

EXTENSION OF THE FUNCTIONAL INDEPENDENCE OF THE RIEMANN ZETA-FUNCTION

ANTANAS LAURINČIKAS
Vilnius University, Lithuania

ABSTRACT. In 1972, Voronin proved the functional independence of the Riemann zeta-function $\zeta(s)$, i. e., if the functions Φ_j are continuous in \mathbb{C}^N and $\Phi_0(\zeta(s), \dots, \zeta^{(N-1)}(s)) + \dots + s^n \Phi_n(\zeta(s), \dots, \zeta^{(N-1)}(s)) \equiv 0$, then $\Phi_j \equiv 0$ for $j = 0, \dots, n$. The problem goes back to Hilbert who obtained the algebraic-differential independence of $\zeta(s)$. In the paper, the functional independence of compositions $F(\zeta(s))$ for some classes of operators F in the space of analytic functions is proved. For example, as a particular case, the functional independence of the function $\cos \zeta(s)$ follows.

1. INTRODUCTION

In the theory of functions, various functional relations occupy an important place. The problem goes back to Hölder and Hilbert. In [2], Hölder proved the algebraic-differential independence of the Euler gamma-function $\Gamma(s)$, i. e., that there is no any polynomial $p(s_1, \dots, s_r) \not\equiv 0$ such that $p(\Gamma(s), \Gamma'(s), \dots, \Gamma^{(r-1)}(s)) \equiv 0$.

We recall that the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

2020 *Mathematics Subject Classification.* 11M06.

Key words and phrases. Algebraic-differential independence, functional independence, Riemann zeta-function, space of analytic functions, universality.

This research is funded by the European Social Fund according to the activity “Improvement of researchers’ qualification by implementing world-class R&D projects” of Measure No. 09.3.3-LMT-K-712-01-0037.

where the product is taken over all prime numbers, and has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1.

Hilbert, presenting the list of the most important problems of mathematics at the International Congress of Mathematicians (Paris, 1900), in the description of the 18th problem, mentioned, see [1], that the function $\zeta(s)$ is algebraically-differentially independent, and this follows from the algebraic-differential independence of the function $\Gamma(s)$ and the functional equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

Moreover, he also conjectured that there is no algebraic-differential equation with partial derivatives which is satisfied by the function

$$\zeta(s, x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}.$$

The latter conjecture was proved by Ostrowski in [8]. The investigations were continued by Postnikov in [9, 10]. For example, in [10] he considered the function

$$L(s, x, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)x^m}{m^s},$$

where $\chi(m)$ is a Dirichlet character, and obtained that the equality

$$p\left(x, s, \frac{\partial^{k+l}L(s, x, \chi)}{\partial s^k \partial x^l}\right) \equiv 0$$

can not be satisfied by any polynomial $p \neq 0$.

The further progress in the field belongs to Voronin. In [11, 12], see also [15, 3], he obtained the functional independence of the function $\zeta(s)$. More precisely, he proved that if $f_0, f_1, \dots, f_n : \mathbb{C}^N \rightarrow \mathbb{C}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$, are continuous functions, and the equality

$$\sum_{k=0}^n s^k f_k\left(\zeta(s), \zeta'(s), \dots, \zeta^{(N-1)}(s)\right) = 0$$

holds identically for s , then $f_k \equiv 0$ for $k = 0, 1, \dots, n$. In [14], the latter result was generalized for a collection of Dirichlet L -functions with pairwise non-equivalent Dirichlet characters.

The aim of this note is a generalization of the functional independence for certain compositions of the function $\zeta(s)$. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $H(G)$ the space of analytic functions on the region G endowed with the topology of uniform convergence on compacta. Let $S = \{g \in H(D) : 1/g(s) \in H(D) \text{ or } g(s) \equiv 0\}$. In this note, we continue investigations of [7] of the compositions $F(\zeta(s))$.

THEOREM 1.1. *Suppose that $F : H(D) \rightarrow H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. If the functions $\Phi_0, \Phi_1, \dots, \Phi_n : \mathbb{C}^N \rightarrow \mathbb{C}$ are continuous, and the equality*

$$\sum_{k=0}^n s^k \Phi_k \left(F(\zeta(s)), (F(\zeta(s)))', \dots, (F(\zeta(s)))^{(N-1)} \right) = 0$$

holds identically for $s \in D$, then $\Phi_k \equiv 0$ for $k = 0, 1, \dots, n$.

Sometimes it is more convenient to deal with operators in the space of analytic functions in a bounded region. Let V be an arbitrary positive number, $D_V = \{s \in \mathbb{C} : 1/2 < \sigma < 1, |t| < V\}$ and $S_V = \{g \in H(D_V) : 1/g(s) \in H(D_V) \text{ or } g(s) \equiv 0\}$. Since $\zeta(s + i\tau) \in H(D_V)$ for all $\tau \in \mathbb{R}$, $F(\zeta(s + i\tau)) = g_\tau(s)$ with some $g_\tau(s) \in H(D_V)$, $\tau \in \mathbb{R}$. Hence, $F(\zeta(\sigma + it)) = g_t(\sigma)$ for $1/2 < \sigma < 1, t \in \mathbb{R}$.

THEOREM 1.2. *Suppose that $F : H(D_V) \rightarrow H(D_V)$ is a continuous operator such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is non-empty. Then the assertion of Theorem 1.1 is true.*

For example, the operator $F : H(D_V) \rightarrow H(D_V)$ given by

$$F(g) = c_1 g' + \dots + c_r g^{(r)}, \quad g \in H(D_V), \quad c_1, \dots, c_r \in \mathbb{C} \setminus \{0\},$$

satisfies the hypotheses of Theorem 1.2. Actually, for each polynomial

$$p(s) = a_k s^k + \dots + a_1 s + a_0, \quad a_k \neq 0,$$

there exists a polynomial

$$q(s) = b_{k+1} s^{k+1} + \dots + b_1 s + b_0, \quad b_{k+1} \neq 0,$$

such that $F(q) = p$ because the coefficients b_{k+1}, \dots, b_1 can be expressed by a_k, \dots, a_0 . Moreover, we may choose b_0 to be $|b_0|$ large enough such that $q(s) \neq 0$ in D_V . Thus, by Theorem 1.2, we have the functional independence for the function

$$c_1 \zeta'(s) + \dots + c_r \zeta^{(r)}(s).$$

For $F : H(D) \rightarrow H(D)$ and $a \in \mathbb{C}$, define the set

$$H_{F(0);a}(D) = \left\{ g \in H(D) : \frac{1}{g(s) - a} \in H(D) \right\} \cup \{F(0)\}.$$

THEOREM 1.3. *Suppose that $F : H(D) \rightarrow H(D)$ is a continuous operator such that $F(S) \supset H_{F(0);a}(D)$. Then the same assertion as in Theorem 1.1 is true.*

For example, the operator $F : g \rightarrow g^N$, $N \in \mathbb{N}$, satisfies the hypotheses of the theorem with $a = 0$.

For $F : H(D) \rightarrow H(D)$, define the set

$$H_{F(0);-1,1}(D) = \{g \in H(D) : g(s) \neq -1, g(s) \neq 1\} \cup \{F(0)\}.$$

THEOREM 1.4. *Suppose that $F : H(D) \rightarrow H(D)$ is a continuous operator such that $F(S) \supset H_{F(0);-1,1}(D)$. Then the same assertion as in Theorem 1.1 is true.*

For example, the operator $F : g \rightarrow \cos g$ satisfies the hypotheses of the theorem. Thus, we have that if the functions $\Phi_0, \Phi_1, \dots, \Phi_n : \mathbb{C}^N \rightarrow \mathbb{C}$ are continuous, and the equality

$$\sum_{k=0}^n s^k \Phi_k \left(\cos \zeta(s), -\zeta'(s) \sin \zeta(s), \dots, (\cos \zeta(s))^{(N-1)} \right) = 0$$

holds identically for s , then $\Phi_k \equiv 0$ for $k = 0, 1, \dots, n$. In other words, we have the functional independence for the function $\cos \zeta(s)$. The same is true for the functions $\sin \zeta(s)$, $\sinh \zeta(s)$, $\cosh \zeta(s)$.

2. UNIVERSALITY

Proofs of Theorems 1.1–1.4 are based on the universality property of compositions $F(\zeta(s))$. We recall that the universality property of the function $\zeta(s)$ was discovered by Voronin in [13], and means that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. More precisely, he proved that if $f(s)$ is a continuous non-vanishing function in the disc $|s| \leq r$, $0 < r < 1/4$, and analytic for $|s| < r$, then, for every $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s| \leq r} |\zeta(s + 3/4 + i\tau) - f(s)| < \varepsilon.$$

The modern version of the Voronin theorem, see, for example, [4], uses the following notation. Let \mathcal{K} be the class of compact subsets of the strip D with connected complements and $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K . Then we have that, for every $\varepsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Here $\text{meas}A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Generalizations of the Voronin universality theorem for composite functions $F(\zeta(s))$ were obtained in [5, 6]. Let $H(K)$ with $K \in \mathcal{K}$ be the class of continuous functions on K that are analytic in the interior of K . Thus, $H_0(K) \subset H(K)$.

LEMMA 2.1. *Suppose that the operator $F : H(D) \rightarrow H(D)$ satisfies the hypotheses of Theorem 1.1. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + i\tau)) - f(s)| < \varepsilon \right\} > 0.$$

The proof of the lemma is given in [5, Theorem 5].

LEMMA 2.2. *Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Suppose that $V > 0$ is such that $K \subset D_V$, and that the operator $F : H(D_V) \rightarrow H(D_V)$ satisfies the hypotheses of Theorem 1.2. Then the same assertion as in Lemma 2.1 is true.*

The proof of the lemma can be found in [5, Theorem 6].

LEMMA 2.3. *Suppose that the operator $F : H(D) \rightarrow H(D)$ satisfies the hypotheses of Theorem 1.3. Let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_{F(0);a}(D)$. Then the same assertion as in Lemma 2.1 is true.*

PROOF. In [6], a stronger statement with $K \in \mathcal{K}$ and $f(s) \neq a$ on K , was proved. If K and $f(s)$ are as in the lemma, then the proof is the same as that of the case $r \geq 2$ of [6, Theorem 4.4]. \square

LEMMA 2.4. *Suppose that the operator $F : H(D) \rightarrow H(D)$ satisfies the hypotheses of Theorem 1.4. Let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_{F(0);-1,1}(D)$. Then the same assertion as in Lemma 2.1 is true.*

Proof of the lemma is given in [6, Theorem 4.4] where a more general case of the set $H_{F(0);a_1,\dots,a_r}(D) = \{g \in H(D) : g(s) \neq a_j, j = 1, \dots, r\}$ with arbitrary distinct $a_1, \dots, a_r \in \mathbb{C}$ is considered.

In [7], the above universality theorems were applied for estimation of the number of zeros of compositions $F(\zeta(s))$.

3. DENSENESS THEOREMS

In this section, we will prove the denseness in \mathbb{C}^N for some sets defined by means of the compositions $F(\zeta(s))$.

THEOREM 3.1. *Suppose that $\sigma, 1/2 < \sigma < 1$, is fixed, and the operator $F : H(D) \rightarrow H(D)$ satisfies the hypotheses of Theorem 1.1. Then the set*

$$\left\{ F(\zeta(\sigma + it)), (F(\zeta(\sigma + it)))', \dots, (F(\zeta(\sigma + it)))^{(N-1)} : t \in \mathbb{R} \right\}$$

is everywhere dense in \mathbb{C}^N .

PROOF. It is sufficient to show that, for any arbitrary collection $(a_0, a_1, \dots, a_{N-1}) \in \mathbb{C}^N$ and every $\varepsilon > 0$, there exists a real number τ such that

$$\left| (F(\zeta(\sigma + i\tau)))^{(j)} - a_j \right| < \varepsilon$$

for all $j = 0, 1, \dots, N-1$. Define the polynomial

$$p_N(s) = a_0 + \frac{a_1 s}{1!} + \dots + \frac{a_{N-1} s^{N-1}}{(N-1)!}.$$

Then we have that

$$(3.1) \quad p_N^{(j)}(0) = a_j$$

for all $j = 0, 1, \dots, N-1$. We fix $\hat{\sigma}$, $1/2 < \hat{\sigma} < 1$, and take $K \in \mathcal{K}$ such that the number $\hat{\sigma}$ was an interior point of K . Let δ be the distance of $\hat{\sigma}$ from the boundary of the set K . Then, in view of Lemma 2.1, there exists $\tau \in \mathbb{R}$ (actually, there exists a sequence $\tau_m \rightarrow \infty$) such that

$$\sup_{s \in K} |F(\zeta(s + i\tau)) - p_N(s - \hat{\sigma})| < \frac{\varepsilon \delta^{N-1}}{2^{N-1}(N-1)!}.$$

Therefore, the application of the Cauchy integral formula and (3.1) shows that, for $j = 0, 1, \dots, N-1$,

$$\left| (F(\zeta(\hat{\sigma} + i\tau)))^{(j)} - a_j \right| = \frac{j!}{2\pi} \left| \int_{|z-\hat{\sigma}|=\delta/2} \frac{F(\zeta(z + i\tau)) - p_N(z - \hat{\sigma})}{(z - \hat{\sigma})^{j+1}} dz \right| < \varepsilon,$$

and the theorem is proved. \square

THEOREM 3.2. *Suppose that σ , $1/2 < \sigma < 1$, is fixed, $V > 0$, and the operator $F : H(D_V) \rightarrow H(D_V)$ satisfies the hypotheses of Theorem 1.2. Then the same assertion as in Theorem 3.1 is true.*

PROOF. We repeat the proof of Theorem 3.1 with using of Lemma 2.2. \square

THEOREM 3.3. *Suppose that the operator $F : H(D) \rightarrow H(D)$ satisfies the hypotheses of Theorem 1.3. Then the same assertion as in Theorem 3.1 is true.*

PROOF. We slightly modify the method of [3]. Let $(a_0, a_1, \dots, a_{N-1}) \in \mathbb{C}^N$, $a_0 \neq a$. We will prove that there exists a collection $(b_0, b_1, \dots, b_{N-1}) \in \mathbb{C}^N$ such that

$$(3.2) \quad e^{b_0 + b_1 s + \dots + b_{N-1} s^{N-1}} \equiv a_0 - a + \frac{a_1 s}{1!} + \dots + \frac{a_{N-1} s^{N-1}}{(N-1)!} \pmod{s^N}.$$

Really, (3.2) is true for $N = 1$, since $a_0 - a \neq 0$ and $b_0 = \log(a_0 - a)$. Now, suppose that (3.2) true for $N = m > 1$. Then, with some $c \in \mathbb{C}$

$$e^{b_0 + b_1 s + \dots + b_m s^m} \equiv a_0 - a + \frac{a_1 s}{1!} + \dots + \frac{a_m s^m}{m!} + c s^{m+1} \pmod{s^{m+2}}.$$

Hence, since

$$\begin{aligned} e^{b s^{m+1}} &\equiv 1 + b s^{m+1} \pmod{s^{m+2}}, \\ e^{b_0 + b_1 s + \dots + b_m s^m + b s^{m+1}} &\equiv \left(a_0 - a + \frac{a_1 s}{1!} + \dots + c s^{m+1} \right) \\ &\quad \times (1 + b s^{m+1}) \pmod{s^{m+2}}. \end{aligned}$$

Therefore, putting

$$b(a_0 - a) + c = \frac{a_{m+1}}{(m+1)!},$$

we find

$$b = \frac{1}{a_0 - a} \left(\frac{a_{m+1}}{(m+1)!} - c \right),$$

and this shows that with $b_{m+1} = b$

$$e^{b_0 + b_1 s + \dots + b_{m+1} s^{m+1}} \equiv a_0 - a + \frac{a_1 s}{1!} + \dots + \frac{a_{m+1} s^{m+1}}{(m+1)!} \pmod{s^{m+2}}.$$

Thus, (3.2) follows by induction.

Now define the function

$$f(s) = e^{b_0 + b_1 s + \dots + b_{N-1} s^{N-1}} + a \equiv a_0 + \frac{a_1 s}{1!} + \dots + \frac{a_{N-1} s^{N-1}}{(N-1)!} \pmod{s^N}.$$

Then, obviously, $f(s) \in H(D)$ and $f(s) \neq a$, thus, $f(s) \in H_{F(0);a}(D)$. Moreover,

$$(3.3) \quad f^{(j)}(0) = a_j$$

for all $j = 0, 1, \dots, N-1$. Let $\hat{\sigma}$, $1/2 < \hat{\sigma} < 1$, be a fixed interior point of the set $K \in \mathcal{K}$, $K \subset D$. By Lemma 2.3, there exists $\tau \in \mathbb{R}$ such that, for every $\varepsilon > 0$,

$$(3.4) \quad \sup_{s \in K} |F(\zeta(s + i\tau)) - f(s)| < \frac{\varepsilon \delta^{N-1}}{2^{N-1}(N-1)!},$$

where δ is the distance of $\hat{\sigma}$ from the set K . Now, taking into account (3.3) and (3.4), we find by using the Cauchy integral formula that, for all $j = 0, 1, \dots, N-1$,

$$\left| (F(\zeta(\hat{\sigma} + i\tau)))^{(j)} - a_j \right| \leq \frac{j!}{2\pi} \int_{|z - \hat{\sigma}| = \delta/2} \frac{|F(\zeta(z + i\tau)) - f(z)|}{|z - \hat{\sigma}|^{j+1}} |dz| < \varepsilon.$$

This inequality implies the assertion of the theorem. \square

THEOREM 3.4. *Suppose that the operator $F : H(D) \rightarrow H(D)$ satisfies the hypotheses of Theorem 1.4. Then the same assertion as in Theorem 3.1 is true.*

PROOF. Let $(a_0, a_1, \dots, a_{N-1}) \in \mathbb{C}^N$ with $a_0 \neq \pm 1$. We will prove that there exists a collection $(b_0, b_1, \dots, b_{N-1}) \in \mathbb{C}^N$ such that

$$(3.5) \quad \begin{aligned} & \frac{1}{2} \left(e^{b_0 + b_1 s + \dots + b_{N-1} s^{N-1}} + e^{-(b_0 + b_1 s + \dots + b_{N-1} s^{N-1})} \right) \\ & \equiv a_0 + \frac{a_1 s}{1!} + \dots + \frac{a_{N-1} s^{N-1}}{(N-1)!} \pmod{s^N}. \end{aligned}$$

As in the case of Theorem 3.3, we apply the inductive method. For $N = 1$, we have $e^{b_0} + e^{-b_0} = 2a_0$. Hence, $b_0 = \log(a_0 \pm \sqrt{a_0^2 - 1})$. Now suppose that

(3.5) is true with $N = m > 1$. Then, with some $c \in \mathbb{C}$,

$$\frac{1}{2} \left(e^{b_0+b_1s+\dots+b_ms^m} + e^{-(b_0+b_1s+\dots+b_ms^m)} \right) \equiv a_0 + \frac{a_1s}{1!} + \dots + \frac{a_ms^m}{m!} + cs^{m+1} \pmod{s^{m+2}}.$$

Hence,

$$\begin{aligned} & \frac{1}{2} \left(e^{b_0+b_1s+\dots+b_ms^m+bs^{m+1}} + e^{-(b_0+b_1s+\dots+b_ms^m+bs^{m+1})} \right) \\ & \equiv \frac{1}{2} \left(e^{b_0+b_1s+\dots+b_ms^m} (1+bs^{m+1}) \right. \\ & \quad \left. + e^{-(b_0+b_1s+\dots+b_ms^m)} (1-bs^{m+1}) \right) \pmod{s^{m+2}} \\ & \equiv \frac{1}{2} \left(e^{b_0+b_1s+\dots+b_{m-1}s^{m-1}} + e^{-(b_0+b_1s+\dots+b_{m-1}s^{m-1})} \right) \\ & \quad + \frac{1}{2} (e^{b_0} - e^{-b_0}) bs^{m+1} \pmod{s^{m+2}} \\ & \equiv a_0 + \frac{a_1s}{1!} + \dots + \frac{a_ms^m}{m!} + cs^{m+1} \\ & \quad + \frac{1}{2} (e^{b_0} - e^{-b_0}) bs^{m+1} \pmod{s^{m+2}}. \end{aligned}$$

We take

$$\frac{1}{2} (e^{b_0} - e^{-b_0}) b + c = \frac{a_{m+1}}{(m+1)!}.$$

Since $a_0 \neq \pm 1$, we have that $e^{b_0} - e^{-b_0} \neq 0$. Therefore, taking

$$b_{m+1} = 2 \left(\frac{a_{m+1}}{(m+1)!} - c \right) (e^{b_0} - e^{-b_0})^{-1},$$

we obtain (3.5) with $N = m + 1$. Thus, by induction, (3.5) is true for all $N \in \mathbb{N}$.

Now, consider the function

$$\begin{aligned} f(s) &= \frac{1}{2} \left(e^{b_0+b_1s+\dots+b_{N-1}s^{N-1}} + e^{-(b_0+b_1s+\dots+b_{N-1}s^{N-1})} \right) \\ &\equiv a_0 + \frac{a_1s}{1!} + \dots + \frac{a_{N-1}s^{N-1}}{(N-1)!} \pmod{s^N}. \end{aligned}$$

Clearly, $f(s) \in H(D)$, and

$$(3.6) \quad f^{(j)}(0) = a_j, \quad j = 0, 1, \dots, N-1.$$

Moreover, since

$$e^{b_0+b_1s+\dots+b_{N-1}s^{N-1}} = f(s) \pm \sqrt{f^2(s) - 1},$$

$f(s)$ can't take values -1 and 1 for $s \in D$. Thus, $f(s) \in H_{F(0); -1, 1}(D)$. Therefore, an application of Lemma 2.4, equality (3.6) and the Cauchy integral formula complete the proof. \square

4. PROOF OF THE MAIN THEOREMS

PROOF OF THEOREM 1.1. Let $\Phi : \mathbb{C}^N \rightarrow \mathbb{C}$ be a continuous function. We will prove that if the equality

$$(4.1) \quad \Phi \left(F(\zeta(s)), (F(\zeta(s)))', \dots, (F(\zeta(s)))^{(N-1)} \right) = 0$$

holds identically for s , then $\Phi \equiv 0$. Suppose, on the contrary, that $\Phi \not\equiv 0$. Then there exists a collection $(a_0, a_1, \dots, a_{N-1}) \in \mathbb{C}^N$ such that $\Phi(a_0, a_1, \dots, a_{N-1}) \neq 0$. The continuity of Φ implies the existence of an open set $G \subset \mathbb{C}^N$ such that $(a_0, a_1, \dots, a_{N-1}) \in G$ and, for all points $\underline{a} \in G$,

$$(4.2) \quad |\Phi(\underline{a})| > c > 0.$$

Let σ , $1/2 < \sigma < 1$, be fixed. Then, by Theorem 3.1, there exist real number t such that

$$\left(F(\zeta(\sigma + it)), (F(\zeta(\sigma + it)))', \dots, (F(\zeta(\sigma + it)))^{(N-1)} \right) \in G.$$

However, this and (4.2) contradict the equality (4.1).

We may suppose that $\Phi_0 \not\equiv 0$ because, in the opposite case, the equality of the theorem becomes

$$\sum_{k=1}^n s^{k-1} \Phi_k \left(F(\zeta(s)), (F(\zeta(s)))', \dots, (F(\zeta(s)))^{(N-1)} \right) \equiv 0.$$

Then there exists a bounded region $G_0 \subset \mathbb{C}^N$ such that

$$|\Phi_0(\underline{a})| > c_0 > 0$$

for all points $\underline{a} \in G_0$. Let

$$k_0 = \max \left(0 \leq k \leq n : \sup_{\underline{a} \in G_0} |\Phi_k(\underline{a})| \neq 0 \right).$$

If $k_0 = 0$, then the assertion of the theorem follows from the first part of the proof.

Now, let $k_0 \geq 1$. Then there exists a region $G \subset G_0$ such that

$$(4.3) \quad \inf_{\underline{a} \in G} |\Phi_{k_0}(\underline{a})| > c > 0.$$

By the proof of Theorem 3.1, there exists a sequence $\{\tau_m\} \subset \mathbb{R}$, $\lim_{m \rightarrow \infty} \tau_m = +\infty$, such that, for fixed σ , $1/2 < \sigma < 1$,

$$\left(F(\zeta(\sigma + i\tau_m)), (F(\zeta(\sigma + i\tau_m)))', \dots, (F(\zeta(\sigma + i\tau_m)))^{(N-1)} \right) \in G.$$

This together with (4.3) shows that

$$\begin{aligned} & \lim_{m \rightarrow \infty} |\sigma + i\tau_m|^{k_0} \\ & \times \left| \Phi_{k_0} \left(F(\zeta(\sigma + i\tau_m)), (F(\zeta(\sigma + i\tau_m)))', \dots, (F(\zeta(\sigma + i\tau_m)))^{(N-1)} \right) \right| \\ & = +\infty. \end{aligned}$$

However, this contradicts the equality

$$\sum_{k=0}^n s^k \Phi_k \left(F(\zeta(s)), (F(\zeta(s)))', \dots, (F(\zeta(s)))^{(N-1)} \right) \equiv 0.$$

The contradiction shows that $\Phi_k \equiv 0$ for all $k = 0, 1, \dots, n$. \square

Proofs of Theorems 1.2, 1.3 and 1.4 repeat the proof of Theorem 1.1 with using in the corresponding places Theorems 3.2, 3.3 and 3.4, respectively.

REFERENCES

- [1] D. Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc. **8** (1902), 437–479.
- [2] O. Hölder, *Über die Eigenschaft der Gammafunktion keiner algebraischen Differentialgleichung zu genügen*, Math. Ann. **28** (1887), 1–13.
- [3] A. A. Karatsuba and S. M. Voronin, Walter de Gruyter, Berlin, 1992.
- [4] A. Laurinčikas, *Limit theorems for the Riemann Zeta-function*, Kluwer, Dordrecht, Boston, London, 1996.
- [5] A. Laurinčikas, *Universality of the Riemann zeta-function*, J. Number Theory **130** (2010), 2323–2331.
- [6] A. Laurinčikas, *Universality of composite functions*, in: Functions in number theory and their probabilistic aspects, RIMS Kôkyûroku Bessatsu **B34** Res. Inst. Math. Sci. (RIMS), Kyoto, 2012. 191–204.
- [7] A. Laurinčikas, *On zeros of some analytic functions related to the Riemann zeta-function*, Glasn. Mat. Ser. III. **48** (2013), 59–65.
- [8] A. Ostrowski, *Über Dirichletsche Reihen und algebraische Differentialgleichungen*, Math. Z. **8** (1920), 241–298.
- [9] A. G. Postnikov, *On the differential independence of Dirichlet series*, Doklady Akad. Nauk SSSR (N.S.) **66** (1949) 561–564 (Russian).
- [10] A. G. Postnikov, *A generalization of one of the Hilbert problems*, Doklady Akad. Nauk SSSR (N.S.) **107** (1956), 512–515 (Russian).
- [11] S. M. Voronin, *The distribution of the nonzero values of the Riemann ζ -function*, Trudy Math. Inst. Steklov **128** (1972), 131–150 (Russian).
- [12] S. M. Voronin, *The differential independence of ζ -functions*, Doklady Akad. Nauk SSSR (N.S.) **209** (1973), 1264–1266 (Russian).
- [13] S. M. Voronin, *A theorem on the “universality” of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 475–486 (Russian).
- [14] S. M. Voronin, *The functional independence of Dirichlet L -functions*, Acta Arith. **27** (1975), 493–503 (Russian).
- [15] S. M. Voronin, *Selected works: Mathematics*, (ed. A. A. Karatsuba), Publishing House MGTU Im. N. E. Baumana, Moscow, 2006 (Russian).

A. Laurinčikas
Institute of Mathematics
Faculty of Mathematics and Informatics
Vilnius University
Naugarduko str. 24, LT-03225 Vilnius
Lithuania
E-mail: antanas.laurincikas@mif.vu.lt
Received: 16.9.2019.