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New applications of concave operators to existence and uniqueness of solutions for fractional differential equations

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Abstract. Recently, Feng and Zhai have studied some results of positive solutions to fractional differential equations. By using mixed monotone operators on cones and the concept of γ-concavity, we study an application for fractional differential equations. An example is also provided illustrating the obtained results.
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1. Introduction

In 2017, Feng and Zhai investigated the following problem:

$$\begin{aligned} &D_t^{\kappa}u(t)+f(t,u(t))+g(t,u(t))=0, \quad 0<t<1, \qquad (1)\\ &u(0)=u'(0)=0, \quad u(1)=\int_0^1 \theta(\xi)u(\xi)d\xi, \end{aligned}$$

where $2<\kappa\leq 3$, D_t^{κ} is the standard Riemann-Liouville fractional derivative of order κ. The authors obtained one positive solution to this problem (see [4, 14]).

The function θ satisfies the following conditions:

$$\begin{aligned} &\theta:[0,1]\to[0,\infty) \quad \text{with} \quad \theta\in L^1[0,1] \quad \text{and}\\ &\sigma_1=\int_0^1 \xi^{\kappa-1}(1-\xi)\theta(\xi)d\xi>0, \quad \sigma_2=\int_0^1 \xi^{\kappa-1}\theta(\xi)d\xi<1. \end{aligned}$$

Motivated by [4], in this paper we establish the existence of a positive solution to the following problem:

$$\begin{aligned} &D_t^{\kappa}u(s,t)+f(t,u(s,t),\frac{\partial}{\partial s}u(s,t))+g(t,u(s,t),\frac{\partial}{\partial s}u(s,t))=0, \qquad (2)\\ &0<s,t<1, \quad u(s,0)=\frac{\partial}{\partial t}u(s,0)=0, \quad u(s,1)=\int_0^1 \varphi(s,\xi)u(s,\xi)d\xi, \end{aligned}$$

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where $2 < \kappa \leq 3$, f, g are continuous and increasing with respect to the second argument and decreasing with respect to the third argument. D_t^κ is the standard Riemann-Liouville fractional derivative of order κ. The function $\varphi(t)$ satisfies the following conditions:

$$(\Phi) \quad \varphi : [0,1] \times [0,1] \to [0,\infty) \quad \text{with} \quad \varphi \in L^1([0,1] \times [0,1]) \quad \text{and}$$
$$\zeta_1 = \int_0^1 \xi^{\kappa-1}(1-\xi)\varphi(s,\xi)d\xi > 0, \quad \zeta_2 = \int_0^1 \xi^{\kappa-1}\varphi(s,\xi)d\xi < 1.$$

Definition 1 (see [7, 8])**.** *The Riemann-Liouville fractional derivative of order κ for a continuous function f is defined by:*

$$D_t^\kappa f(t) = \frac{1}{\Gamma(n-\kappa)}(\frac{d}{dt})^n \int_0^t \frac{f(\xi)}{(t-\xi)^{\kappa-n+1}}d\xi, \qquad (n = [\kappa]+1)$$

where the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2 (see [7, 8])**.** *Let $[a,b]$ be an interval in $\mathbb{R}$ and $\kappa > 0$. The Riemann-Liouville fractional order integral of a function $f \in L^1([a,b],\mathbb{R})$ is defined by:*

$$I_t^\kappa f(t) = \frac{1}{\Gamma(\kappa)} \int_a^t \frac{f(\xi)}{(t-\xi)^{1-\kappa}}d\xi,$$

whenever the integral exists.

It exists extensively in the research of nonlinear fractional differential and integral equations (see [1, 2, 3, 6, 13, 12]).

In this paper, we present some basic concepts in ordered Banach spaces and a fixed-point theorem which will be used later. For the convenience of readers, we suggest that one refers to [5] for details. Suppose that $(E, \| \,.\, \|)$ is a Banach space, which is partially ordered by a cone $P \subseteq E$, that is, $z \leq w$ if and only if $w - z \in P$. If $z \neq w$, then we denote $z < w$ or $z > w$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $z \in P,\ \lambda \geq 0 \Longrightarrow \lambda z \in P$, and (ii) $z \in P,\ -z \in P \Longrightarrow z = \theta$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq z \leq w$ implies $\| z \| \leq N \| w \|$. We also define the ordered interval $[z_1, z_2] = \{z \in E | z_1 \leq z \leq z_2\}$ for all $z_1, z_2 \in E$.

Definition 3 (see [5])**.** *$A : P \times P \to P$ is said to be a mixed monotone operator if $A(z,w)$ is increasing in z and decreasing in w, i.e., $u_i, v_i (i = 1,2) \in P, u_1 \leq u_2, v_1 \geq v_2$ imply $A(u_1,v_1) \leq A(u_2,v_2)$, $z \in P$ is called a fixed point of A if $A(z,z) = z$ and for $h > \theta$, $P_h = \{z \in P | \exists \lambda, \mu > 0$ such that $\lambda h \leq z \leq \mu h\}$.*

Definition 4. *Let γ be a real number with $0 < \gamma < 1$. An operator $A : P \to P$ is said to be γ-concave if it satisfies $A(tz) \geq t^\gamma A(z)$ for all $t > 0, z \in P$. An operator $A : P \to P$ is said to be homogeneous if it satisfies $A(tz) = tA(z)$ for all $t > 0, z \in P$. An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies $A(tz) \geq tA(z)$ for all $t > 0, z \in P$.*

We point out that $C[0,1] = \{z : [0,1] \to \mathbb{R} \text{ is continuous}\}$, $\| z \| = \sup\{|z(t)| : t \in [0,1]\}$ is a Banach space. Let $P = \{z \in C[0,1] : z(t) \geq 0, t \in [0,1]\}$, then it is a normal cone in $C[0,1]$ and the normality constant is 1. We know that this space can be equipped with a partial order given by:

$$z \leq w, \quad z, w \in C[0,1] \Leftrightarrow z(t) \leq w(t),\ t \in [0,1].$$

Theorem 1 (see [10]). *Let P be a normal cone in a real Banach space E, $\gamma \in (0,1)$ $A : P \to P$ an increasing sub-homogeneous operator, $B : P \to P$ a decreasing operator, $C : P \times P \to P$ a mixed monotone operator and let the following conditions:*

$$B(\frac{1}{t}z) \geq tBw, \quad C(tz, \frac{1}{t}w) \geq t^{\gamma} C(z,w), \quad t \in (0,1), z, w \in P,$$

be satisfied. Assume that

(*i*) *there is $h_0 \in P_h$ such that $Ah_0 \in P_h, Bh_0 \in P_h, C(h_0, h_0) \in P_h$;*

(*ii*) *there exists a constant $\delta_0 > 0$ such that $C(z,w) \geq \delta_0(Az+Bz)$ for all $z, w \in P$.*

Then

(*1*) *$A : P_h \to P_h$, $B : P_h \to P_h$ and $C : P_h \times P_h \to P_h$;*

(*2*) *there are $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$ru_0 \leq u_0 < v_0, u_0 \leq Au_0 + Bv_0 + C(u_0, v_0) \leq Av_0 + Bu_0 + C(v_0, u_0) \leq v_0;$$

(*3*) *the operator equation $Az + Bz + C(z,z) = z$ has a unique solution z^* in P_h;*

(*4*) *for $z_0, w_0 \in P_h$, construct*

$$z_n = Az_{n-1} + Bw_{n-1} + C(z_{n-1}, w_{n-1}), n = 1, 2, \dots,$$
$$w_n = Aw_{n-1} + Bz_{n-1} + C(w_{n-1}, z_{n-1}), n = 1, 2, \dots.$$

We have $z_n \to z^*$ and $w_n \to z^*$ as $n \to \infty$.

Lemma 1 (see [11]). *If*

$$G_1(t,\xi) = \frac{1}{\Gamma(\kappa)} \begin{cases} t^{\kappa-1}(1-\xi)^{\kappa-1} - (t-\xi)^{\kappa-1}, & 0 \leq \xi \leq t \leq 1, \\ t^{\kappa-1}(1-\xi)^{\kappa-1}, & 0 \leq t \leq \xi \leq 1. \end{cases} \tag{3}$$

Then for $G_1(t,\xi)$ the following property holds:

$$\frac{t^{\kappa-1}(1-t)\xi(1-\xi)^{\kappa-1}}{\Gamma(\kappa)} \leq G_1(t,\xi) \leq \frac{\xi(1-\xi)^{\kappa-1}}{\Gamma(\kappa-1)}, \quad t, \xi \in [0,1].$$

From [9] and Lemma 1, we have

$$\frac{\zeta_1 \xi(1-\xi)^{\kappa-1} t^{\kappa-1}}{(1-\zeta_2)\Gamma(\kappa)} \leq G(t,\xi) \leq \frac{t^{\kappa-1}(1-\xi)^{\kappa-1}}{(1-\zeta_2)\Gamma(\kappa)}, \quad t, \xi \in [0,1], \tag{4}$$

where $G(t,\xi)$ is given as follow:

$$G(t,\xi) = G_1(t,\xi) + G_2(t,\xi), \quad (t,\xi) \in [0,1] \times [0,1], \tag{5}$$

where

$$G_2(t,\xi) = \frac{t^{\kappa-1}}{1-\zeta_2} \int_0^1 G_1(\tau,\xi)\varphi(\xi,\tau)d\tau. \tag{6}$$

In 2017, Feng and Zhai established the following theorem.

Theorem 2 (see [4]). *Assume* (Φ) *and*

(H_1) $f, g : [0,1] \times [0,\infty) \to [0,\infty)$ *are continuous and increasing with respect to the second argument,* $g(t,0) \not\equiv 0$*;*

(H_2) $g(t,\lambda z) \geq \lambda g(t,z)$ *for* $\lambda \in (0,1), t \in [0,1], z \in [0,\infty)$*, and there exists a constant* $\gamma \in (0,1)$ *such that* $f(t,\lambda z) \geq \lambda^\gamma f(t,z)$ *for all* $t \in [0,1], \lambda \in (0,1), z \in [0,\infty)$*;*

(H_3) $\exists\ \delta_0 > 0$ *such that* $f(t,z) \geq \delta_0 g(t,z), t \in [0,1], z \geq 0$.

Then problem (1) has a unique positive solution u^* *in* P_h*, where* $h(t) = t^{\kappa-1}, t \in [0,1]$ *and for* $u_0 \in P_h$ *construct*

$$u_{n+1}(t) = \int_0^1 G(t,\xi)[f(\xi,u_n(\xi)) + g(\xi,u_n(\xi))]d\xi, \quad n = 0,1,2,\dots.$$

We have $u_n(t) \to u^*(t)$ *as* $n \to \infty$*, where* $G(t,\xi)$ *is given as (5).*

2. Main result

As a prompt consequence of Theorem 1 we have the following result.

Proposition 1. *Let* P *be a normal cone in a real Banach space* E*,* $\gamma \in (0,1)$*,* $T, C : P \times P \to P$ *mixed monotone operators and let the following conditions*

$$T(tz, \frac{1}{t}w) \geq tT(z,w), \quad t \in (0,1), z, w \in P,$$

$$C(tz, \frac{1}{t}w) \geq t^\gamma C(z,w), \quad t \in (0,1), z, w \in P,$$

be satisfied. Assume that

(*i*) *there is* $h_0 \in P_h$ *such that* $T(h_0,h_0) \in P_h$*,* $C(h_0,h_0) \in P_h$*;*

(*ii*) *there exists a constant* $\delta_0 > 0$ *such that* $C(z,w) \geq \delta_0 T(z,w)$ *for all* $z, w \in P$.

Then

(*1*) $T : P_h \times P_h \to P_h$ *and* $C : P_h \times P_h \to P_h$*;*

(2) there are $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$ru_0 \leq u_0 < v_0, u_0 \leq T(u_0, v_0) + C(u_0, v_0) \leq T(v_0, u_0) + C(v_0, u_0) \leq v_0;$$

(3) the operator equation $T(z,z) + C(z,z) = z$ *has a unique solution* z^* *in* P_h*;*

(4) for $z_0, w_0 \in P_h$*, construct*

$$\begin{aligned} z_n &= T(z_{n-1}, w_{n-1}) + C(z_{n-1}, w_{n-1}) \\ w_n &= T(w_{n-1}, z_{n-1}) + C(w_{n-1}, z_{n-1}), n = 1, 2, \ldots. \end{aligned}$$

We have $z_n \to z^*$ *and* $w_n \to z^*$ *as* $n \to \infty$*.*

Definition 5. *An operator* $A : P \times P \to P$ *is said to be* γ*-concave if*

$$A(tz, \frac{1}{t}w) \geq t^\gamma A(z,w), \quad t \in (0,1), (z,w) \in P \times P, \quad 0 \leq \gamma < 1.$$

Definition 6. *An operator* $B : P \times P \to P$ *is said to be sub-homogeneous if it satisfies the following:*

$$B(tz, \frac{1}{t}w) \geq tB(z,w), \quad t \in (0,1), \quad z, w \in P.$$

Definition 7. *Let* γ *be a real number with* $0 < \gamma < 1$*. An operator* $A : P \times P \to P$ *is said to be* γ*-concave if it satisfies* $A(tz, \frac{1}{t}w) \geq t^\gamma A(z,w)$ *for all* $t > 0, z, w \in P$*. An operator* $B : P \times P \to P$ *is said to be sub-homogeneous if* $B(tz, \frac{1}{t}w) \geq tB(z,w)$ *for all* $t > 0, z, w \in P$*.*

Lemma 2. *Assume* (Φ) *holds and* $y : [0,1] \times [0,1] \to \mathbb{R}$ *is continuous. Then the problem*

$$\begin{aligned} &D_t^\kappa u(s,t) + y(s,t) = 0, \quad 2 < \kappa \leq 3, \\ &s, t \in [0,1], \ u(s,0) = \frac{\partial}{\partial t} u(s,0) = 0, \quad u(s,1) = \int_0^1 \varphi(s,\xi) u(s,\xi) d\xi, \end{aligned} \tag{7}$$

has the solution

$$u(s,t) = \int_0^1 G(t,\xi) y(s,\xi) d\xi,$$

where $G(t,\xi)$ *is given as (5).*

Proof. By (7), the following inequality holds:

$$u(s,t) = -I_t^\kappa y(s,t) + c_1 t^{\kappa-1} + c_2 t^{\kappa-2} + c_3 t^{\kappa-3} \quad c_1, c_2, c_3 \in \mathbb{R}.$$

Hence

$$u(s,t) = -\int_0^t \frac{(t-\xi)^{\kappa-1}}{\Gamma(\kappa)} y(t,\xi) d\xi + c_1 t^{\kappa-1} + c_2 t^{\kappa-2} + c_3 t^{\kappa-3}.$$

From $u(s,0) = \frac{\partial}{\partial t} u(s,0) = 0$ and $u(s,1) = \int_0^1 \varphi(s,\xi) u(s,\xi) d\xi$, we obtain

$$c_1 = \int_0^1 \frac{(1-\xi)^{\kappa-1}}{\Gamma(\kappa)} y(s,\xi) d\xi + \int_0^1 \varphi(s,\xi) u(s,\xi) d\xi, \quad c_2 = c_3 = 0.$$

Therefore

$$\begin{aligned}u(s,t)=&-\int_0^t \frac{(t-\xi)^{\kappa-1}}{\Gamma(\kappa)}y(s,\xi)d\xi+\frac{t^{\kappa-1}}{\Gamma(\kappa)}\int_0^1(1-\xi)^{\kappa-1}y(s,\xi)d\xi\\&+t^{\kappa-1}\int_0^1\varphi(s,\xi)u(s,\xi)d\xi\\=&\int_0^1 G_1(t,\xi)y(s,\xi)d\xi+t^{\kappa-1}\int_0^1\varphi(s,\xi)u(s,\xi)d\xi.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_0^1\varphi(s,t)u(s,t)dt=&\int_0^1\varphi(s,t)(\int_0^1 G_1(t,\xi)y(s,\xi)d\xi)dt\\&+\int_0^1(\varphi(s,t)t^{\kappa-1}\int_0^1\varphi(s,\xi)u(s,\xi)d\xi)dt\\=&\int_0^1(\int_0^1\varphi(s,t)G_1(t,\xi)dt)y(s,\xi)d\xi\\&+(\int_0^1 t^{\kappa-1}\varphi(s,t)dt)(\int_0^1\varphi(s,\xi)u(s,\xi)d\xi),\\\int_0^1\varphi(s,\xi)u(s,\xi)d\xi=&\frac{1}{1-\zeta_2}\int_0^1(\int_0^1 G_1(t,\xi)\varphi(s,t)dt)y(s,\xi)d\xi\\=&\frac{1}{1-\zeta_2}\int_0^1(\int_0^1 G_1(\tau,\xi)\varphi(s,\tau)d\tau)y(s,\xi)d\xi.\end{aligned}$$

Clearly we get

$$\begin{aligned}u(s,t)=&\int_0^1 G_1(t,\xi)y(s,\xi)d\xi+\frac{t^{\kappa-1}}{1-\zeta_2}\int_0^1(\int_0^1 G_1(\tau,\xi)\varphi(s,\tau)d\tau)y(s,\xi)d\xi\\=&\int_0^1 G_1(t,\xi)y(s,\xi)d\xi+\int_0^1 G_2(t,\xi)y(s,\xi)d\xi\\=&\int_0^1 G(t,\xi)y(s,\xi)d\xi.\end{aligned}$$

□

Now we consider the new Banach space E_1 as follows:

$$E_1=\{u(s,t)\in C([0,1]\times[0,1])|\frac{\partial}{\partial s}u(s,t)\in C([0,1]\times[0,1])\}.$$

E_1 is a Banach space with the norm

$$\|u\|=\max\{\max_{s,t\in[0,1]}|u(s,t)|,\max_{s,t\in[0,1]}|\frac{\partial}{\partial s}u(s,t)|\}.$$

E_1 is endowed with an order relation

$$u(s,t)\preceq v(s,t)\text{ if and only if }u(s,t)\le v(s,t),\frac{\partial}{\partial s}u(s,t)\le\frac{\partial}{\partial s}v(s,t),$$

for all $u(s,t), v(s,t) \in E_1$.

Moreover, let $P_1 \subseteq E_1$ be defined by:

$$P_1 = \{u \in E_1 : u(s,t) \geq 0, \frac{\partial}{\partial s}u(s,t) \geq 0,\ s,t \in [0,1]\}.$$

We point out P_1 is a normal cone. Indeed, for $u(s,t), v(s,t) \in P_1$, with $u(s,t) \preceq v(s,t)$ we have

$$u(s,t) \leq v(s,t) \text{ and } \frac{\partial}{\partial s}u(s,t) \leq \frac{\partial}{\partial s}v(s,t).$$

Then obviously for $M = 1$ the following conditions hold:

$$|u(s,t)| \leq M|v(s,t)| \text{ and } |\frac{\partial}{\partial s}u(s,t)| \leq M|\frac{\partial}{\partial s}v(s,t)|.$$

So we have four items below:

(*i*) $\| u(s,t) \| = \max|u(s,t)|$, $\| v(s,t) \| = \max|v(s,t)|$ and $M = 1$, then we have

$$\max|u(s,t)| \leq M \max|v(s,t)|,$$

therefore

$$\| u(s,t) \| \leq \| v(s,t) \|,$$

(*ii*) $\|u(s,t)\| = \max|\frac{\partial}{\partial s}u(s,t)|$ and $\| v(s,t) \| = \max|\frac{\partial}{\partial s}v(s,t)|$, then we have

$$\| u(s,t) \| = \max|\frac{\partial}{\partial s}u(s,t)| \leq \max|\frac{\partial}{\partial s}v(s,t)| = \|v(s,t)\|,$$

(*iii*) $\|u(s,t)\| = \max|\frac{\partial}{\partial s}u(s,t)|$ and $\| v(s,t) \| = \max|v(s,t)|$, then we have

$$\| u(s,t) \| = \max|\frac{\partial}{\partial s}u(s,t)| \leq \max|\frac{\partial}{\partial s}v(s,t)| \leq \max|v(s,t)| = \|v(s,t)\|,$$

(*iv*) $\|u(s,t)\| = \max|u(s,t)|$ and $\| v(s,t) \| = \max|\frac{\partial}{\partial s}v(s,t)|$, then we have

$$\| u(s,t) \| = \max|u(s,t)| \leq \max|v(s,t)| \leq \max|\frac{\partial}{\partial s}v(s,t)| = \|v(s,t)\|,$$

therefore P_1 is a normal cone.

Now here, continuing the work of Feng and Zhai, we establish the existence and uniqueness of solution to fractional differential equation (2).

Theorem 3. *Assume* (Φ) *and*

(H_1) $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *are continuous and increasing with respect to the second argument, but also decreasing with respect to third argument.* $g(t,0,1) \not\equiv 0$*;*

(H_2) $g(t, \lambda z, \frac{1}{\lambda}w) \geq \lambda g(t,z,w)$ *for* $\lambda \in (0,1), t \in [0,1], z,w \in [0,\infty)$*, and there exists a constant* $\gamma \in (0,1)$ *such that* $f(t, \lambda z, \frac{1}{\lambda}w) \geq \lambda^{\gamma} f(t,z,w)$ *for all* $t \in [0,1], \lambda \in (0,1), z,w \in [0,\infty)$*;*

(H_3) *there exists a constant* $\delta_0 > 0$ *such that* $f(t,z,w) \geq \delta_0 g(t,z,w)$, $t \in [0,1]$ *and* $z, w \geq 0$.

(H_4) $y(s,t) \leq y'(s,t)$ *implies that* $\frac{\partial}{\partial s} y(s,t) \leq \frac{\partial}{\partial s} y'(s,t)$.

Then problem (2) has a unique positive solution u^* *in* P_{1_h}, *where* $h(t) = t^{\kappa-1}, t \in [0,1]$ *and for* $u_0 \in P_{1_h}$, *construct*

$$\begin{aligned} u_{n+1}(s,t) = \int_0^1 & G(t,\xi)[f(\xi, u_n(s,\xi), \frac{\partial}{\partial s} u_n(s,\xi)) \\ & + g(\xi, u_n(s,\xi), \frac{\partial}{\partial s} u_n(s,\xi))] d\xi, \quad n = 0,1,2,\ldots. \end{aligned}$$

We have $u_n(s,t) \to u^*(s,t)$ *as* $n \to \infty$, *where* $G(t,\xi)$ *is given as (5).*

Proof. From Lemma 2, problem (2) has an integral formulation given by

$$u(s,t) = \int_0^1 G(t,\xi)[f(\xi, u(s,\xi), \frac{\partial}{\partial s} u(s,\xi)) + g(\xi, u(s,\xi), \frac{\partial}{\partial s} u(s,\xi))] d\xi.$$

Define $A : P_1 \times P_1 \to P_1$ and $B : P_1 \times P_1 \to P_1$ by:

$$\begin{aligned} A(u(s,t), v(s,t)) &= \int_0^1 G(t,\xi) f(\xi, u(s,\xi), \frac{\partial}{\partial s} v(s,\xi)) d\xi, \\ B(u(s,t), v(s,t)) &= \int_0^1 G(t,\xi) g(\xi, u(s,\xi), \frac{\partial}{\partial s} v(s,\xi)) d\xi. \end{aligned}$$

Then u is the solution to problem (2) if and only if

$$u = A(u,u) + B(u,u).$$

Firstly, we show that A, B are two increasing operators with respect to the second argument, but also decreasing with respect to third argument. For $(u,v), (u',v') \in P_1 \times P_1$ with $u \succeq u'$ and $v \preceq v'$, we have

$$\begin{aligned} A(u(s,t), v(s,t)) &= \int_0^1 G(t,\xi) f(\xi, u(s,\xi), \frac{\partial}{\partial s} v(s,\xi)) d\xi \\ &\geq \int_0^1 G(t,\xi) f(\xi, u'(s,\xi), \frac{\partial}{\partial s} v'(s,\xi)) d\xi \\ &= A(u'(s,t), v'(s,t)). \end{aligned}$$

By (H_4), it is easy to see that

$$\frac{\partial}{\partial s} A(u(s,t), v(s,t)) \geq \frac{\partial}{\partial s} A(u'(s,t), v'(s,t)).$$

So

$$A(u(s,t), v(s,t)) \succeq A(u'(s,t), v'(s,t)).$$

Similarly, $B(u,v) \succeq B(u',v')$. Secondly, we prove that A is a γ-concave operator and B is a sub-homogeneous operator. For any $\lambda \in (0,1)$ with $(u,v) \in P_1 \times P_1$, from (H_2) we obtain:

$$\begin{aligned} A(\lambda u(s,t), \frac{1}{\lambda} v(s,t)) &= \int_0^1 G(t,\xi) f(\xi, \lambda u(s,\xi), \frac{1}{\lambda}\frac{\partial}{\partial s} v(s,\xi)) d\xi \\ &\geq \lambda^\gamma \int_0^1 G(t,\xi) f(\xi, u(s,\xi), \frac{\partial}{\partial s} v(s,\xi)) d\xi \\ &= \lambda^\gamma A(u(s,t), v(s,t)). \end{aligned}$$

By (H_4), we have $\frac{\partial}{\partial s} A(\lambda u(s,t), \frac{1}{\lambda} v(s,t)) \geq \lambda^\gamma \frac{\partial}{\partial s} A(u(s,t), v(s,t))$, therefore

$$\begin{aligned} A(\lambda u(s,t), \frac{1}{\lambda} v(s,t)) &\succeq \lambda^\gamma \ A(u(s,t), v(s,t)) \\ B(\lambda u(s,t), \frac{1}{\lambda} v(s,t)) &= \int_0^1 G(t,\xi) g(\xi, \lambda u(s,\xi), \frac{1}{\lambda}\frac{\partial}{\partial s} v(s,\xi)) d\xi \\ &\geq \lambda \int_0^1 G(t,\xi) g(\xi, u(s,\xi), \frac{\partial}{\partial s} v(s,\xi)) d\xi \\ &= \lambda B(u(s,t), v(s,t)), \end{aligned}$$

and also

$$\frac{\partial}{\partial s} B(\lambda u(s,t), \frac{1}{\lambda} v(s,t)) \geq \lambda \frac{\partial}{\partial s} B(u(s,t), v(s,t)),$$

hence

$$B(\lambda u(s,t), \frac{1}{\lambda} v(s,t)) \succeq \lambda B(u(s,t), v(s,t)).$$

So A is γ-concave and B is sub-homogeneous.

Next, we prove that $A(h,h) \in P_{1_h}$ and $B(h,h) \in P_{1_h}$. From (H_1), (3), (6) and (4), we have

$$\begin{aligned} A(h(t), h(t)) &= \int_0^1 G(t,\xi) f(\xi, \xi^{\kappa-1}, 0) d\xi \\ &\leq \frac{t^{\kappa-1}}{(1-\zeta_2)\Gamma(\kappa)} \int_0^1 (1-\xi)^{\kappa-1} f(\xi, 1, 0) d\xi, \\ A(h(t), h(t)) &= \int_0^1 G(t,\xi) f(\xi, \xi^{\kappa-1}, 0) d\xi \\ &\geq \frac{\zeta_1 t^{\kappa-1}}{(1-\zeta_2)\Gamma(\kappa)} \int_0^1 \xi(1-\xi)^{\kappa-1} f(\xi, 0, 1) d\xi. \end{aligned}$$

From (H_3) and (H_1) we have

$$f(\xi, 1, 0) \geq f(\xi, 0, 1) \geq \delta_0 g(\xi, 0, 1) > 0.$$

Because $\kappa-1>0$ and $g(\xi,0,1)\not\equiv 0$, we can get

$$\int_0^1(1-\xi)^{\kappa-1}f(\xi,1,0)d\xi \geq \int_0^1 \xi(1-\xi)^{\kappa-1}f(\xi,0,1)d\xi$$
$$\geq \delta_0\int_0^1 \xi(1-\xi)^{\kappa-1}g(\xi,0,1)d\xi>0.$$

Let

$$l_1 := \frac{\zeta_1}{(1-\zeta_2)\Gamma(\kappa)}\int_0^1 \xi(1-\xi)^{\kappa-1}f(\xi,0,1)d\xi.$$
$$l_2 := \frac{1}{(1-\zeta_2)\Gamma(\kappa)}\int_0^1 (1-\xi)^{\kappa-1}f(\xi,1,0)d\xi.$$

Then $l_2\geq l_1>0$ and thus $l_1h(t)\leq A(h(t),h(t))\leq l_2h(t), t\in[0,1]$; similarly,

$$l_1\frac{\partial}{\partial s}h(t)\leq \frac{\partial}{\partial s}A(h(t),h(t))\leq l_2\frac{\partial}{\partial s}h(t),$$

hence

$$l_1h(t)\preceq A(h(t),h(t))\preceq l_2h(t),$$

Thus $A(h,h)\in P_{1_h}$.

Also

$$B(h(t),h(t))=\int_0^1 G(t,\xi)g(\xi,\xi^{\kappa-1},0)d\xi$$
$$\leq \frac{t^{\kappa-1}}{(1-\zeta_2)\Gamma(\kappa)}\int_0^1(1-\xi)^{\kappa-1}g(\xi,1,0)d\xi,$$

$$B(h(t),h(t))=\int_0^1 G(t,\xi)g(\xi,\xi^{\kappa-1},0)d\xi$$
$$\geq \frac{\zeta_1t^{\kappa-1}}{(1-\zeta_2)\Gamma(\kappa)}\int_0^1\xi(1-\xi)^{\kappa-1}g(\xi,0,1)d\xi.$$

We can easily get $B(h,h)\in P_{1_h}$, from $g(t,0,1)\not\equiv 0$ and similarly to operator A. That is, condition (i) of Theorem 1 holds.

Further, we prove that condition (ii) of Theorem 1 is also satisfied.

For $(u,u)\in P_1\times P_1$, by (H_3),

$$A(u(t),u(t))=\int_0^1 G(t,\xi)f(\xi,u(s,\xi),\frac{\partial}{\partial s}v(s,\xi))d\xi$$
$$\geq \delta_0\int_0^1 G(t,\xi)g(\xi,u(s,\xi),\frac{\partial}{\partial s}v(s,\xi))d\xi$$
$$=\delta_0B(u(t),u(t))$$

and

$$\frac{\partial}{\partial s}A(u(t),u(t)) \geq \delta_0 \frac{\partial}{\partial s}B(u(t),u(t)).$$

Hence we get $A(u,u) \succeq \delta_0 B(u,u)$.

Finally, from Theorem 1 we know that $A(u,u)+B(u,u)=u$ has a unique solution $u^* \in P_1$; for $u_0 \in P_{1_h}$, construct $u_n = A(u_{n-1},u_{n-1}) + B(u_{n-1},u_{n-1}),\ n=1,2,\ldots$. We have $u_n \to u^*$. That is, problem (2) has a unique positive solution $u^* \in P_{1_h}$ for the sequence

$$\begin{aligned} u_{n+1}(s,t) = \int_0^1 & G(t,\xi)[f(\xi,u_n(s,\xi),\frac{\partial}{\partial s}u_n(s,\xi)) \\ & + g(\xi,u_n(s,\xi),\frac{\partial}{\partial s}u_n(s,\xi))]d\xi, \quad n=0,1,2,\ldots. \end{aligned}$$

We have $u_n(s,t) \to u^*(s,t)$. □

Corollary 1. *Assume* (Φ) *and*

(H_1) *Let* $f:[0,1]\times[0,\infty)\times[0,\infty)\to[0,\infty)$ *be a continuous and increasing with respect to the second argument, but also decreasing with respect to the third argument.* $f(t,0,1)\not\equiv 0$*;*

(H_2) *there exists a constant* $\gamma\in(0,1)$ *such that* $f(t,\lambda z,\frac{1}{\lambda}w)\geq\lambda^\gamma f(t,z,w)$ *for all* $t\in[0,1],\lambda\in(0,1),z,w\in[0,\infty)$*;*

(H_3) $y(s,t)\leq y'(s,t)$ *implies that* $\frac{\partial}{\partial s}y(s,t)\leq\frac{\partial}{\partial s}y'(s,t)$*.*

Then

$$\begin{aligned} & D_t^\kappa u(s,t) + f(t,u(s,t),\frac{\partial}{\partial s}u(s,t)) = 0, \quad 2<\kappa\leq 3, \\ & 0<s,t<1, \quad u(s,0)=\frac{\partial}{\partial t}u(s,0)=0, \quad u(s,1)=\int_0^1\varphi(s,\xi)u(s,\xi)d\xi, \end{aligned}$$

has a unique positive solution u^* *in* P_{1_h}*, where* $h(t)=t^{\kappa-1},t\in[0,1]$*. For* $u_0\in P_{1_h}$*, construct*

$$u_{n+1}(s,t)=\int_0^1 G(t,\xi)f(\xi,u_n(s,\xi),\frac{\partial}{\partial s}u_n(s,\xi))d\xi \quad n=0,1,2,\ldots.$$

We have $u_n(s,t)\to u^*(s,t)$ *as* $n\to\infty$*, where* $G(t,\xi)$ *is given as (5).*

Example 1. *Consider the problem*

$$\begin{aligned} & D_t^{2.3}u(s,t) + (\frac{u(s,t)}{\frac{\partial}{\partial s}u(s,t)})^{\frac{1}{2}} + \frac{\sqrt{u(s,t)}}{\sqrt{u(s,t)}+\sqrt{\frac{\partial}{\partial s}u(s,t)}}e^t + a = 0, \qquad (8) \\ & 0<s<\frac{1}{2}, \quad 0<t<1, \\ & u(s,0)=\frac{\partial}{\partial t}u(s,0)=0, \quad u(s,1)=\int_0^1\varphi(s,\xi)u(s,\xi)d\xi, \end{aligned}$$

where $a > 0$ *is a constant.*

Here, $\varphi(s,t) = (t+s)^2$*. Then* $\varphi : [0,1]\times[0,1] \to [0,\infty)$ *with* $\varphi \in L^1([0,1]\times[0,1])$,

$$\zeta_1 = \int_0^1 \xi^{1.3}(1-\xi)(\xi+s)^2 d\xi > 0 \text{ and } \zeta_2 = \int_0^1 \xi^{\kappa-1}(\xi+s)^2 d\xi < 1.$$

Suppose also

$$u(s,t) \leq u'(s,t) \text{ implies that } \frac{\partial}{\partial s}u(s,t) \leq \frac{\partial}{\partial s}u'(s,t).$$

Take $0 < b < a$ *and* $f, g : [0,1] \times (0,\infty) \times (0,\infty) \to [0,\infty)$ *defined by:*

$$f(t,z,w) = (\frac{z}{w})^{\frac{1}{2}} + b, \quad g(t,z,w) = \frac{\sqrt{z}}{\sqrt{z}+\sqrt{w}}e^t + a - b, \quad \gamma = \frac{1}{2}.$$

f *and* g *are increasing with respect to the second argument, but also decreasing with respect to the third argument,* $g(t,0,1) = a - b > 0$ *for* $\lambda \in (0,1)$, $t \in (0,1)$, $z, w \in (0,\infty)$ *and*

$$g(t,\lambda z, \frac{1}{\lambda}w) \geq \lambda g(t,z,w),$$
$$f(t,\lambda z, \frac{1}{\lambda}w) \geq \lambda f(t,z,w).$$

Moreover, for $\delta_0 \in (0, \frac{b}{e+a-b})$,

$$f(t,z,w) = (\frac{z}{w})^{\frac{1}{2}} + b \geq b = \frac{b}{e+a-b}.(e+a-b)$$
$$\geq \delta_0(\frac{\sqrt{z}}{\sqrt{z}+\sqrt{w}}e^t + a - b) = \delta_0 g(t,z,w).$$

By Theorem 3, problem (8) has a unique positive solution in P_{1_h}*, where*

$$h(s,t) = (t+s)^{1.3}, \quad 0 < s < \frac{1}{2} \quad \text{and} \quad 0 < t < 1.$$

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