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MARKOFF-ROSENBERGER TRIPLES WITH FIBONACCI COMPONENTS

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Abstract. We characterize the solutions of the Markoff-Rosenberger equation

$$ax^2 + by^2 + cz^2 = dxyz$$

with $a, b, c, d \in \mathbb{Z}$, $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$ and $a, b, c \mid d$, for which $(x, y, z) = (F_i, F_j, F_k)$, where F_n denotes the n-th Fibonacci number for any integer $n \geq 0$.

1. Introduction

Markoff ([6]) obtained many nice results related to the equation

$$x^2 + y^2 + z^2 = 3xyz.$$

He showed that there exist infinitely many integral solutions. The so-called Markoff equation defined above has been generalized in many directions by several authors. In this article we focus on the generalization considered by Rosenberger ([7])

$$ax^2 + by^2 + cz^2 = dxyz. \tag{1.1}$$

Rosenberger proved that if $a, b, c, d \in \mathbb{N}$ are integers such that $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$ and $a, b, c|d$, then non-trivial solutions exist only if

$$(a, b, c, d) \in \{(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 2, 4), (1, 1, 5, 5), (1, 2, 3, 6)\}.$$

Silverman ([8]) studied equation (1.1) with $a = b = c = 1$ over imaginary quadratic number fields. Baer and Rosenberger ([1]) considered solutions of equation (1.1) over imaginary quadratic number fields. González-Jiménez and Tornero ([4]) looked for solutions of equation (1.1) in arithmetic progression

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that lie in the ring of integers of a number field. González-Jiménez ([3]) studied solutions of (1.1) whose coordinates belong to the ring of integers of a number field and form a geometric progression. A well-known identity related to the Fibonacci numbers

$$1+F_{2n-1}^2+F_{2n+1}^2=3F_{2n-1}F_{2n+1}$$

shows that $(x,y,z)=(1,F_{2n-1},F_{2n+1})$ is a solution of the Markoff equation for any $n\in\mathbb{N}$. Luca and Srinivasan ([5]) proved that there are infinitely many solutions (F_i,F_j,F_k) to the classical Markoff equations (given by the above identity). In this paper we extend the result of Luca and Srinivasan, we determine the solutions $(x,y,z)=(F_i,F_j,F_k)$ of equation (1.1) for

$$(a,b,c,d)\in\{(1,1,1,1),(1,1,2,2),(1,1,2,4),(1,1,5,5),(1,2,3,6)\}.$$

In the proofs, we simplify the strategy described by Luca and Srinivasan, by providing a direct way to get a bound for $k-j$ from above.

2. Main result

THEOREM 1. *If* $(x,y,z)=(F_i,F_j,F_k)$ *is a solution of equation* (1.1) *and* $(a,b,c,d)\in\{(1,1,1,1),(1,1,2,2),(1,1,2,4),(1,1,5,5),(1,2,3,6)\}$, *then the complete list of solutions are given by*

(a,b,c,d)	*solutions*
$(1,1,1,1)$	$\{(3,3,3)\}$
$(1,1,2,2)$	$\{(2,2,2)\}$
$(1,1,2,4)$	$\{(1,1,1),(1,3,1),(1,3,5),(3,1,1),(3,1,5)\}$
$(1,1,5,5)$	$\{(1,2,1),(1,3,1),(1,3,2),(2,1,1),(3,1,1),(3,1,2)\}$
$(1,2,3,6)$	$\{(1,1,1),(1,2,1),(1,2,3),(5,1,1)\}$

PROOF. A well-known fact is that the n-th Fibonacci number can be written as follows

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\ \text{where } \alpha=\frac{1+\sqrt{5}}{2} \text{ and } \beta=\frac{1-\sqrt{5}}{2}.$$

We also have that for all $n\geq 1$

$$\alpha^{n-2}\leq F_n\leq\alpha^{n-1}.$$

We note that in the Markoff case, $a=b=c$ and the equation is fully symmetric in (x,y,z). This symmetry is no longer present in the case of the Rosenberg equation. In the proof we assume that $x\leq y\leq z$ hence we need to consider not only the equation $ax^2+by^2+cz^2=dxyz$ but also all the permutations of (a,b,c). We provide a bound for i for general (a,b,c,d) and we use it to get an upper bound for $k-j$. Based on inequalities from ([5]) we have

(2.1)
$$\frac{aF_i^2+bF_j^2}{F_k}\leq(a+b)\alpha^j\,,\quad\left|\frac{\beta^k}{\sqrt{5}}\right|\leq\frac{\alpha^j}{5}\,,\quad\left|\alpha^i\beta^j+\alpha^j\beta^i-\beta^{i+j})\right|\leq 3\alpha^j.$$

Suppose $(x,y,z)=(F_i,F_j,F_k)$ for $i\le j\le k$ is a solution of

$$aF_i^2+bF_j^2+cF_k^2=dF_iF_jF_k.$$

We obtain that

$$c\frac{\alpha^k}{\sqrt{5}}-d\frac{\alpha^{i+j}}{5}=-\frac{aF_i^2+bF_j^2}{F_k}+c\frac{\beta^k}{\sqrt{5}}-\frac{d}{5}(\alpha^i\beta^j+\alpha^j\beta^i-\beta^{i+j}).$$

Taking absolute values and using the inequalities at (2.1) we obtain:

$$\left|c\frac{\alpha^k}{\sqrt{5}}-d\frac{\alpha^{i+j}}{5}\right|\le\frac{\alpha^j}{5}(5a+5b+c+3d),$$

and dividing by $\frac{\alpha^{i+j}}{\sqrt{5}}$:

$$\left|c\,\alpha^{k-i-j}-\frac{d}{\sqrt{5}}\right|\le\frac{5a+5b+c+3d}{\sqrt{5}\alpha^i}. \tag{2.2}$$

Now define $f(n)=\left|c\,\alpha^n-\frac{d}{\sqrt{5}}\right|$ and let $t_0\in\mathbb{Z}$ such that $f(t_0)\le f(n)$ for any $n\in\mathbb{Z}$. Then

$$\alpha^i\le\frac{5a+5b+c+3d}{\sqrt{5}f(t_0)}. \tag{2.3}$$

For a given tuple (a,b,c,d) equation (2.3) provides an upper bound for i, denote it by $\mathfrak{ub}(a,b,c,d)$. For a given i equation (2.2) yields an upper bound for $k-j$. For the concrete equations we consider these bounds are as follows:

$$\begin{aligned}
&\mathfrak{ub}(1,1,1,1)=9,\\
&\mathfrak{ub}(1,1,2,2)=8, \mathfrak{ub}(1,2,1,2)=\mathfrak{ub}(2,1,1,2)=9,\\
&\mathfrak{ub}(1,1,2,4)=\mathfrak{ub}(1,2,1,4)=\mathfrak{ub}(2,1,1,4)=8,\\
&\mathfrak{ub}(1,2,3,6)=\mathfrak{ub}(2,1,3,6)=8, \mathfrak{ub}(1,3,2,6)=\mathfrak{ub}(3,1,2,6)=7,\\
&\mathfrak{ub}(2,3,1,6)=\mathfrak{ub}(3,2,1,6)=11,\\
&\mathfrak{ub}(1,1,5,5)=7, \mathfrak{ub}(1,5,1,5)=\mathfrak{ub}(5,1,1,5)=8.
\end{aligned}$$

For each (a,b,c,d) and any $i\le\mathfrak{ub}(a,b,c,d)$ one needs to compute the (finitely many) possibilities for $m=k-j$. That is, fixing (a,b,c,d), i and m we study the equation

$$aF_i^2+bF_j^2+cF_{j+m}^2-dF_iF_jF_{j+m}=0.$$

We note that the equation above only depends on j. To deal with the concrete cases we use the following arguments.

(I) We eliminate as many values of i as possible by checking solvability of quadratic equations

$$aF_i^2+by^2+cz^2-F_iyz=0.$$

(II) For fixed m we eliminate equations $aF_i^2+bF_j^2+cF_{j+m}^2-dF_iF_jF_{j+m}=0$ modulo p, where p is a prime.

(III) We consider the equation $aF_i^2 + bF_j^2 + cF_{j+m}^2 = dF_iF_jF_{j+m}$ as a quadratic in F_j. Then its discriminant $d^2F_i^2F_{j+m}^2 - 4b(aF_i^2 + cF_{j+m}^2)$ must be a square. A fundamental identity for the Fibonacci and Lucas numbers (denoted by L_n, defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$) says that

$$L_n^2 = 5F_n^2 \pm 4.$$

That is we have the system of equations

$$\begin{aligned} Y_1^2 &= 5X^2 \pm 4, \\ Y_2^2 &= d^2F_i^2X^2 - 4b(aF_i^2 + cX^2), \end{aligned}$$

where $X = F_{j+m}$. Multiplying these equations together yields

$$Y^2 = (5X^2 \pm 4)(d^2F_i^2X^2 - 4b(aF_i^2 + cX^2)).$$

Therefore we reduce our problem to obtain integral points on the above quartic genus 1 curves. This will be realized using the Magma ([2]) function `SIntegralLjunggrenPoints`.

We implemented the above procedure in SageMath ([9]) and the code can be downloaded from the URL address `http://shrek.unideb.hu/~tengely/MarkoffSolver.sage`. Detailed computations can be found at `http://shrek.unideb.hu/~tengely/Markoff-Rosenberger-Fibonacci.pdf`.

2.1. *The case with* $d = 1$. We have that $2 \leq i \leq 9$. In this range the Diophantine equation $F_i^2 + y^2 + z^2 = F_iyz$ is solvable only for $i = 4$. If $i = 4$, then we have that $0 \leq k - j \leq 4$. The equation $9 + F_j^2 + F_{j+m}^2 - 3F_jF_{j+m} = 0$ has no solution modulo 3 for $m = 1, 2, 3$, and it is not solvable modulo 11 for $m = 4$. It remains to consider the case $m = 0$. We have that $k = j$, therefore the equation is simply $9 = F_j^2$. Hence, we get the solution $(x, y, z) = (3, 3, 3)$.

2.2. *Cases with* $d = 2$. Consider the tuple $(a, b, c, d) = (1, 1, 2, 2)$. The bound for i is 8, however only the quadratic equation related to $i = 3$ is solvable in integers. If $i = 3$, then $0 \leq k - j \leq 3$. We eliminate the cases $m = 1, 2$ modulo 7 and the case $m = 3$ modulo 23. If $k = j$, then we get that $4 = F_j^2$. Hence, we obtain the solution $(x, y, z) = (2, 2, 2)$. There are 2 other subcases here, $(a, b, c, d) = (1, 2, 1, 2)$ and $(2, 1, 1, 2)$ having the same upper bound for i, namely 9. In case of $(a, b, c, d) = (1, 2, 1, 2)$ we can eliminate all values of i except $i = 3$ and 9. If $i = 3$ we have

$$4 + 2F_j^2 + F_{j+m}^2 - 4F_jF_{j+m} = 0,$$

where $0 \leq m \leq 5$. Congruence arguments eliminate the cases with $m \in \{1, 2, 3, 4, 5\}$ as follows:

m	1	2	3	4	5
mod	17	7	19	3	13

.

The remaining value of m is 0, that yields the equation $4 = F_j^2$, so we obtain the solution $(x, y, z) = (2, 2, 2)$. If $i = 9$, then the corresponding equation is

$$1156 + 2F_j^2 + F_{j+m}^2 - 68F_jF_{j+m} = 0,$$

where $0 \leq m \leq 9$. The following table contains the primes used to get a contradiction

m	0	1	2	3	4	5	6	7	8	9
mod	3	7	11	19	11	5	11	7	3	29

In case of $(a, b, c, d) = (2, 1, 1, 2)$ we only need to handle $i = 3$ for which we get that $0 \leq m \leq 5$. The equation is given by

$$8 + F_j^2 + F_{j+m}^2 - 4F_jF_{j+m} = 0,$$

and we can eliminate all these (except $m = 0$) as the table below shows

m	1	2	3	4	5
mod	11	7	11	3	13

If $m = 0$, then we have $8 = 2F_j^2$ and the only solution is $(x, y, z) = (2, 2, 2)$.

2.3. *Cases with $d = 4$.* If $(a, b, c, d) = (1, 1, 2, 4)$, then if follows that $i = 2$ or 4. If $(a, b, c, d) = (1, 2, 1, 4)$, then we obtain that $i = 2$ or 4. The last tuple to consider here is $(a, b, c, d) = (2, 1, 1, 4)$ and we get that $i = 2$ or 5. We need to handle the equations

$$\begin{aligned} 1 + F_j^2 + 2F_{j+m}^2 - 4F_jF_{j+m} &= 0, \\ 9 + F_j^2 + 2F_{j+m}^2 - 12F_jF_{j+m} &= 0, \\ 1 + 2F_j^2 + F_{j+m}^2 - 4F_jF_{j+m} &= 0, \\ 9 + 2F_j^2 + F_{j+m}^2 - 12F_jF_{j+m} &= 0, \\ 2 + F_j^2 + F_{j+m}^2 - 4F_jF_{j+m} &= 0, \\ 50 + F_j^2 + F_{j+m}^2 - 20F_jF_{j+m} &= 0. \end{aligned}$$

We provide details in the case of the first equation, the other 5 can be solved in a similar way. We consider the equation as a quadratic in F_j and follow the argument described in (III). It remains to solve the quartic Diophantine equations

$$y^2 = 10x^4 - 13x^2 + 4, \qquad y^2 = 10x^4 + 3x^2 - 4.$$

The integral solutions of these equations can be completely determined using the Magma ([2]) procedure `SIntegralLjunggrenPoints`. In the former case we get that $x \in \{0, \pm 1, \pm 5\}$. In case of the latter equation we have that $x \in \{\pm 1\}$. It follows that $F_{j+m} = 1$ or 5 and we get the solutions $(x, y, z) = (1, 1, 1)$ and $(x, y, z) = (1, 3, 5)$.

2.4. *Cases with* $d = 5$. Here, we get the following possibilities for i for the 3 tuples

(a,b,c,d)	i
$(1,1,5,5)$	$\{2,3,4\}$
$(1,5,1,5)$	$\{2,3,4\}$
$(5,1,1,5)$	$\{2,3,5,7\}$

.

Consider the tuple $(5,1,1,5)$. If $i = 5$, then $0 \leq m \leq 7$ and if $i = 7$, then $0 \leq m \leq 9$. All these cases can be eliminated using congruence arguments: if $i = 5$, then we have

m	0	1	2	3	4	5	6	7
mod	7	11	11	11	3	11	17	11

and if $i = 7$, then we obtain

m	0	1	2	3	4	5	6	7	8	9
mod	3	11	13	29	11	19	11	29	3	11

.

It remains to check the solutions for $i = 2$ and 3. The equations can be written as follows

$$5 + F_j^2 + F_{j+m}^2 - 5F_jF_{j+m} = 0,$$
$$20 + F_j^2 + F_{j+m}^2 - 10F_jF_{j+m} = 0.$$

As before we reduce the problem to genus 1 curves, we obtain the following 4 equations

$$y^2 = 105x^4 - 184x^2 + 80,$$
$$y^2 = 105x^4 - 16x^2 - 80,$$
$$y^2 = 30x^4 - 49x^2 + 20,$$
$$y^2 = 30x^4 - x^2 - 20.$$

The complete set of possible values for F_j is given by $\{1,2,3,987\}$. We also know that $F_i \in \{1,2\}$, hence one can easily determine F_k. The solutions of the equation $x^2 + y^2 + 5z^2 = 5xyz$ from these cases are given by $(x,y,z) = (1,2,1),(2,1,1),(1,3,1),(3,1,1),(1,3,2)$ and $(3,1,2)$.

2.5. *Cases with* $d = 6$. Let us consider the equation $x^2+2y^2+3z^2 = 6xyz$. Here we can eliminate many quadratic equations. In the table below we collect the remaining cases.

(a,b,c,d)	i
$(1,2,3,6)$	$\{2,5\}$
$(2,1,3,6)$	$\{2,3\}$
$(1,3,2,6)$	$\{2,5\}$
$(3,1,2,6)$	$\{2,4\}$
$(2,3,1,6)$	$\{2,3\}$
$(3,2,1,6)$	$\{2,4,11\}$

.

We provide details in case of the tuple $(3,2,1,6)$ only, the remaining ones can be treated in a similar way. We have three values for i, these correspond to the equations

$$3+2F_j^2+F_{j+m}^2-6F_jF_{j+m}=0,$$
$$27+2F_j^2+F_{j+m}^2-18F_jF_{j+m}=0,$$
$$23763+2F_j^2+F_{j+m}^2-534F_jF_{j+m}=0.$$

The last equation corresponds to $i=11$. Here, we do not expect any solution so we compute the possible values of m and try to get a contradiction modulo some prime. It turns out that $0\le m\le 13$ and all these cases can be handled using congruence arguments. We summarize the computation in the following table

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13
mod	5	17	19	7	13	5	17	13	7	17	13	13	17	29

.

Solving the remaining two equations as described in (III) we get that we need to find the integral solutions of the Diophantine equations

$$y^2=35x^4-43x^2+12,$$
$$y^2=35x^4+13x^2-12,$$
$$y^2=395x^4-451x^2+108,$$
$$y^2=395x^4+181x^2-108.$$

We use the Magma function `SIntegralLjunggrenPoints` to determine the integral solutions and we get that $F_j\in\{1,2\}$. The tuple we consider is given by $(3,2,1)$ and the corresponding equation is $3F_i^2+2F_j^2+F_k^2=6F_iF_jF_k$. Since $i=2$ or 4 we have $F_i\in\{1,3\}$. These possibilities yield the solutions $(x,y,z)=(1,1,1),(1,1,5),(1,2,1)$ and $(3,2,1)$.

□

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