + B_{n+2}(B_{n+2}-B_n)(B_{n+2}^2U^2 - V^2) = 0$$

upon solving for Z, we get $Z = B_{n+2}U \pm V$. We further observe that

$$B_nB_{n+2}[B_{n+2}U \pm V]^2 + B_{n+2}(B_{n+2}-B_n) = B_{n+2}^2[V \pm B_nU]^2.$$

Therefore,

$$Z = B_{n+2}U \pm V, \quad Y = V \pm B_nU. \tag{4}$$

Using (4) in (1), we get $X = 6B_{n+1}U^2 \pm 2UV$. Thus, for a fixed positive integer n, if $\{B_n, B_{n+2}, X\}$ is a Diophantine triple, then there are two classes of choices for X given by $X = 6B_{n+1}{U_j^{(n)}}^2 + 2U_j^{(n)}V_j^{(n)}$ and $X = 6B_{n+1}{U_j^{(n)}}^2 - 2U_j^{(n)}V_j^{(n)}$, for $j \geq 1$. The above discussion proves the following theorem:

Theorem 4. *For any fixed positive integer n, if $\{B_n, B_{n+2}, X\}$ is a Diophantine triple, then the possible values of X may be realized in multiple classes. Two such classes are given by*

$$X = 6B_{n+1}{U_j^{(n)}}^2 + 2U_j^{(n)}V_j^{(n)} \quad \text{and} \quad X = 6B_{n+1}{U_j^{(n)}}^2 - 2U_j^{(n)}V_j^{(n)},$$

where $U_j^{(n)}$ and $V_j^{(n)}$ are solutions to Pell's equation $V^2 - B_nB_{n+2}U^2 = 1$ with $j \geq 1$.

Observe that the case $j = 1$ gives two Diophantine triples, i.e, $\{B_n, B_{n+2}, 4B_{n+1}\}$ and $\{B_n, B_{n+2}, 8B_{n+1}\}$. In the next section, we will see that if $j > 1$, then X will not be of this form.

4. The Diophantine triples $\{B_n, B_{n+2}, (6 \pm 2)B_k\}$

In this section, we will find the possible value(s) of k such that $\{B_n, B_{n+2}, 4B_k\}$ and $\{B_n, B_{n+2}, 8B_k\}$ are Diophantine triples.

In view of Theorem 4, it is clear that if $\{B_n, B_{n+2}, 4B_k\}$ is a Diophantine triple, then

$$B_k = \frac{1}{4}\left[6B_{n+1}{U_j^{(n)}}^2 \pm 2U_j^{(n)}V_j^{(n)}\right]. \tag{5}$$

Consider

$$C_j^{(\pm)} = \frac{1}{4}\left[6B_{n+1}{U_j^{(n)}}^2 \pm 2U_j^{(n)}V_j^{(n)}\right], \quad \text{for } j = 1, 2, \ldots. \tag{6}$$

Therefore, we need to solve $C_j^{(\pm)} = B_k$. For $j = 1$, one can obtain $C_1^{(-)} = B_{n+1}$ as the only solution, and hence, to prove Theorem 1, it is sufficient to prove that $C_j^{(\pm)} = B_k$ has no solution for $j \geq 2$. Throughout the remaining part of the proof, we assume that $j \geq 2$.

The Binet formula of a balancing number is given by

$$B_k = \frac{\alpha^k - \overline{\alpha}^k}{4\sqrt{2}}, \quad k \geq 1, \tag{7}$$

where $\alpha = 3 + \sqrt{8}$ and $\overline{\alpha} = 3 - \sqrt{8}$. Let

$$\beta_n := B_{n+1} + \sqrt{B_{n+1}^2 - 1},$$

and thus,

$$V_j^{(n)} = \frac{\beta_n^j + \beta_n^{-j}}{2}, \quad U_j^{(n)} = \frac{\beta_n^j - \beta_n^{-j}}{2\sqrt{B_{n+1}^2 - 1}}.$$

Let

$$\gamma_n^{(\pm)} := \frac{1}{4}\left[\frac{6B_{n+1}}{4(B_{n+1}^2 - 1)} \pm \frac{2}{4\sqrt{B_{n+1}^2 - 1}}\right].$$

Using (6), we get

$$C_j^{(\pm)} = \beta_n^{2j}\gamma_n^{(\pm)} - \frac{6B_{n+1}}{8(B_{n+1}^2 - 1)} + \beta_n^{-2j}\gamma_n^{(\mp)}. \tag{8}$$

Thus, (7), (8) and $C_j^{(\pm)} = B_k$ together imply

$$\beta_n^{2j}\gamma_n^{(\pm)} - \frac{6B_{n+1}}{8(B_{n+1}^2 - 1)} + \beta_n^{-2j}\gamma_n^{(\mp)} = \frac{\alpha^k - \overline{\alpha}^k}{4\sqrt{2}}. \tag{9}$$

Next, we will define a linear form in three logarithms and find some upper and lower bounds for it. We begin with a lemma which deals with the bounds for $\gamma_n^{(+)}$ and $\gamma_n^{(-)}$.

Lemma 6. *For a fixed positive integer n, the following holds:*

(*i*) $0.48\alpha^{-n} < \gamma_n^{(+)} < 0.58\alpha^{-n}$,

(*ii*) $0.24\alpha^{-n} < \gamma_n^{(-)} < 0.31\alpha^{-n}$.

Proof. In view of the definition of $\gamma_n^{(\pm)}$, we have

$$4\sqrt{\gamma_n^{(\pm)}} = \frac{1}{\sqrt{B_n}} \pm \frac{1}{\sqrt{B_{n+2}}} = 2^{5/4}\alpha^{-n/2}\left[\frac{1}{\sqrt{1 - 1/\alpha^{2n}}} \pm \frac{1}{\alpha\sqrt{1 - 1/\alpha^{2n+4}}}\right]. \tag{10}$$

For $0 < x < 1$, the Taylor series expansion of $\frac{1}{\sqrt{1-x}}$ is

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \cdots,$$

which gives

$$1 + \frac{x}{2} < \frac{1}{\sqrt{1-x}} < 1 + \frac{x}{2(1-x)},$$

and so

$$1 \pm \frac{1}{\alpha} < \frac{1}{\sqrt{1-1/\alpha^{2n}}} \pm \frac{1}{\alpha\sqrt{1-1/\alpha^{2n+4}}} < 1.1 \pm \frac{1}{\alpha}. \tag{11}$$

From (10) and (11), we have

$$1 \pm \frac{1}{\alpha} < \frac{2^{3/4}\sqrt{\gamma_n^{(\pm)}}}{\alpha^{n/2}} < 1.1 \pm \frac{1}{\alpha},$$

which gives the desired bounds for $\gamma_n^{(+)}$ and $\gamma_n^{(-)}$ by putting the value of α. □

In view of (9), we define the following linear form in three logarithms:

$$\Lambda := 2j\log\beta_n - k\log\alpha + \log(4\sqrt{2}\cdot\gamma_n^{(\pm)}).$$

In the following lemma, we determine an upper bound for Λ.

Lemma 7. *If $j \geq 2$, then $0 < \Lambda < 13\beta_n^{-2j}$.*

Proof. In view of (9), we have

$$\beta_n^{2j}\gamma_n^{(\pm)} - \frac{\alpha^k}{4\sqrt{2}} = \frac{6B_{n+1}}{8(B_{n+1}^2-1)} - \beta_n^{-2j}\gamma_n^{(\mp)} - \frac{\overline{\alpha}^k}{4\sqrt{2}}.$$

If $\beta_n^{2j}\gamma_n^{(\pm)} \leq \frac{\alpha^k}{4\sqrt{2}}$, then

$$\frac{4\sqrt{2}}{\alpha^k} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(\pm)}} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(-)}}. \tag{12}$$

Using (12) in the inequality

$$\begin{aligned}\frac{1}{4B_{n+2}} < \frac{1}{8B_n} + \frac{1}{8B_{n+2}} = \frac{B_n + B_{n+2}}{8(B_{n+1}^2-1)} &< \beta_n^{-2j}\gamma_n^{(\mp)} + \frac{\overline{\alpha}^k}{4\sqrt{2}} \\ &\leq \beta_n^{-2j}\gamma_n^{(+)} + \frac{1}{4\sqrt{2}\alpha^k},\end{aligned}$$

we obtain

$$\frac{1}{4B_{n+2}} < \beta_n^{-2j}\left(\gamma_n^{(+)} + \frac{1}{32\gamma_n^{(-)}}\right). \tag{13}$$

Using Lemma 6 in (13), we get

$$4^j B_n^j B_{n+2}^j < \beta_n^{2j} < 4B_{n+2}\left(\gamma_n^{(+)} + \frac{1}{32\gamma_n^{(-)}}\right) < 4B_{n+2}(0.58\alpha^{-n} + 0.14\alpha^n),$$

which reduces to

$$4^j B_n^j B_{n+2}^{j-1} < 2.32\alpha^{-n} + 0.56\alpha^n. \tag{14}$$

But, (14) implies that $j < 2$, which is a contradiction to the assumption that $j \geq 2$. Therefore,

$$\beta_n^{2j}\gamma_n^{(\pm)} > \frac{\alpha^k}{4\sqrt{2}}, \quad \Lambda > 0,$$

$$\left|\alpha^k 2^{-5/2}\beta_n^{-2j}(\gamma_n^{(\pm)})^{-1}-1\right| < \frac{B_n+B_{n+2}}{8(B_{n+1}^2-1)}\cdot\frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}} < \frac{1}{4B_n}\cdot\frac{1}{\beta_n^{2j}\gamma_n^{(-)}} < 6.1\beta_n^{-2j},$$

and since

$$|\Lambda| < 2|e^{\Lambda}-1| \quad \text{whenever} \quad |e^{\Lambda}-1| < \frac{1}{2}, \tag{15}$$

we have $\Lambda < 13\beta_n^{-2j}$. □

Let us now determine a bound for j by proving the following result.

Lemma 8. *If* (5) *has a solution pair* (j,k) *of positive integers with* $j>1$*, then*

$$j < 1.93\cdot 10^{12}(8n+12)\log(78j(n+2)).$$

Proof. To apply Lemma 1, we take

$$N=3,\ D=4,\ b_1=2j,\ b_2=-k,\ b_3=1,\ \alpha_1=\beta_n,\ \alpha_2=\alpha,\ \alpha_3=4\sqrt{2}\cdot\gamma_n^{(\pm)}.$$

Observe that $\alpha_2\in\mathbb{Q}(\sqrt{2})$ and $\alpha_1,\alpha_3^2\in\mathbb{Q}(\sqrt{B_nB_{n+2}})$. But, $\alpha_1,\alpha_2,\alpha_3$ are multiplicatively independent if B_nB_{n+2} is not a square or twice a square. In view of Lemma 5, $B_n=x^2$ and $B_n=2x^2$ has no integer solution x for $n>1$ and if $n=1$, then $B_1B_3=35$ is neither a square nor twice a square. Thus, if $B_nB_{n+2}=du^2$ for an integer u and a square-free integer d, then $d>2$. Moreover, since no integer power of α_2 belongs to $\mathbb{Q}(\sqrt{d})$, α_1 and α_3^2 are multiplicatively dependent if $\alpha_1,\alpha_2,\alpha_3$ are multiplicatively dependent. Further, α_1 is a unit in $\mathbb{Q}(\sqrt{d})$, and hence, $\alpha_3^2=32(\gamma_n^{(\pm)})^2$ is also a unit. But, the norm of $32(\gamma_n^{(\pm)})^2$ is $1024(\gamma_n^{(+)}\gamma_n^{(-)})^2=\frac{C_{n+1}^4}{4B_n^4B_{n+2}^4}$, which is not an integer for any $n\geq 1$, and hence, α_3^2 cannot be a unit.

Next, we determine the heights of α_i. Clearly, we see that

$$h(\alpha_1)=h(\beta_n)=\frac{1}{2}\log\beta_n \quad\text{and}\quad h(\alpha_2)=h(\alpha)=\frac{1}{2}\log\alpha.$$

Since $\gamma_n^{(+)},\gamma_n^{(-)}$ are conjugate to each other in $\mathbb{Q}(\sqrt{d})$ and are roots of the polynomial

$$256B_n^2B_{n+2}^2X^2-32(B_n^2B_{n+2}+B_nB_{n+2}^2)X+(B_{n+2}-B_n)^2,$$

$$|\gamma_n^{(\pm)}|\leq|\gamma_n^{(+)}|=\frac{1}{16}\left(\frac{1}{\sqrt{B_n}}+\frac{1}{\sqrt{B_{n+2}}}\right)^2<1$$

and since $B_l<\frac{\alpha^l}{4\sqrt{2}}$ for every positive integer l, it follows that

$$h(\gamma_n^{(\pm)})\leq\frac{1}{2}\log(256B_n^2B_{n+2}^2)=\log(16B_nB_{n+2})<(2n+2)\log\alpha+\log(1/2).$$

This yields

$$\begin{aligned}h(\alpha_3)=h(4\sqrt{2}\cdot\gamma_n^{(\pm)})&\leq h(4\sqrt{2})+h(\gamma_n^{(\pm)})\\&<\log(4\sqrt{2})+(2n+2)\log\alpha+\log(1/2)\\&=\log(2\sqrt{2})+(2n+2)\log\alpha<(2n+3)\log\alpha.\end{aligned}$$

Let $A_1 = 2\log\beta_n$, $A_2 = 2\log\alpha$, $A_3 = (8n+12)\log\alpha$. Observe that $\alpha^{l-1} < B_l < \alpha^l$ for all $l \geq 1$ and since $\alpha > 4$, we get

$$\beta_n < 2B_{n+1} < \alpha^{n+1.5}.$$

For $n \geq 1$, we have

$$\begin{aligned}\alpha^{k-1} < B_k &\leq \frac{1}{4}\Big[6B_{n+1}{U_j^{(n)}}^2 + 2U_j^{(n)}V_j^{(n)}\Big]\\ &= \frac{1}{4}\Big[(B_n + B_{n+2}){U_j^{(n)}}^2 + 2U_j^{(n)}V_j^{(n)}\Big]\\ &< \Big(V_j^{(n)} + U_j^{(n)}\sqrt{B_nB_{n+2}}\Big)^2 = \Big(B_{n+1} + \sqrt{B_nB_{n+2}}\Big)^{2j}\\ &< (2B_{n+1})^{2j} < (\alpha^{n+1.5})^{2j} < \alpha^{2j(n+2)}.\end{aligned}$$

Let

$$E = \max\left\{\frac{2j\log\beta_n}{\log\alpha}, 4n+6, k\right\} \leq 2j(n+2).$$

Using lemmas 1 and 7, we get

$$\begin{aligned}C(3) &= \frac{8}{2!}(3+2)(6+3)(4^2e)^4 < 6.45\cdot 10^8\\ C_0 &= \log(e^{4.4\cdot 3+7}3^{5.5}4^2\log(4e)) < 30,\\ W_0 &= \log(1.5eE4\log(4e)) < \log(78j(n+2)),\\ \Omega &= (2\log\beta_n)(2\log\alpha)((8n+12)\log\alpha),\end{aligned}$$

and thus,

$$2j\log\beta_n - \log 13 < -\log|\Lambda| < 12384\cdot 10^8\cdot\log(78j(n+2))\log(\beta_n)(\log\alpha)^2(8n+12)$$

yielding

$$j < 1.93\cdot 10^{12}(8n+12)\log(78j(n+2)).$$

□

Next, we define the following linear form in logarithms

$$\Lambda_0 := 2\log\beta_n - (n+1)\log\alpha + \log(4\sqrt{2}\cdot\gamma_n^{(\pm)}). \tag{16}$$

Substituting $(j,k) = (1, n+1)$ in (9), we get

$$\beta_n^2\gamma_n^{(\pm)} - \frac{\alpha^{n+1}}{4\sqrt{2}} = \frac{B_n + B_{n+2}}{8(B_{n+1}^2 - 1)} - \beta_n^{-2}\gamma_n^{(\mp)} - \frac{\alpha^{-(n+1)}}{4\sqrt{2}}, \quad \text{for } n > 1.$$

The case $n = 1$ is well known. If $\beta_n^2\gamma_n^{(\pm)} \leq \alpha^{(n+1)}/(4\sqrt{2})$, then

$$\alpha^{-(n+1)}/(4\sqrt{2}) \leq 1/(32\beta_n^2\gamma_n^{(\pm)})$$

and hence,

$$|\alpha^{(n+1)}2^{-5/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1| < \frac{\beta_n^{-2}\gamma_n^{(\mp)} + \alpha^{-(n+1)}/(4\sqrt{2})}{\beta_n^2\gamma_n^{(\pm)}}$$
$$< \frac{\gamma_n^{(\mp)} + 1/(32\gamma_n^{(\pm)})}{\beta_n^4\gamma_n^{(\pm)}} < \frac{2.42 + 0.55\alpha^2}{\beta_n^4}.$$

Since $\beta_n \geq \alpha^{n+1}$ and $\beta_n \geq 6+\sqrt{35}$, the last inequality implies

$$|\alpha^{(n+1)}2^{-5/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1| < 0.57\beta_n^{-2}.$$

Further, if $\beta_n^2\gamma_n^{(\pm)} > \alpha^{(n+1)}/(4\sqrt{2})$, then

$$|\alpha^{(n+1)}2^{-5/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1| < \frac{1/(8B_n) + 1/(8B_{n+2})}{\beta_n^2\gamma_n^{(\pm)}} < \frac{1}{4B_n\beta_n^2\gamma_n^{(\pm)}} < 6.08\beta_n^{-2}.$$

In both cases, we have

$$|\alpha^{(n+1)}2^{-5/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1| < 6.08\beta_n^{-2}. \tag{17}$$

Since for $n \geq 1$, $\beta_n \geq 6+\sqrt{35}$, we have $6.08\beta_n^{-2} < 1/2$, and hence, inequalities (15) and (17) together imply $|\Lambda_0| < 2\cdot 6.08\beta_n^{-2} < 13\beta_n^{-2}$.

The above discussion proves the following result:

Lemma 9. *It holds,* $|\Lambda_0| < 13\beta_n^{-2}$.

Consider $K := (2j-1)(n+1) - k$ and

$$\Lambda_1 := K\log\alpha - 3(j-1)\log 2. \tag{18}$$

Observe that

$$\beta_n = B_{n+1} + \sqrt{B_{n+1}^2 - 1} = 2B_{n+1} - \frac{1}{B_{n+1} + \sqrt{B_{n+1}^2 - 1}}$$
$$= 2B_{n+1}\left(1 - \frac{1}{2B_{n+1}\big(B_{n+1} + \sqrt{B_{n+1}^2 - 1}\big)}\right)$$

and

$$2B_{n+1} = \frac{\alpha^{n+1} - \overline{\alpha}^{n+1}}{2\sqrt{2}} = \frac{\alpha^{n+1}}{2\sqrt{2}}\left(1 - \frac{1}{\alpha^{2n+2}}\right).$$

Define

$$\delta_n = \left(1 - \frac{1}{2B_{n+1}\big(B_{n+1} + \sqrt{B_{n+1}^2 - 1}\big)}\right)\left(1 - \frac{1}{\alpha^{2n+2}}\right),$$

and hence,

$$\log\beta_n = \log\Big(\frac{1}{2\sqrt{2}}\Big) + (n+1)\log\alpha + \log\delta_n.$$

Therefore,

$$\begin{aligned}\Lambda - \Lambda_0 &= (2j-2)\log\beta_n - (k-(n+1))\log\alpha\\ &= (2j-2)\log\Big(\frac{1}{2\sqrt{2}}\Big) + (2j-2)(n+1)\log\alpha\\ &\qquad + (2j-2)\log\delta_n - (k-(n+1))\log\alpha\\ &= (2j-2)\log\delta_n + K\log\alpha - 3(j-1)\log 2,\end{aligned}$$

which yields

$$\Lambda_1 = \Lambda - \Lambda_0 - (2j-2)\log\delta_n.$$

Using lemmas 7 and 9 and the inequality

$$\begin{aligned}|\log\delta_n| &\le \left|\log\left(1 - \frac{1}{2B_{n+1}\big(B_{n+1}+\sqrt{B_{n+1}^2-1}\big)}\right)\right| + \left|\log\Big(1-\frac{1}{\alpha^{2n+2}}\Big)\right|\\ &< \frac{1}{B_{n+1}\big(B_{n+1}+\sqrt{B_{n+1}^2-1}\big)} + \frac{2}{\alpha^{2n+2}} < \frac{20}{\alpha^{2n+2}} < \frac{6}{10\alpha^{2n}},\end{aligned}$$

we get

$$|\Lambda_1| \le |\Lambda| + |\Lambda_0| + |2j-2|\cdot|\log\delta_n| < \frac{26}{\beta_n^2} + \frac{12(j-1)}{10\alpha^{2n}}. \tag{19}$$

Further, we obtain

$$\beta_n = B_{n+1}\left(1+\sqrt{1-\frac{1}{B_{n+1}^2}}\right) \ge B_{n+1}\Big(1+\frac{\sqrt{35}}{6}\Big) > \frac{\alpha^n}{4\sqrt{2}}\Big(1+\frac{\sqrt{35}}{6}\Big)$$

and thus,

$$\beta_n^2 > \alpha^{2n}\cdot\frac{(1+\sqrt{35}/6)^2}{32} > \frac{\alpha^{2n}}{10}. \tag{20}$$

Inequalities (19) and (20) together imply $|\Lambda_1| < (2j+258)/\alpha^{2n}$, and hence, we have the following result:

Lemma 10. *We have,* $|\Lambda_1| < (2j+258)\alpha^{-2n}$.

For applying Lemma 2 to Λ_1, we require to check that $\Lambda_1 \neq 0$. We assume to the contrary that $\Lambda_1 = 0$. Then, (18) implies $\alpha^K = 2^{3(j-1)} \in \mathbb{Q}$. Conjugating this in $\mathbb{Q}(\sqrt{2})$, we get $\overline{\alpha}^K = 2^{3(j-1)}$, which is a contradiction since $\overline{\alpha}^K < 1$ and $j \ge 2$. So we substitute

$$D = 2,\ \gamma_1 = 2,\ \gamma_2 = \alpha,\ b_1 = 3(j-1),\ b_2 = K$$

and obviously $h_1 = \log 2$ and $h_2 = \log\alpha/2$. In view of Lemma 10, we get

$$K < \frac{3(j-1)\log 2 + (2j+258)\alpha^{-2n}}{\log\alpha} < 1.18(j-1) + 0.04j + 4.31 = 1.22j + 3.13.$$

Thus,

$$b' = 1.74j - 1.62 > \frac{3(j-1)}{2\log\alpha} + \frac{K}{2\log 2} = \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

Lemma 2 implies that

$$\log|\Lambda_1| > -17.9 \cdot 8\log 2(\max\{\log(1.74j - 1.62) + 0.38, 15\})^2, \tag{21}$$

and Lemma 10 gives

$$\log|\Lambda_1| < -2n\log\alpha + \log(2j + 258). \tag{22}$$

Inequalities (21) and (22) together imply

$$n < 28.2 \cdot (\max\{\log(1.74j - 1.62) + 0.38, 15\})^2 + 0.3\log(2j + 258).$$

If

$$\log(1.74j - 1.62) + 0.38 \leq 15,$$

then

$$j < 1284803,$$

and evidently,

$$n < 28.2 \cdot 15^2 + 0.3\log(2 \cdot 1284803 + 258) < 28095.$$

Otherwise,

$$n < 28.2 \cdot (\log(1.74j - 1.62) + 0.38)^2 + 0.3\log(2j + 258). \tag{23}$$

Using (23) in Lemma 8 yields

$$\begin{aligned} j <& 1.93 \cdot 10^{12}(8(28.2 \cdot (\log(1.74j - 1.62) + 0.38)^2 + 0.3\log(2j + 258)) + 12) \\ &\times \log(78j((28.2 \cdot (\log(1.74j - 1.62) + 0.38)^2 + 0.3\log(2j + 258)) + 2)), \end{aligned}$$

and hence, $j < 5.72 \cdot 10^{19}$, and (23) implies that $n < 60798$. So, we have the following result:

Lemma 11. *If* (5) *has a solution pair* (j, k) *of positive integers with* $j > 1$*, then*

$$j < 5.72 \cdot 10^{19} \quad \text{and} \quad n < 60798.$$

Now, we will try to obtain better bounds on j and n. Using inequality (22), we get

$$|K\log\alpha - 3(j-1)\log 2| < (2j + 258)\alpha^{-2n},$$

and hence,

$$\left|\frac{3\log 2}{\log\alpha} - \frac{K}{j-1}\right| < \frac{2j + 258}{(j-1)\alpha^{2n}\log\alpha}. \tag{24}$$

If

$$\frac{2j + 258}{(j-1)\alpha^{2n}\log\alpha} < \frac{1}{2(j-1)^2}, \tag{25}$$

then

$$\left|\frac{3\log 2}{\log\alpha} - \frac{K}{j-1}\right| < \frac{1}{2(j-1)^2}.$$

By using Legendre's criterion, it can be seen that $K/(j-1)$ is a convergent in the simple continued fraction expansion of $3\log 2/\log\alpha$. Using Mathematica, we obtain

$$\begin{aligned}\frac{3\log 2}{\log\alpha} =&[1,5,1,1,3,3,1,1,7,3,1,1,2,12,1,1,4,2,1,11,2,1,1,1,1,\\ &2,17,4,1,66,3,1,2,2,2,1,1,13,6,1,1,15,7,6,2,4,33,29,9,5,\ldots].\end{aligned}$$

The 42^{nd} convergent is

$$\frac{132989060139139716955}{112735119136364899428}$$

and its denominator is greater than the upper bound $5.72\cdot 10^{19}$. The 41^{st} convergent

$$\frac{18825356247280428882}{15958295946307445189}$$

provides the lower bound

$$\left|\frac{3\log 2}{\log\alpha} - \frac{K}{j-1}\right| > 5.5\cdot 10^{-40}. \tag{26}$$

Combining (24) and (26), we get

$$5.5\cdot 10^{-40} < \frac{2j+258}{(j-1)\alpha^{2n}\log\alpha} < 262\alpha^{-2n}(\log\alpha)^{-1},$$

which yields $n < 28$. It is known that if p_r/q_r is the rth convergent of $3\log 2/\log\alpha$, then

$$\left|\frac{3\log 2}{\log\alpha} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1}+2)q_r^2},$$

where a_{r+1} is the $(r+1)$st partial quotient of $3\log 2/\log\alpha$ (see[10]). Thus,

$$\min_{2\le r\le 41}\left[\frac{1}{(a_{r+1}+2)(j-1)^2}\right] < \frac{2j+258}{(j-1)\alpha^{2n}\log\alpha}. \tag{27}$$

Since $\max\limits_{2\le r\le 41} a_{r+1} = 66$, inequality (27) implies that

$$\alpha^{2n} < 68(j-1)(2j+258)(\log\alpha)^{-1},$$

whenever (25) holds. Otherwise,

$$\alpha^{2n} \le 2(j-1)(2j+258)(\log\alpha)^{-1}.$$

The last two inequalities imply

$$\alpha^{2n} < 68(j-1)(2j+258)(\log\alpha)^{-1} < 78j(j+129) < 10140j^2,$$

and hence, $n < 0.57\log j + 2.62$, which is an improved bound.

The above discussion proves the following result:

Lemma 12. *If* (5) *has a solution pair* (j,k) *of positive integers with* $j>1$*, then*

$$n<0.57\log j+2.62.$$

Lemma 8 and 12 together imply

$$j<1.93\cdot 10^{12}(8(0.57\log j+2.62)+12)\times\log(78j(0.57\log j+4.62)),$$

and hence, $j<1.8\cdot 10^{16}$, and consequently $n<24$. Thus, we have the following result:

Lemma 13. *If* (5) *has a solution pair* (j,k) *of positive integers with* $j>1$*, then*

$$j<1.8\cdot 10^{16}\quad\text{and}\quad n<24.$$

To handle the remaining cases for $2\le n\le 23$, we first use the Baker-Davenport reduction method to reduce the bounds of both j and n. Since

$$0<2j\log\beta_n-k\log\alpha+\log(4\sqrt{2}\cdot\gamma_n^{(\pm)})<13\beta_n^{-2j},$$

we use Lemma 3 with

$$\kappa=\frac{2\log\beta_n}{\log\alpha},\quad \mu=\frac{\log(4\sqrt{2}\cdot\gamma_n^{(\pm)})}{\log\alpha},\quad A=\frac{13}{\log\alpha},\quad B=\beta_n^2,\quad M=1.8\cdot 10^{16}.$$

For each n in the interval $[2,23]$, we take $q=q_{47}$ to be the denominator of the 47^{th} convergent to κ. For all $n\in[2,23]$, we have $q>6M$ and $\varepsilon>0$, so we may apply Lemma 3. In all cases, the new bound of j is 8 obtained when $n=2$. For example, if $n=23$ with the sign $+$, then the terms of the continued fraction of κ are

$$[46,29,2,117,5,1,6,1,13,2,19,1,1,1,3,8,4,1,1,3,39,1,25,4,1,6,2,1,\dots].$$

The denominator of its 47^{th} convergent is

$$q_{47}=3313499951634952449281736\,7$$

and the corresponding ε is

$$\varepsilon=.39999999988699520023410803289042287563365630659703325\,29.$$

Therefore, the corresponding bound of j is 1.

From Lemma 12 and as $j\le 8$, we deduce that $n\le 3$. We set $M=8$ to check again for $n=2,3$. The second run of the reduction method yields $j\le 8$ and then $n=2,3$. So we have the following result:

Lemma 14. *If* (5) *has a positive integer solution pair* (j,k) *with* $j>1$*, then*

$$j\le 8\quad\text{and}\quad n\le 3.$$

Now, for $2 \leq j \leq 8$ and $1 \leq n \leq 3$, we need to see whether any of $C_j^{(\pm)}$ is a balancing number. However, direct verification shows no such $C_j^{(\pm)}$ is a balancing number. Therefore, (5) has no solution for $j \geq 2$. Thus, if $\{B_n, B_{n+2}, 4B_k\}$ is a Diophantine triple, then $k = n+1$.

Replacing the left hand side of (5) and (6) by $2B_k$ and $2C_j^{(\pm)}$, respectively, and defining $\gamma_n^{(\pm)}$ as

$$\gamma_n^{(\pm)} := \frac{1}{8}\left[\frac{6B_{n+1}}{4(B_{n+1}^2-1)} \pm \frac{2}{4\sqrt{B_{n+1}^2-1}}\right],$$

the coefficients of $\frac{B_{n+1}}{(B_{n+1}^2-1)}$ in (8) and (9) will be $\frac{6}{16}$. Consequently, the lower and upper bounds for $\gamma_n^{(+)}$ and $\gamma_n^{(-)}$ in Lemma 6 will be modified as

$$0.24\alpha^{-n} < \gamma_n^{(+)} < 0.29\alpha^{-n}, \quad 0.12\alpha^{-n} < \gamma_n^{(-)} < 0.16\alpha^{-n}.$$

The new values of $\gamma_n^{(\pm)}$ and $C_j^{(\pm)}$ result in the same bound for the linear form in logarithms $\Lambda, \Lambda_0, \Lambda_1$ defined on pages 8, 11 and 12. These changes will not affect the bounds for j and n obtained from the Baker-Davenport reduction method, and consequently, $\{B_n, B_{n+2}, 8B_k\}$ is a Diophantine triple only for $k = n+1$. This completes the proof of Theorem 1.

5. Pronic and triangular triples of balancing numbers

We devote this section to exploring pronic and triangular triples of balancing numbers. In particular, given two consecutive balancing numbers, we find some third number X (which is not necessarily a balancing number) to construct a pronic or triangular triple.

The following theorem, the proof of which is similar to that of Theorem 4, helps us to find the third number X.

Theorem 5. *For any fixed positive integer n, if $\{B_n, B_{n+1}, X\}$ is a pronic triple, then the possible values of X may be realized in multiple classes. Two such classes are given by*

$$X = \frac{1}{4}\left[(B_{n+1}+B_n){x_j^{(n)}}^2 + 2x_j^{(n)}y_j^{(n)}\right], X = \frac{1}{4}\left[(B_{n+1}+B_n){x_j^{(n)}}^2 - 2x_j^{(n)}y_j^{(n)}\right],$$

where $x_j^{(n)}$ and $y_j^{(n)}$ are solutions of the Pell's equation $y^2 - B_nB_{n+1}x^2 = 1$, with $j \geq 1$.

But, in view of [9, Theorem 8], the values of X in Theorem 5 partition into just two classes and are precisely those that are mentioned in the statement of Theorem 5. The solutions $x_j^{(n)}$ of Pell's equation $y^2 - B_nB_{n+1}x^2 = 1$ are all even, and hence X is a positive integer. Moreover,

$$\gcd\left(x_j^{(n)}, 4\right) = \begin{cases} 2, & \text{if } j \text{ is odd} \\ 4, & \text{if } j \text{ is even.} \end{cases}$$

Observe that when $j = 1$, Theorem 5 gives two pronic triples $\{B_n, B_{n+1}, 2B_n\}$ and $\{B_n, B_{n+1}, 2B_{n+1}\}$.

Let n be any fixed positive integer such that $\{B_n, B_{n+1}, 2B_k\}$ is a pronic triple for some positive integer k. Since $P_{2n} = 2B_n$, it follows that $\{P_{2n}, P_{2n+2}, 2P_{2k}\}$ is a Diophantine triple. By virtue of [17, Theorem 1.1], $2k \in \{2n, 2n + 2\}$, and hence $k = n$ or $k = n + 1$. This proves Theorem 2.

The following theorem, the proof of which is also similar to that of Theorem 4, helps us to find the third number X such that $\{B_n, B_{n+1}, X\}$ is a triangular triple.

Theorem 6. *For any fixed positive integer n, if $\{B_n, B_{n+1}, X\}$ is a triangular triple, then the possible values of X may be realized in two or more classes. Two such classes are given by*

$$X = \frac{1}{8}\left[(B_{n+1}+B_n){x_j^{(n)}}^2+2x_j^{(n)}y_j^{(n)}\right], X = \frac{1}{8}\left[(B_{n+1}+B_n){x_j^{(n)}}^2-2x_j^{(n)}y_j^{(n)}\right], j \geq 1,$$

where $x_j^{(n)}$ and $y_j^{(n)}$ are solutions of Pell's equation $y^2 - B_nB_{n+1}x^2 = 1$.

Observe that when $j = 1$, Theorem 6 gives $X \in \{B_n, B_{n+1}\}$ and [9, Theorem 8] tells us that the possible values of X in Theorem 6 partition into exactly two classes, and are precisely those that are mentioned in the statement of Theorem 6.

Theorem 3 can be proved by using arguments similar to those used to prove Theorem 1. So, we prefer to omit the details of the proof. However, below we give some crucial steps required for the proof.

In view of Theorem 6, if $\{B_n, B_{n+1}, B_k\}$ is a triangular triple, then

$$B_k = \frac{1}{8}\left[(B_{n+1} + B_n){x_j^{(n)}}^2 \pm 2x_j^{(n)}x_j^{(n)}\right].$$

Consider $C_j^{(\pm)} = B_k$. For $j = 1$, one can obtain $C_1^{(-)} = B_n, C_1^{(+)} = B_{n+1}$ as the only solution. Let $\beta_n := (B_{n+1} - B_n) + 2\sqrt{B_nB_{n+1}}$, and thus,

$$y_j^{(n)} = \frac{\beta_n^j + \beta_n^{-j}}{2}, \quad x_j^{(n)} = \frac{\beta_n^j - \beta_n^{-j}}{2\sqrt{B_nB_{n+1}}}.$$

Defining $\gamma_n^{(\pm)}$ as

$$\gamma_n^{(\pm)} := \frac{1}{32}\left[\frac{B_{n+1} + B_n}{B_nB_{n+1}} \pm \frac{2}{\sqrt{B_nB_{n+1}}}\right],$$

we get

$$\beta_n^{2j}\gamma_n^{(\pm)} - \frac{B_{n+1} + B_n}{16B_nB_{n+1}} + \beta_n^{-2j}\gamma_n^{(\mp)} = \frac{\alpha^k - \overline{\alpha}^k}{4\sqrt{2}}.$$

The lower and upper bounds for $\gamma_n^{(+)}$ and $\gamma_n^{(-)}$ are

$$0.35\alpha^{-n} < \gamma_n^{(+)} < 0.41\alpha^{-n} \quad \text{and} \quad 0.06\alpha^{-n} < \gamma_n^{(-)} < 0.09\alpha^{-n}.$$

Using the same Λ as in the proof of Theorem 1 and defining

$$\Lambda_0 := 2\log\beta_n - \frac{1}{2}((2n + 1) \pm 1)\log\alpha + \log(4\sqrt{2} \cdot \gamma_n^{(\pm)}),$$

$$\Lambda_1 := K\log\alpha - (j-1)\log 2 \quad \text{with} \quad K := [(2j-1)(2n+1) - 2k \pm 1]/2,$$

we obtain the bounds

$$0 < \Lambda < 25\beta_n^{-2j}, \quad |\Lambda_0| < 25\beta_n^{-2}, \quad |\Lambda_1| < (12j+113)\alpha^{-(2n+1)}.$$

Correspondingly, the bounds for j and n are $j < 1.7 \cdot 10^{16}$ and $n < 24$. Further, the Baker-Davenport reduction method can be applied to reduce the bounds of j and n and the remaining cases can be verified by direct computation.

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