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THE HYPERSPACE OF TOTALLY DISCONNECTED SETS

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Abstract. In this paper we study the hyperspace of all nonempty closed totally disconnected subsets of a space, equipped with the Vietoris topology. We show results of compactness, connectedness and local connectedness for this hyperspace. We also include a study of path connectedness, particularly we prove that for a smooth dendroid this hyperspace is pathwise connected, and we present a general result which implies that for an Euclidean space this hyperspace has uncountably many arc components.

1. Introduction

In this paper the spaces are T_1, we mean the one-point sets are closed sets. For a space X we denote by 2^X the collection of all nonempty closed subsets of X, equipped with the Vietoris topology, [8, Definition 1.7] and [10, Definition (0.12)]. A subspace of 2^X is called a hyperspace of the space X. A problem treated by many authors is to investigate topological properties of a space through its hyperspaces and vice versa, see references in the classical books in this matter [5] and [10]. Here we begin the study of the hyperspace of closed and totally disconnected subsets of a space X, that we denote by $TD(X)$. After Definitions, in Section 3, we show some basic facts concerning the topological structure of this hyperspace, and we present conditions for compactness, connectedness and local connectedness for it. In Section 4, we prove that for every smooth dendroid X the hyperspace $TD(X)$ is contractible, and we include an example of a (non-smooth) dendroid for which this hyperspace is not pathwise connected. We also prove that for a Hausdorff,

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locally compact, not compact and Lindelöf space X, the hyperspace $TD(X)$ has uncountably many arc components.

2. Definitions

Given a subset A of a space X, the interior and the closure of A in X are denoted by $int(A)$ and $\overline{A}$, respectively. The cardinality of the set A is denoted by $|A|$. We use the symbols $\mathbb{R}$ and $\mathbb{N}$ to denote the set of all real numbers and the set of all positive integers, respectively. Also $|\mathbb{R}|$ es denoted by $\mathfrak{c}$. A *cellular family* in a space X is a family of pairwise disjoint nonempty open subsets of X. The collection of all finite cellular families of a space X is denoted by $\mathfrak{C}(X)$. The dimension of a space X is denoted by $\dim(X)$.

A subset A of a space X is *totally disconnected* provided that no connected subset of A consists of more than one point. We recall that if p is a point of a space X, then X is said to be *locally connected* (*connected im kleinen*) at p provided that for each open subset U of X such that $p \in U$, there exists a connected open subset (a connected subset) V of X such that $p \in V$ ($p \in int(V)$) and $V \subset U$. The space X is said to be *locally connected* provided that it is locally connected at each of its points. It is clear that if a space is locally connected at a point p, then it is connected im kleinen at p; the converse is false (see the example in [11, Figure 5.22, p. 84]). Nevertheless, it is easy to prove that these two notions are globally equivalent, i.e., a space is locally connected if and only if it is connected im kleinen at each of its points.

For a space X, let

$$2^X = \{A \subseteq X : A \text{ is closed in } X \text{ and } A \neq \emptyset\},$$

$$TD(X) = \{A \in 2^X : A \text{ is totally disconnected}\}, \text{ and}$$

$$K(X) = \{A \in 2^X : A \text{ is compact}\}.$$

Given a finite family of open subsets $U_1, \ldots, U_n$ of X, we define $\langle U_1, \ldots, U_n \rangle$ as the subset of 2^X consisting of those elements A such that $A \subseteq U_1 \cup \cdots \cup U_n$ and $U_i \cap A \neq \emptyset$ for each $i \in \{1, \ldots, n\}$. The *Vietoris topology* is the topology in 2^X generated by the base consisting of all sets of the form $\langle U_1, \ldots, U_n \rangle$ where $n \in \mathbb{N}$ and U_i is an open subset of X, see [8, Proposition 2.1, p. 155]. The hyperspace $TD(X)$ will be considered as subspace of 2^X. In particular, a base for the topology of $TD(X)$ consists of all sets of the form $\langle U_1, \ldots, U_n \rangle \cap TD(X)$.

An *arc* is a space homeomorphic to the closed interval $[0, 1]$. A *continuum* is a nondegenerate compact connected metric space. A continuum X is *unicoherent* provided that whenever A and B are closed connected subsets of X such that $X = A \cup B$, then $A \cap B$ is connected; X is *hereditarily unicoherent* if each subcontinuum of X is unicoherent. A *dendroid* is a hereditarily unicoherent and arcwise connected continuum. Given different points a and b in a dendroid X, the unique arc in X with end points a and b is denoted by ab. A dendroid is *smooth* if there exists some point $p \in X$ such that given any sequence $\{a_n\}_{n\in\mathbb{N}}$ in X with $\lim a_n = a$, it holds that $\lim a_n p = ap$.

3. Basic facts, compactness, connectedness and local connectedness

In this section we prove some basic facts about interior, compactness, connectedness, local connectedness and compactness of the hyperspace $TD(X)$.

Lemma 3.1. *If A is a nonempty compact and totally disconnected subset of a Hausdorff space X, then for each finite collection of open subsets of X, $V_1, \dots, V_n$, such that A belongs to $\langle V_1, \dots, V_n\rangle$, there exist a finite collection of pairwise disjoint nonempty closed subsets of X, $A_1, \dots, A_k$, and a finite collection of pairwise disjoint open subsets of X, $U_1, ..., U_k$, satisfying the following conditions:*

(1) $A = A_1 \cup \dots \cup A_k$;
(2) For each $i \in \{1, \dots, k\}$, $A_i \subseteq U_i$;
(3) For each $i \in \{1, \dots, k\}$, there exists $j_i \in \{1, \dots, n\}$ such that $U_i \subseteq V_{j_i}$; and
(4) For each $j \in \{1, \dots, n\}$, there exists $r_j \in \{1, \dots, k\}$ such that $U_{r_j} \subseteq V_j$.

Proof. Suppose that $A \in \langle V_1, \dots, V_n\rangle$ satisfies the hypothesis. Then A is a compact zero-dimensional space (it is well-known that every locally compact totally disconnected Hausdorff space is zero-dimensional and completely regular). Then we can find open subsets $W_1, \dots, W_k$ of X such that for every $i \le k$ there is $j \le n$ such that $\overline{W_i} \subseteq U_j$, $\{A \cap W_i : i \le k\}$ is pairwise disjoint and $A \cap W_i = A \cap \overline{W_i}$ for every $i \le k$. Thus, we define $U_i = W_i - \bigcup\{\overline{W_j} : i \neq j\}$ and $A_i = A \cap \overline{W_i}$ for every $i \le k$. Clearly $\{A_i : A_i \neq \emptyset\}$ and $\{U_i : U_i \neq \emptyset\}$ satisfies the Lemma. □

Next proposition follows from Lemma 3.1, see also [8, Lemma 2.3.1].

Proposition 3.2. *If X is a compact Hausdorff space, then the collection $\{\langle V_1, \dots, V_n\rangle : n \in \mathbb{N}$ and $\{V_1, ..., V_n\} \in \mathfrak{C}(X)\}$ is a base for the Vietoris topology on $TD(X)$.*

Proposition 3.3. *If X is a regular space, then $TD(X)$ has nonempty interior in 2^X if and only if X contains a nonempty open totally disconnected subset.*

Proof. Suppose that $TD(X)$ has nonempty interior in 2^X. Let $U_1, \dots, U_n$ be nonempty open subsets of X such that $\langle U_1, \dots, U_n\rangle \subset TD(X)$. For each $i \in \{1, \dots, n\}$ let x_i be a point in U_i. Let V be an open subset of X such that $x_1 \in V$ and $\overline{V} \subset U_1$. We assert that V is a totally disconnected subset of X. Let A be a nonempty connected subset of V. Let $B = \overline{A} \cup \{x_1, \dots, x_n\}$. Clearly $B \in \langle U_1, \dots, U_n\rangle$, so $B \in TD(X)$. It follows that A is a one-point set.

Conversely, if U is a nonempty open totally disconnected subset of X, then $\langle U \rangle$ is a nonempty open set contained in $TD(X)$. Thus, $TD(X)$ has nonempty interior in 2^X. □

Corollary 3.4. *If X is a compact connected Hausdorff space, then $TD(X)$ has empty interior in 2^X.*

In [8, Theorem 4.2] it is proved that compactness of a subset of the hyperspace 2^X containing the singletons of X implies compactness of the space X. As a consequence of this fact we have the next proposition. We recall that all the spaces that we consider are T_1-spaces.

Proposition 3.5. *If X is a space such that $TD(X)$ is compact, then X is compact.*

Proposition 3.6. *If X is a Hausdorff space such that $TD(X)$ is compact, then X is totally disconnected.*

Proof. By Proposition 3.5, we have that X is compact. Thus, X is a normal space. Consequently 2^X is a Hausdorff space, [8, Theorem 4.9, 4.9.3]. Since $TD(X)$ is compact, we have that $TD(X)$ is a closed subset of 2^X. We notice that each finite subset of X belongs to $TD(X)$, so this hyperspace is a dense subset of 2^X ([8, Proposition 2.4, 2.4.1]). It follows that $TD(X) = 2^X$. Therefore X is an element of $TD(X)$, which means that X is totally disconnected. □

Remark 3.7. The negation of the last proposition says that if X is a Hausdorff space containing a nondegenerate connected subset, then the hyperspace of totally disconnected subsets of X is not compact. In particular, for each nondegenerate continuum this hyperspace is not compact. In this way we also see that the compactness of X does not imply the compactness of $TD(X)$.

Corollary 3.8. *Let X be a Hausdorff space. Then $TD(X)$ is compact if and only if X is compact and totally disconnected.*

Proof. Necessity follows from Propositions 3.5 and 3.6. For the converse, we notice that the assumptions imply that $TD(X)$ coincides with 2^X. Therefore, by [8, Theorem 4.2], we obtain that $TD(X)$ is compact. □

In [8, Theorem 4.10] it is proved that a subset of the hyperspace 2^X containing every finite set of X is connected if and only if the space X is connected. So, we have next proposition.

Proposition 3.9. *For any space X, $TD(X)$ is connected if and only if X is connected.*

In [8, Theorem 4.12] it is proved that, for a compact space X, a subset of the hyperspace 2^X containing every finite set of X is locally connected if and only if the space X is locally connected. Similarly, we have the next proposition.

Proposition 3.10. *Let p be a point of a space X. If $H(X)$ is a subset of 2^X containing the singletons of X and $H(X)$ is connected im kleinen at $\{p\}$, then X is connected im kleinen at p.*

Proof. Let U be an open subset of X such that $p \in U$. By hypothesis, there exists a connected subset $\mathcal{V}$ of $H(X)$ such that $\{p\} \in int_{H(X)}(\mathcal{V})$ and $\mathcal{V} \subseteq \langle U \rangle$. Let $V = \bigcup\{A \in H(X) : A \in \mathcal{V}\}$. We have that V is a connected subset of X, [10, Lemma (1.43)]. Clearly $p \in V$. We assert that $p \in int_X(V)$ and $V \subseteq U$. Let $U_1, \ldots, U_n$ be open subsets of X such that $\{p\} \in \langle U_1, ..., U_n \rangle \cap H(X) \subset int_{H(X)}(\mathcal{V})$ and let $W = U_1 \cap \cdots \cap U_n$. We have that W is an open subset of X containing p. If x is a point in W then $x \in U_i$ for each $i \in \{1, ..., n\}$, so $\{x\} \in \langle U_1, ..., U_n \rangle \cap H(X) \subseteq int_{H(X)}(\mathcal{V})$. Thus, $\{x\} \in \mathcal{V}$. Hence, $x \in V$. Thus we have that W is an open subset of X such that $p \in W \subseteq V$. So, $p \in int_X(V)$. Finally, if x is a point of V there exists $A \in \mathcal{V}$ such that $x \in A$. Since $\mathcal{V} \subseteq \langle U \rangle$ we have that $A \subset U$, so $x \in U$. Thus, V is contained in U. We have that V is a connected subset of X such that $p \in int_X(V)$ and $V \subseteq U$. □

Corollary 3.11. *If X is a space such that $TD(X)$ is locally connected, then X is locally connected.*

Proposition 3.12. *If X is a compact Hausdorff space, then $TD(X)$ is locally connected if and only if X is locally connected.*

Proof. Necessity follows from the more general fact stated in Corollary 3.11. Conversely, let A be a nonempty closed totally disconnected subset of X and let $W_1, \ldots, W_n$ be open subsets of X such that A belongs to $\langle W_1, \ldots, W_n \rangle \cap TD(X)$. By Lemma 3.1 there exist r nonempty compact subsets of A, $A_1, \ldots, A_r$, such that $A = A_1 \cup \cdots \cup A_r$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and they satisfy the following conditions.

(1) For each $i \in \{1, \ldots, r\}$, there exists $j_i \in \{1, \ldots, n\}$ such that $A_i \subseteq W_{j_i}$.
(2) For each $j \in \{1, \ldots, n\}$, there exists $r_j \in \{1, \ldots, r\}$ such that $A_{r_j} \subseteq W_j$.

Let $U_1, \ldots, U_r$ be pairwise disjoint open subsets of X such that $A_i \subseteq U_i$, for each $i \in \{1, \ldots, r\}$ and we define $V_i = U_i \cap (\cap\{W_j : A_i \subseteq W_j\})$, for each $i \in \{1, \ldots, r\}$. If $i \in \{1, ..., r\}$, for each $x \in A_i$ there exists C_x an open connected subset of X such that $x \in C_x$ and $C_x \subseteq V_i$. Since A_i is compact there exists a finite subset, $\{x_{i1}, ..., x_{ik_i}\}$, such that $A_i \subseteq C_{x_{i1}} \cup \cdots \cup C_{x_{ik_i}}$. Let $D_i = C_{x_{i1}} \cup \cdots \cup C_{x_{ik_i}}$. Observe that D_i has only a finite number of components, $M_1^i, ..., M_{p_i}^i$, and every M_j^i is an open subset of X. Let

$$\mathcal{V} = \langle M_1^1, \ldots, M_{p_1}^1, \ldots, M_1^r, \ldots, M_{p_r}^r \rangle \cap TD(X).$$

Clearly $\mathcal{V}$ is an open subset of $TD(X)$. We assert that $\mathcal{V}$ is connected. We know that $\mathcal{V} \cap F_m(X)$ is connected, for each $m \geq p_1 + \cdots + p_r$, [6, Lemma 1], where $F_m(X)$ denotes the hyperspace of all nonempty subsets of X having at most m points. Let $\mathcal{U} = \cup\{\mathcal{V} \cap F_m(X) : m \geq p_1 + \cdots + p_r\}$, we have that $\mathcal{U}$ is a connected subset of $TD(X)$. Now, since $\cup\{F_m(X) : m \geq p_1 + \cdots + p_r\}$ is a dense subset of $TD(X)$ and $\mathcal{U} = \mathcal{V} \cap (\cup\{F_m(X) : m \geq p_1 + \cdots + p_r\})$, we have that $\mathcal{U}$ is dense in $\mathcal{V}$. Thus, $\mathcal{V}$ is connected. We will see that $A \in \mathcal{V}$. Observe that $A = \bigcup_{i=1}^{r} A_i \subseteq \bigcup_{i=1}^{r} (\bigcup_{j=1}^{k_i} C_{x_{ij}}) = \bigcup_{i=1}^{r} (\bigcup_{j=1}^{p_i} M_j^i)$. For each M_j^i, there exists a point $x_{il} \in \{x_{i1}, \ldots, x_{ik_i}\} \subseteq A_i$ such that $C_{x_{il}} \subseteq M_j^i$, so $A_i \cap M_j^i \neq \emptyset$, thus $A \cap M_j^i \neq \emptyset$. This shows that $A \in \mathcal{V}$. Finally we assert that $\mathcal{V} \subseteq \langle W_1, \ldots, W_n \rangle$. Let $B \in \mathcal{V}$. According to condition (2), if $j \in \{1, \ldots, n\}$, there exists $r_j \in \{1, ..., r\}$ such that $A_{r_j} \subseteq W_j$. Since $B \cap (\bigcup_{l=1}^{p_{r_j}} M_l^{r_j}) \neq \emptyset$ and $B \cap (\bigcup_{l=1}^{p_{r_j}} M_l^{r_j}) \subseteq W_j$ we have that $B \cap W_j \neq \emptyset$, so $B \cap W_j \neq \emptyset$, for each $j \in \{1, \ldots, n\}$. Now, if $i \in \{1, ..., r\}$ and $l \in \{1, ..., p_i\}$ then M_l^i is a component of D_i and by definition of D_i we have that $D_i \subseteq V_i$ and $V_i \subseteq W_{j_i}$, for some $j_i \in \{1, ..., n\}$. Therefore $B \in \langle W_1, ..., W_n \rangle$. We have shown that $\mathcal{V}$ is an open connected subset of $TD(X)$ such that $A \in \mathcal{V} \subseteq \langle W_1, ..., W_n \rangle$. □

4. Path connectedness

In this section, we prove that for every smooth dendroid X the hyperspace $TD(X)$ is contractible, so it is pathwise connected, and we include an example of a (non-smooth) dendroid for which this hyperspace is not pathwise connected. We also prove that for a Hausdorff, locally compact, not compact and Lindelöf space X, the hyperspace $TD(X)$ has uncountably many arc components.

Theorem 4.1. *If X is a smooth dendroid, then $TD(X)$ is contractible.*

Proof. Let p be a point of smoothness of X. By [9, Theorem 1.16] there is a continuous function, $H : X \times [0, 1] \to X$, such that for each $x \in X$ and for each $t \in [0, 1]$, H satisfies the following conditions:

(i) $H((x, 0)) = p$;
(ii) $d(H((x, t)), p) = t$, if $t \leq d(x, p)$; and
(iii) $H((x, t)) = x$, if $d(x, p) \leq t$.

We define $\mathcal{H} : TD(X) \times [0, 1] \to 2^X$ by $\mathcal{H}((A, t)) = H(A \times \{t\})$ for each $(A, t) \in TD(X) \times [0, 1]$. We will prove that $\mathcal{H}$ is the required homotopy. Let $B_t = \{x \in X : d(x, p) = t\}$ and $F_t = \{x \in X : d(x, p) \leq t\}$ for each t in the open interval $(0,1)$. Notice that for each $x \in X$, $H((x,t)) \in F_t$. We will prove that each B_t is a totally disconnected set. Fix $t \in (0,1)$. Let $x \in B_t$ and K be the component of x in B_t. Suppose that there exists $z \in K - \{x\}$. Thus K is a nondegenerate subdendroid of X. We have that $xz \subset K$. Since $t > 0$, we have that p is not in B_t, thus p is not in xz. Let w be the first time that the arc px intersects the arc xz. Thus $w \in xz$ and $pw \cap xz = \{w\}$. Notice that $p \neq w$. Without loss of generality, assume that $z \neq w$. The fact that $p, z \in H(\{z\} \times [0,t])$ ($z \in F_t$) and $H(\{z\} \times [0,t])$ is connected implies that $w \in H(\{z\} \times [0,t])$ ($pz = pw \cup wz$). There exists $r \in [0,t]$ such that $H((z,r)) = w$. If $r = t$, then $w = H((z,t)) = z$ ($z \in B_t \subseteq F_t$) and hence $w = z$ but this is a contradiction. Therefore $r \in [0,t)$ and $H((z,r)) = w$, i.e. $w \in F_r$ with $r < t$, this implies that $w \notin B_t$ but this contradicts the fact that $w \in xz \subseteq K \subseteq B_t$. Hence, we have proved that B_t is totally disconnected.

On the other hand, it is easy to see that $\mathcal{H}$ is a continuous function. Now by conditions (i) and (iii) we have that $\mathcal{H}((A,0)) = \{p\}$ and $\mathcal{H}((A,1)) = A$ for each $A \in TD(X)$ (we are assuming that $maximum\{d(x,y) : x, y \in X\} = 1$). We will prove that $\mathcal{H}((A,t)) \in TD(X)$ for each $t \in (0,1)$ and each $A \in TD(X)$. Let $t \in (0,1)$ and $A \in TD(X)$. Since for each $x \in X$, $H((x,t)) \in F_t$, we have that $H(A \times \{t\}) \subseteq F_t$. Hence according to conditions (ii) and (iii), $H(A \times \{t\}) \subseteq (A \cap F_t) \cup B_t$. In order to prove that $\mathcal{H}((A,t))$ is a totally disconnected set, we will prove that $(A \cap F_t) \cup B_t$ is totally disconnected. Let C be a component of $(A \cap F_t) \cup B_t$. Suppose that $|C| \geq 2$. Since B_t is totally disconnected we have that $C \cap ((A \cap F_t) - B_t) \neq \emptyset$. Let $a \in C \cap (A \cap F_t - B_t)$. By [10, (1.8) and (1.11)], there is an order arc, $\gamma : [0,1] \to \{E \in 2^X : E \text{ is connected}\}$, from $\{a\}$ to C. Since $\gamma(0) = \{a\} \in \langle X - B_t \rangle$ there is $r > 0$ such that $\gamma(r) \in \langle X - B_t \rangle$. We have that $\gamma(r) \subseteq A \cap F_t$ and $\gamma(r)$ is a nondegenerate connected subset of $A \cap F_t$, this contradicts that $A \cap F_t$ is totally disconnected. Thus $|C| \leq 1$. Hence, we have proved that $\mathcal{H}((A,t))$ is totally disconnected. Therefore $TD(X)$ is contractible. □

Corollary 4.2. *If X is a smooth dendroid, then $TD(X)$ is pathwise connected.*

Corollary 4.3. *If X is a dendrite, then $TD(X)$ is pathwise connected.*

Problem 4.1. *Is it true that if X is a contractible dendroid, then $TD(X)$ is contractible?*

Problem 4.2. *Give conditions on X under which $TD(X)$ is contractible.*

Next we give an example of a (non-smooth) dendroid X whose hyperspace $TD(X)$ is not pathwise connected. In order to do this we include the following lemma that can be proved in the same way as Lemma 2.7 of [3].

Lemma 4.4. *Let X be a regular space. If $f : [0,1] \to TD(X)$ is a continuous function, C is a nonempty closed subset of $\bigcup f([0,1])$ and $u = inf\{s \in [0,1] : f(s) \cap C \neq \emptyset\}$, then $f(u) \cap C \neq \emptyset$.*

Also we recall that a continuum X is *uniformly arcwise connected* if it is arcwise connected and for each number $\varepsilon > 0$ there is a positive integer k such that every arc A contained in X contains points $a_0, a_1, \ldots, a_k$ such that $A = a_0a_1 \cup \cdots \cup a_{k-1}a_k$ and $\operatorname{diam}(a_ia_{i+1}) < \varepsilon$ for each $i \in \{0, \ldots, k-1\}$, [1, D5].

Example 4.5. A dendroid X such that $TD(X)$ is not pathwise connected.

Let L_0 be the line segment in the plain from the point $(0, 0)$ to the point $(1, 0)$. For each $n \in \mathbb{N}$, let $x_n = (1, \frac{1}{n})$, $y_{in} = (1, \frac{1}{n} - \frac{i}{2n^2(n+1)})$ with $i \in \{1, \ldots, n\}$ and $z_{in} = (\frac{3}{4}, \frac{\frac{3}{4}}{n} - (\frac{3}{4})\frac{i}{2n^2(n+1)})$ for each $i \in \{1, \ldots, n\}$. For each $n \in \mathbb{N}$ let H_n be the line segment of the point x_n to the point z_{1n}, I_n the line segment of the point $(0, 0)$ to the point x_n, J_{in} the line segment of the point z_{in} to the point y_{in} for each $i \in \{1, \ldots, n\}$ and K_{in} the line segment of the point y_{in} to the point $z_{(i+1)n}$ for each $i \in \{1, \ldots, n-1\}$. For each $n \in \mathbb{N}$, let $L_n = H_n \cup I_n \cup (\bigcup\{J_{in} : i \in \{1, \ldots, n\}\}) \cup (\bigcup\{K_{in} : i \in \{1, \ldots, n-1\}\})$ and $X = L_0 \cup (\bigcup\{L_n : n \in \mathbb{N}\})$, see Figure 1. We will prove that $TD(X)$ is not pathwise connected.

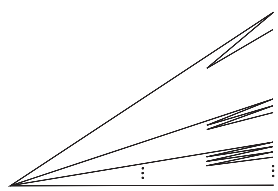

Figure 1. A dendroid with non-pathwise connected hyperspace $TD(X)$

Let $S_0 = \{(0, 0)\}$ and $S_1 = \{x_n : n \in \mathbb{N}\} \bigcup \{(1, 0)\} \bigcup \{y_{in} : n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}\}$. Suppose there is a path $f : [0, 1] \to TD(X)$ such that $f(0) = S_0$ and $f(1) = S_1$. We notice that S_0 is contained in a uniformly arcwise connected subcontinuum of X and S_1 is not. Let r be the least upper bound of the set of all numbers $s \in [0, 1]$ such that $f(t)$ is contained in a uniformly arcwise connected subcontinuum of X for each $t \leq s$. By Proposition 3.2 we can choose pairwise disjoint open sets $U_1, \ldots, U_k$ of X such that $\operatorname{diam}(U_i) < \frac{1}{8}$, for each $i \in \{1, \ldots, k\}$ and $f(r) \in \mathcal{U} = \langle U_1, \ldots, U_k \rangle$. We have to consider two cases:

Case 1. $f(r)$ is contained in a uniformly arcwise connected subcontinuum of X. Clearly $0 \leq r < 1$. Let $t_0 \in (r, 1]$ such that $f([r, t_0]) \subseteq \mathcal{U}$ and $f(t_0)$ is not contained in a uniformly arcwise connected subcontinuum of X. We have that for each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ with $N > n$ such that $f(t_0) \cap J_{iN} \neq \emptyset$ or $f(t_0) \cap K_{iN} \neq \emptyset$ for some $i > N$. Then $f(t_0)$ has a limit point in the line segment from the point $(\frac{3}{4}, 0)$ to the point $(1, 0)$. Let x_0 be a point limit of $f(t_0)$ in the line segment from the point $(\frac{3}{4},0)$ to the point $(1,0)$. Without loss of generality assume that $x_0 \in U_1$. Since $f(r)$ is contained in a uniformly arcwise connected subcontinuum of X there exists $m \in \mathbb{N}$ such that $f(r) \cap J_{in} = \emptyset$ for each $n \geq m$ and for each $i \in \{m, \ldots, n\}$; and $f(r) \cap K_{in} = \emptyset$ for each $n \geq m$ and for each $i \in \{m, \ldots, n-1\}$. Now choose a connected component C of U_1 such that $f(t_0) \cap C \cap (J_{jN} \cup K_{lM}) \neq \emptyset$ with $N, M > 2m$, $j \in \{2m, \ldots, N\}$ and $l \in \{2m, \ldots, M\}$. We notice that $f(r) \cap C = \emptyset$. Let $D = C \cap [\bigcup f([r, t_0])]$. Observe that D is a closed subset of $\bigcup f([r, t_0])$. Since $f_{|_{[r,t_0]}}$ and D satisfy the conditions of Lemma 4.4, $f(u) \cap D \neq \emptyset$ where $u = \inf\{s \in [r, t_0] : f(s) \cap D \neq \emptyset\}$. Since $f(r) \cap C = \emptyset$ and $D \subseteq C$ we have that $r < u$. Consider the open sets C and $\mathcal{V} = \langle C, U_1, \ldots, U_m \rangle$. Since $f(u) \cap D \neq \emptyset$ and $f(u) \in f([r, t_0]) \subseteq \mathcal{U}$ we have that $f(u) \in \mathcal{V}$. Then $f^{-1}(\mathcal{V})$ is an open subset of $[0,1]$ such that $u \in f^{-1}(\mathcal{V})$ and $f^{-1}(\mathcal{V}) \cap [r, u) = \emptyset$, which contradicts the continuity of f.

Case 2. $f(r)$ is not contained in a uniformly arcwise connected subcontinuum of X. In this case, we notice that $0 < r$ and for each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ with $N > n$ such that either $f(r) \cap J_{iN} \neq \emptyset$ or $f(r) \cap K_{iN} \neq \emptyset$ for some $i > N$. We have that $f(r)$ has a limit point in the line segment from the point $(\frac{3}{4},0)$ to the point $(1,0)$. Let x_0 be a limit point of $f(r)$ in the line segment from the point $(\frac{3}{4},0)$ to the point $(1,0)$. Without loss of generality assume that $x_0 \in U_1$. Let t_0 be a point in $[0, r)$ such that $f([t_0, r]) \subseteq \mathcal{U}$. Observe that $f(t_0)$ is contained in a uniformly arcwise connected subcontinuum of X. Then, we may take $m \in \mathbb{N}$ such that $f(t_0) \cap J_{in} = \emptyset$ for each $n \geq m$ and for each $i \in \{m, \ldots, n\}$; and $f(t_0) \cap K_{in} = \emptyset$ for each $n \geq m$ and for each $i \in \{m, \ldots, n-1\}$. We choose a connected component C of U_1 such that $f(r) \cap C \cap (J_{jN} \cup K_{lM}) \neq \emptyset$ with $N, M > 2m$, $j \in \{2m, ..., N\}$ and $l \in \{2m, ..., M\}$. We have that $f(t_0) \cap C = \emptyset$. Let $D = C \cap [\bigcup f([t_0, r])]$. Observe that D is a closed subset of $\bigcup f([t_0, r])$. By Lemma 4.4, if $u = \inf\{s \in [t_0, r] : f(s) \cap D \neq \emptyset\}$, then $f(u) \cap D \neq \emptyset$. Since $f(t_0) \cap C = \emptyset$ and $D \subseteq C$ we have that $t_0 < u$. Consider the open sets C and $\mathcal{V} = \langle C, U_1, ..., U_m \rangle$. Since $f(u) \cap D \neq \emptyset$ and $f(u) \in f([t_0, r]) \subseteq \mathcal{U}$ we have that $f(u) \in \mathcal{V}$. Then $f^{-1}(\mathcal{V})$ is an open subset of $[0,1]$ such that $u \in f^{-1}(\mathcal{V})$ and $f^{-1}(\mathcal{V}) \cap [t_0, u) = \emptyset$, but this also contradicts the continuity of f. Therefore S_0 and S_1 cannot be connected by a path in $TD(X)$. This proves that for the dendroid in Figure 1 the hyperspace of totally disconnected sets is not pathwise connected.

Note that the dendroid in the previous example is not uniformly arcwise connected and that smooth dendroids are uniformly arcwise connected, see [2, Corollary 16]. In this context it is interesting to ask:

Problem 4.3. *Is $TD(X)$ pathwise connected for every uniformly arcwise connected continuum X?*

PROBLEM 4.4. *Let X be a continuum that is not uniformly arcwise connected. Is it true that $TD(X)$ has $\mathfrak{c}$ arc components?*

Next we will present a class of spaces, that include the Euclidean spaces $\mathbb{R}^n$, for which the hyperspace of totally disconnected subsets has at least $\mathfrak{c}$ path components, see Theorem 4.11 below. First we state several technical lemmas.

LEMMA 4.6. *Let X be a normal space such that $dim(X) = 0$. If $Y = \{y_n : n \in \mathbb{N}\}$ is a discrete, infinite and closed subset of X and $\{Q_n : n \in \mathbb{N}\}$ is a cellular family of closed-open subsets of X such that $y_n \in Q_n$ for each $n \in \mathbb{N}$, then there exists a cellular family of closed-open subsets of X, $\{V_n : n \in \mathbb{N}\}$, such that $y_n \in V_n$ for each $n \in \mathbb{N}$ and $\bigcup\{V_n : n \in \mathbb{N}\} \in 2^X$. Moreover, if $\{P_n : n \in \mathbb{N}\} \subseteq 2^X$ satisfies that $P_n \subseteq V_n$ for each $n \in \mathbb{N}$, then $\bigcup\{P_n : n \in \mathbb{N}\} \in 2^X$.*

PROOF. Let $L = \overline{\bigcup\{Q_n : n \in \mathbb{N}\}} - \bigcup\{Q_n : n \in \mathbb{N}\}$. If $L = \emptyset$, then the Lemma is true. Suppose that L is not empty. Since X is a normal space, there exists an open subset of X, W_0 such that $Y \subseteq W_0 \subseteq \overline{W_0} \subseteq X - L$. Observe that for each $n \in \mathbb{N}, y_n \in Q_n \cap W_0$ and $Q_n \cap W_0$ is an open subset of X. Since $\dim(X) = 0$, for each $n \in \mathbb{N}$, there exists a closed-open subset of X, V_n, such that $y_n \in V_n \subseteq Q_n \cap W_0$. Clearly $\{V_n : n \in \mathbb{N}\}$ is a cellular family in X. We assert that $\bigcup\{V_n : n \in \mathbb{N}\} \in 2^X$. Since $V_n \subseteq Q_n \cap W_0$, for each $n \in \mathbb{N}$, we have that $\bigcup\{V_n : n \in \mathbb{N}\} \subseteq (\bigcup\{Q_n : n \in \mathbb{N}\}) \cap W_0$, consequently $\overline{\bigcup\{V_n : n \in \mathbb{N}\}} \subseteq \overline{\bigcup\{Q_n : n \in \mathbb{N}\}} \cap \overline{W_0} \subseteq \overline{\bigcup\{Q_n : n \in \mathbb{N}\}} \cap (X - L)$. So $\overline{\bigcup\{V_n : n \in \mathbb{N}\}} \subseteq \bigcup\{Q_n : n \in \mathbb{N}\}$. Now, we take a point x in $\overline{\bigcup\{V_n : n \in \mathbb{N}\}}$. We have that $x \in Q_n$, for some $n \in \mathbb{N}$. If $x \in Q_n - V_n$, then $(Q_n - V_n) \cap (\bigcup\{V_n : n \in \mathbb{N}\}) = \emptyset$ ($Q_n \cap Q_m = \emptyset$, if $n \neq m$), which is a contradiction. Therefore, $x \in V_n$. This proves that $\bigcup\{V_n : n \in \mathbb{N}\} \in 2^X$. Similarly one can prove that if $\{P_n : n \in \mathbb{N}\} \subseteq 2^X$ satisfies that $P_n \subseteq V_n$ for each $n \in \mathbb{N}$, then $\bigcup\{P_n : n \in \mathbb{N}\} \in 2^X$. □

LEMMA 4.7. *Let X be a normal space. If $\{P_n : n \in \mathbb{N}\}$ is a family of subsets of X such that $P_n \cap P_m = \emptyset$ if $n \neq m$, $\bigcup\{P_n : n \in \mathbb{N}\} \in 2^X$ and P_n is closed-open subset of $\bigcup\{P_n : n \in \mathbb{N}\}$ for each $n \in \mathbb{N}$, then there exists a cellular family, $\{U_n : n \in \mathbb{N}\}$, such that $P_n \subseteq U_n$ for each $n \in \mathbb{N}$.*

PROOF. Observe that for each $m \in \mathbb{N}$, P_m and $\bigcup\{P_n : n \in \mathbb{N} - \{m\}\}$ are closed subsets of $\bigcup\{P_n : n \in \mathbb{N}\}$, so P_m and $\bigcup\{P_n : n \in \mathbb{N} - \{m\}\}$ are closed disjoint subsets of X. Since X is a normal space, for each $m \in \mathbb{N}$, there exists an open subset of X, V_m such that $P_m \subseteq V_m$ and $\overline{V_m} \cap \bigcup\{P_n : n \in \mathbb{N} - \{m\}\} = \emptyset$. Let $U_1 = V_1$ and $U_m = V_m - (\overline{V_1} \cup \cdots \cup \overline{V_{m-1}})$. Clearly $\{U_m : m \in \mathbb{N}\}$ satisfies the Lemma. □

LEMMA 4.8. *Let X be a locally compact and normal space and let $E, B \in TD(X)$. If Y is a discrete, countably infinite and closed subset of $\overline{E - B}$,* *then there exists an open subset of X, V, and for each $n \in \mathbb{N}$ there exists an open subset of X, U_n, satisfying the following conditions:*

(1) For each $n \in \mathbb{N}$ we have that $V \cap U_n = \emptyset$ and $U_n \cap U_m = \emptyset$, if $n \neq m$.
(2) $B \subseteq V$; and for each $n \in \mathbb{N}$, $U_n \cap E \neq \emptyset$.
(3) $E \subseteq V \cup (\bigcup\{U_n : n \in \mathbb{N}\})$.

PROOF. Let $Y = \{y_n : n \in \mathbb{N}\}$. We know that for each $n \in \mathbb{N}$ there exists an open subset of X, W_n, such that $y_n \in W_n$ and $W_n \cap W_m = \emptyset$ if $n \neq m$. We also know that E has dimension zero, consequently, for each $n \in \mathbb{N}$ there exists a closed-open subset of E, O_n, such that $y_n \in O_n \subseteq W_n$. Now, by Lemma 4.6, there exists a cellular family of closed-open subsets of E, $\{V_n : n \in \mathbb{N}\}$, such that $y_n \in V_n$ for each $n \in \mathbb{N}$ and $\bigcup\{V_n : n \in \mathbb{N}\} \in 2^E$. Observe that $\bigcup\{V_n : n \in \mathbb{N}\} \in 2^X$. Since for each $n \in \mathbb{N}$, $y_n \in \overline{E - B}$ and $y_n \in V_n$ there exists a point x_n in $V_n \cap (E - B) = V_n - B$. Since $\dim(E) = 0$ we have that for each $n \in \mathbb{N}$ there exists a closed-open subset of E, P_n, such that $x_n \in P_n \subseteq V_n - B$. Now, by Lemma 4.6 we may assume $\bigcup\{P_n : n \in \mathbb{N}\} \in 2^E$ so $\bigcup\{P_n : n \in \mathbb{N}\} \in 2^X$. Observe that $\{P_n : n \in \mathbb{N}\}$ satisfies the assumptions of the Lemma 4.7, consequently for each $n \in \mathbb{N}$ there exists an open subset of X, Q_n, such that $P_n \subseteq Q_n$ and $Q_n \cap Q_m = \emptyset$ if $n \neq m$. On the other hand, observe that $\bigcup\{P_n : n \in \mathbb{N}\} \subseteq E - B$. Thus $B \cup (E - \bigcup\{P_n : n \in \mathbb{N}\})$ and $\bigcup\{P_n : n \in \mathbb{N}\}$ are disjoint closed subsets of X. Since X is a normal space there exist disjoint open subsets of X, V and W, such that $B \cup (E - \bigcup\{P_n : n \in \mathbb{N}\}) \subseteq V$ and $\bigcup\{P_n : n \in \mathbb{N}\} \subseteq W$. For each $n \in \mathbb{N}$ let $U_n = Q_n \cap W$. Clearly $\{V\} \cup \{U_n : n \in \mathbb{N}\}$ satisfies conditions (1) and (3). Finally in order to see condition (2), observe that for each $n \in \mathbb{N}$, $x_n \in P_n \subseteq Q_n \cap W \cap E = U_n \cap E$. ☐

LEMMA 4.9. *Let X be a space and let $\mathcal{A}$ be a connected subset of 2^X. If K is a nonempty open and closed subset of $\bigcup \mathcal{A}$, then $A \cap K \neq \emptyset$ for each $A \in \mathcal{A}$.*

PROOF. Assume that there exists $A \in \mathcal{A}$ such that $A \cap K = \emptyset$. Since K is an open subset of $\bigcup \mathcal{A}$, there exists an open subset U of X such that $K = U \cap (\bigcup \mathcal{A})$. We have that $\langle K, X\rangle \cap \mathcal{A} = \langle U, X\rangle \cap \mathcal{A}$, which is an open subset of $\mathcal{A}$. Now, since K is a closed subset of $\bigcup \mathcal{A}$ we have that $(\bigcup \mathcal{A}) - K$ is an open subset of $\bigcup \mathcal{A}$, consequently there exists an open subset V of X such that $(\bigcup \mathcal{A}) - K = V \cap (\bigcup \mathcal{A})$. We have that $\langle (\bigcup \mathcal{A}) - K\rangle \cap \mathcal{A} = \langle V\rangle \cap \mathcal{A}$, which is an open subset of $\mathcal{A}$. Note that $\mathcal{A} = [\langle (\bigcup \mathcal{A}) - K\rangle \cap \mathcal{A}] \cup [\langle K, X\rangle \cap \mathcal{A}]$ and that $A \in \langle \bigcup(\mathcal{A}) - K\rangle \cap \mathcal{A}$. Also, since $K \neq \emptyset$ and $K \subseteq \bigcup \mathcal{A}$ we have that $\langle K, X\rangle \cap \mathcal{A} \neq \emptyset$. Observe that $[\langle (\bigcup \mathcal{A}) - K\rangle \cap \mathcal{A}] \cap [\langle K, X\rangle \cap \mathcal{A}] = \emptyset$. Therefore $\mathcal{A}$ is not connected. ☐

LEMMA 4.10. *Let X be a locally compact and Hausdorff space. Let K be a compact subset of X, $P \in 2^X$ and $B \subseteq X$ such that $P - K \subseteq B$. Suppose that $P - K = \{c_n : n \in \mathbb{N}\} \in 2^X$ is discrete and $c_n \neq c_m$ for each $n \neq m$. If* *there exists* $\{P_k : k \in \mathbb{N}\} \subseteq 2^X$ *such that* $P_k \to P$ *and* $\overline{P_k - B}$ *is not compact for each* $k \in \mathbb{N}$, *then there exist an open set* V, $M \in \mathbb{N}$, *and for each* $m \in \mathbb{N}$, *an open set* U_m *and* $n_m \in \mathbb{N}$ *such that:*

(1) *The elements of* $\{U_m : m \in \mathbb{N}\} \cup \{V\}$ *are pairwise disjoint.*
(2) $K \subseteq V$ *and* $c_m \in U_m$, *for each* $m \in \mathbb{N}$.
(3) $n_m < n_{m+1}$, *for each* $m \in \mathbb{N}$.
(4) *For each* $m \in \mathbb{N}$, *there exists* $x_m \in P_{M+m} \cap (U_{n_m} - \{c_{n_m}\})$.

Proof. There exists an open subset V of X such that $K \subseteq V \subseteq \overline{V} \subseteq X - (P - K)$ and $\overline{V}$ is compact. We know that for each $m \in \mathbb{N}$, there exists an open subset V_m of X such that $c_m \in V_m$ and $V_n \cap V_m = \emptyset$, if $n \neq m$. Now, for each $m \in \mathbb{N}$, there exist open subsets W_m and W'_m of X such that $c_m \in W_m$, $\overline{V} \subseteq W'_m$ and $W_m \cap W'_m = \emptyset$. For each $m \in \mathbb{N}$, we define $U'_m = W_m \cap V_m$. For each $m \in \mathbb{N}$ there exists an open subset U_m of X such that $c_m \in U_m \subseteq \overline{U_m} \subseteq U'_m$ and $\overline{U_m}$ is compact. It is easy to verify that $\{U_m : m \in \mathbb{N}\} \cup \{V\}$ satisfies conditions (1) and (2). In order to prove (3) and (4) let $U = V \cup (\bigcup\{U_m : m \in \mathbb{N}\})$. Observe that $P \in \langle U \rangle$. Since $P_k \to P$, there exists $M \in \mathbb{N}$ such that $P_k \in \langle U \rangle$, for each $k > M$. Suppose that $\overline{P_{M+1} - B} \subseteq \overline{V}$. Since $\overline{V}$ is compact, we have that $\overline{P_{M+1} - B}$ is compact, which is a contradiction. Then, there is $y_1 \in \overline{P_{M+1} - B} - \overline{V} \subseteq P_{M+1} - V$. Let $n_1 \in \mathbb{N}$ such that $y_1 \in U_{n_1}$. Hence, there is $x_1 \in (P_{M+1} - B) \cap U_{n_1}$. Observe that $x_1 \neq c_{n_1}$ because $c_{n_1} \in P - K \subseteq B$, thus $x_1 \in (P_{M+1} - B) \cap (U_{n_1} - \{c_{n_1}\}) \subseteq P_{M+1} \cap (U_{n_1} - \{c_{n_1}\})$. Suppose that there are $n_1 < \cdots < n_j$ and points $x_1, \ldots, x_j \in X$ such that $x_i \in P_{M+i} \cap (U_{n_i} - \{c_{n_i}\})$ for each $i \in \{1, \ldots, j\}$. Then, since $\overline{P_{M+j+1} - B} \subseteq P_{M+j+1}$ is not compact and $\overline{V} \cup \overline{U_{n_1}} \cup \cdots \cup \overline{U_{n_j}}$ is compact, there exists $y_{j+1} \in \overline{P_{M+j+1} - B} - (\overline{V} \cup \overline{U_{n_1}} \cup \cdots \cup \overline{U_{n_j}})$. Then, there is $n_{j+1} \in \mathbb{N}$ such that $y_{j+1} \in U_{n_{j+1}}$. Let $x_{j+1} \in (P_{M+j+1} - B) \cap (U_{n_{j+1}} - \{c_{n_{j+1}}\}) \subseteq P_{M+j+1} \cap (U_{n_{j+1}} - \{c_{n_{j+1}}\})$. □

Theorem 4.11. *Let* X *be a locally compact, Hausdorff and Lindelöf space. If* C *is a closed, discrete and countably infinite subset of* X, *and* $A, B \in TD(X)$ *are subsets of* C *such that* $A - B$ *is infinite, then there is no path from* A *to* B *in* $TD(X)$.

Proof. Suppose that there exists a path in $TD(X)$ such that $\alpha(0) = B$ and $\alpha(1) = A$. Let $T = \{t \in [0,1] : \overline{\alpha(t) - B}$ is compact $\}$ and $t_0 = \sup T$. Observe that $0 \in T$ and $1 \notin T$. Suppose first that $t_0 \notin T$, i.e. $\overline{\alpha(t_0) - B}$ is not compact, so it is not countably compact. Then there exists a subset Y of $\overline{\alpha(t_0) - B}$ such that Y is closed, discrete and countably infinite. By Lemma 4.8 there exists an open subset, V, of X and for each $n \in \mathbb{N}$ there exists an open subset, U_n, of X such that:

(1) For each $n \in \mathbb{N}$, $U_n \cap V = \emptyset$ and $U_n \cap U_m = \emptyset$ if $n \neq m$.
(2) For each $n \in \mathbb{N}$, $U_n \cap \alpha(t_0) \neq \emptyset$.
(3) $\alpha(t_0) \subseteq V \cup (\bigcup\{U_n : n \in \mathbb{N}\})$ and $B \subseteq V$.

Let $U = V \cup (\bigcup\{U_n : n \in \mathbb{N}\})$. Observe that $\alpha(t_0) \in \langle U \rangle$ and $t_0 > 0$. Since α is continuous there exists $\delta > 0$ such that $\alpha([t_0 - \delta, t_0]) \subseteq \langle U \rangle$. Let $t_1 \in [t_0 - \delta, t_0] \cap T$. Since $\overline{\alpha(t_1) - B}$ is compact there exists a finite subset F of $\mathbb{N}$ such that $\overline{\alpha(t_1) - B} \subseteq V \cup (\bigcup\{U_n : n \in F\})$. Observe that $\alpha([t_0 - \delta, t_0]) \subseteq 2^X$ is connected. Let $N \in \mathbb{N} - F$. We have that $U_N \cap (\bigcup \alpha([t_0 - \delta, t_0]))$ is a nonempty, open and closed subset of $\bigcup \alpha([t_0 - \delta, t_0])$. Now, by Lemma 4.9, $\alpha(t) \cap U_N \neq \emptyset$, for each $t \in [t_0 - \delta, t_0]$, in particular $\alpha(t_1) \cap U_N \neq \emptyset$. Since $B \subseteq V$, $U_N \cap V = \emptyset$ and $\alpha(t_1) \subseteq U$ we have that $(\overline{\alpha(t_1) - B}) \cap U_N \neq \emptyset$, a contradiction. This proves that $t_0 \in T$. Now, since $1 \notin T$, we have that $t_0 < 1$. We assert that $\alpha(t_0)$ is not compact. Suppose that $\alpha(t_0)$ is compact and let U be an open subset of X such that $\alpha(t_0) \in \langle U \rangle$. Since X is locally compact there exists an open subset V of X such that $\alpha(t_0) \subseteq V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact. There exists $\varepsilon > 0$ such that $\alpha((t_0, t_0 + \varepsilon)) \subseteq \langle V \rangle$. Then, for each $t \in (t_0, t_0 + \varepsilon)$ we have that $\alpha(t) \subseteq V \subseteq \overline{V}$. Since $\overline{V}$ is compact, $\alpha(t)$ is compact too, for each $t \in (t_0, t_0 + \varepsilon)$. Now $\overline{\alpha(t) - B}$ is a closed subset of $\alpha(t)$, so $\overline{\alpha(t) - B}$ is compact, for each $t \in (t_0, t_0 + \varepsilon)$, a contradiction. This proves that $\alpha(t_0)$ is not compact, consequently $\alpha(t_0)$ is infinite. We define $K = \overline{\alpha(t_0) - B}$. Observe that $\alpha(t_0) - K \subseteq B$, hence it is closed, discrete and countably infinite. Assume that $\alpha(t_0) - K = \{c_n : n \in \mathbb{N} \text{ and } c_n \neq c_m \text{ if } n \neq m\}$. Now, take a sequence $\{t_k\}_{k \in \mathbb{N}}$ of points in $[0, 1] - T$ such that $\{t_k\}_{k \in \mathbb{N}}$ converges to t_0. Since α is continuous, we have that $\{\alpha(t_k)\}_{k \in \mathbb{N}}$ converges to $\alpha(t_0)$. Observe that $\overline{\alpha(t_k) - B}$ is not compact, for each $k \in \mathbb{N}$. Then K, $\alpha(t_0)$ and B satisfy the conditions of Lemma 4.10. Thus there is an open subset of X, V, and there exists $M \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ there exists an open subset of X, U_m, and there exists $n_m \in \mathbb{N}$ such that:

(1) The elements of $\{U_m : m \in \mathbb{N}\} \cup \{V\}$ are pairwise disjoint.
(2) $K \subseteq V$ and $c_m \in U_m$, for each $m \in \mathbb{N}$.
(3) $n_m < n_{m+1}$, for each $m \in \mathbb{N}$.
(4) For each $m \in \mathbb{N}$, there exists $x_m \in \alpha(t_{M+m}) \cap (U_{n_m} - \{c_{n_m}\})$.

For each $m \in \mathbb{N}$, we define

$$V_m = \begin{cases} U_m, & \text{if } m \neq n_k, \text{ for each } k \in \mathbb{N} \\ U_{n_k} - \{x_k\}, & \text{if } m = n_k, \text{ for some } k \in \mathbb{N}. \end{cases}$$

Let $W = V \cup (\bigcup\{V_m : m \in \mathbb{N}\})$. Clearly $\alpha(t_0) \in \langle W \rangle$. Since $\{\alpha(t_k)\}_{k \in \mathbb{N}}$ converges to $\alpha(t_0)$, there exists $N \in \mathbb{N}$ such that $\alpha(t_{M+N}) \in \langle W \rangle$. Observe that $x_N \in \alpha(t_{M+N}) \cap (U_{n_N} - \{c_{n_N}\})$. The fact that $U_r \cap U_s = \emptyset$, if $r \neq s$, implies that $x_N \in V_{n_N} = U_{n_N} - \{x_N\}$, a contradiction. Hence, there is no path from A to B in $TD(X)$. ∎

Proposition 4.12. *We define a relation $\sim$ on $\mathcal{P}(\mathbb{N})$ as follows: If $A, B \subseteq \mathbb{N}$, we set $A \sim B$ if and only if there exists $N \in \mathbb{N}$ such that $A \cap [N, \infty) = B \cap [N, \infty)$. The following conditions hold:*

(1) *The relation* $\sim$ *is an equivalence relation.*
(2) *If* $A, B \subseteq \mathbb{N}$ *are finite, then* $A \sim B$.
(3) *The equivalence class* $[A]_\sim$ *is countably infinite, for each* $A \in \mathcal{P}(\mathbb{N})$.
(4) $\sim$ *has* $\mathfrak{c}$ *equivalence classes.*

Theorem 4.13. *If* X *is a locally compact, not compact, Hausdorff and Lindelöf space, then* $TD(X)$ *has at least* $\mathfrak{c}$ *path components.*

Proof. Since X is a Lindelöf space but it is not compact, we have that X contains a discrete, countably infinite and closed subset, C. Assume that $C = \{x_n : n \in \mathbb{N}\}$ and let $N, M \subseteq \mathbb{N}$ such that N is not related to M in the sense of Proposition 4.12. We define $A = \{x_n : n \in M\}$ and $B = \{x_n : n \in N\}$. Without loss of generality assume that $A - B$ is infinite. By Theorem 4.11 we have that there is no path in $TD(X)$ from A to B. Since this happens whenever $M, N \subseteq \mathbb{N}$ are such that $[M] \neq [N]$, from Proposition 4.12, $\sim$ has $\mathfrak{c}$ equivalence classes. Consequently $TD(X)$ has at least $\mathfrak{c}$ path components. □

Corollary 4.14. *If* X *is locally compact, not compact, Hausdorff, second countable space, then* $TD(X)$ *has exactly* $\mathfrak{c}$ *path components.*

Proof. Since X has a base of cardinality ω, then X has at most 2^ω open sets. In other words, X has at most $\mathfrak{c}$ closed set. Thus $|TD(X)| \leq \mathfrak{c}$. The result follows from Theorem 4.13. □

Corollary 4.15. *If* X *is a noncompact manifold, then* $TD(X)$ *has exactly* $\mathfrak{c}$ *path components.*

Corollary 4.16. *Let* X *be a locally compact, Hausdorff and Lindelöf space. If* $TD(X)$ *is pathwise connected, then* X *is compact.*

We note that the converse of Corollary 4.16 is not true: for this it is enough to consider the dendroid of Example 4.5.

Theorem 4.17. *Let* X *be a locally compact, Lindelöf and Hausdorff space. If* L *and* M *can be connected by a path in* $TD(X)$*, then* L *is compact if and only if* M *is compact.*

Proof. Let $L, M \in TD(X)$ and let $\alpha : [0,1] \to TD(X)$ be a continuous function such that $\alpha(0) = L$ and $\alpha(1) = M$. It is enough to prove that if L is compact, then M is compact. Suppose that L is compact and M is not compact. Let $r = \sup\{t \in [0,1] : \alpha(t) \text{ is compact}\}$. We assert that $\alpha(r)$ is not compact. If $r = 1$, then $\alpha(r)$ is not compact. Now, suppose that $r < 1$ and $\alpha(r)$ is compact. Since X is locally compact, there exists an open subset, V, of X such that $\alpha(r) \subseteq V \subseteq \overline{V} \subseteq X$ and $\overline{V}$ is compact. Since α is continuous, there exists $\varepsilon > 0$ such that $\alpha((r, r+\varepsilon)) \subseteq \langle V \rangle$. Consequently $\alpha(t)$ is compact, for each $t \in (r, r+\varepsilon)$, a contradiction. Therefore $\alpha(r)$ is not compact and so $r > 0$. On the other hand, since X is a Lindelöf space and

$\alpha(r)$ is a closed subset of X, we have that $\alpha(r)$ is a Lindelöf space. So $\alpha(r)$ is not countably compact, consequently $\alpha(r)$ contains a closed, discrete and countably infinite subset A. Assume that $A = \{x_n : n \in \mathbb{N}\}$. Now, for each $n \in \mathbb{N}$ there is an open subset V'_n of X such that $x_n \in V'_n$ and $V'_n \cap V'_m = \emptyset$ if $n \neq m$. We know that $\alpha(r)$ is 0-dimensional, then for each $n \in \mathbb{N}$ there is an open-closed subset W_n of $\alpha(r)$ such that $x_n \in W_n \subset V'_n$. By Lemma 4.6 there exists a cellular family $\{V_n : n \in \mathbb{N}\}$ of open-closed subsets of $\alpha(r)$ such that $x_n \in V_n \subset W_n \subset V'_n$ and $\bigcup\{V_n : n \in \mathbb{N}\} \in 2^{\alpha(r)} \subset 2^X$. Note that the family $\{V_n : n \in \mathbb{N}\} \cup \{\alpha(r) - \bigcup\{V_n : n \in \mathbb{N}\}\}$ satisfies the conditions of Lemma 4.7, so there exists a cellular family $\{U_n : n \in \mathbb{N} \cup \{0\}\}$ in X such that $(\alpha(r) - \bigcup\{V_n : n \in \mathbb{N}\}) \subseteq V_0$ and for each $n \in \mathbb{N}$, $V_n \subseteq U_n$. Observe that $\alpha(r) \subseteq \bigcup\{U_n : n \in \mathbb{N} \cup \{0\}\}$, i.e. $\alpha(r) \in \langle \bigcup\{U_n : n \in \mathbb{N} \cup \{0\}\}\rangle$. Since α is continuous, there is $\varepsilon > 0$ such that $\alpha([r-\varepsilon, r+\varepsilon]) \subseteq \langle \bigcup\{U_n : n \in \mathbb{N} \cup \{0\}\}\rangle$. Take $t_0 \in [r-\varepsilon, r)$ such that $\alpha(t_0)$ is compact. So there exists $N \in \mathbb{N}$ such that $\alpha(t_0) \cap U_N = \emptyset$. Since $(\bigcup \alpha([r-\varepsilon, r+\varepsilon])) \cap U_N = (\bigcup \alpha([r-\varepsilon, r+\varepsilon])) \cap \overline{U}_N$ we have that $(\bigcup \alpha([r-\varepsilon, r+\varepsilon])) \cap U_N$ is a nonempty open and closed subset of $\bigcup \alpha([r-\varepsilon, r+\varepsilon])$. Since $\alpha([r-\varepsilon, r+\varepsilon])$ is a connected subset of 2^X, by Lemma 4.9, we have that $\alpha(t) \cap ((\bigcup \alpha([r-\varepsilon, r+\varepsilon])) \cap U_N) \neq \emptyset$, for each $t \in [r-\varepsilon, r+\varepsilon]$, so $\alpha(t_0) \cap U_N \neq \emptyset$, a contradiction. This proves that M is compact. □

The next Theorem can be proved in the same way as Theorem 3.2 of [4].

THEOREM 4.18. *If X is a Hausdorff space, $\alpha : [0,1] \to TD(X) \cap K(X)$ is a path and $p \in \alpha(0)$, then there exists a continuous function $f : [0,1] \to X$ such that $f(0) = p$ and $f(t) \in \alpha(t)$ for each $t \in [0,1]$.*

THEOREM 4.19. *If X is a locally compact, Lindelöf and Hausdorff space such that $TD(X)$ is pathwise connected, then X is pathwise connected.*

PROOF. Let $x, y \in X$ and $\alpha : [0,1] \to TD(X)$ be a path such that $\alpha(0) = \{x\}$ and $\alpha(1) = \{y\}$. Note that $\{x\}$ is compact, so by Theorem 4.17 we have that $\alpha([0,1]) \subseteq TD(X) \cap K(X)$. Now, by Theorem 4.18 there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) \in \{y\}$. □

Note that the converse of Theorem 4.19 is not true, for this it is enough to consider the dendroid of Example 4.5.

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REFERENCES

[1] J. J. Charatonik, *Two invariants under continuity and the incomparability of fans,* Fund. Math. **53** (1963/64), 187–204.

[2] J. J. Charatonik and C. Eberhart, *On smooth dendroids,* Fund. Math. **67** (1970), 297–322.

[3] S. García-Ferreira and Y. F. Otriz-Castillo, *The hyperspace of convergent sequences,* Topol. Appl. **196** (2015), 795–804.
[4] S. García-Ferreira and R. Rojas Hernández, *Connectedness like properties on the hyperspace of convergent sequences,* Topol. Appl. **230** (2017), 639–647.
[5] A. Illanes and S. B. Nadler, Jr., *Hyperspaces. Fundamental and recent advences,* Marcel Dekker, Inc., New York, 1999.
[6] J. M. Martinez-Montejano, *Mutual aposyndesis of symmetric products,* Topology Proc. **24** (1999), 203–213.
[7] D. Maya, P. Pellicer-Covarrubias and R. Pichardo-Mendoza, *General properties of the hyperspace of convergent sequences*, Topology Proc. **51** (2018), 143–168.
[8] E. Michael, *Topologies on spaces of subsets,* Trans. Amer. Math. Soc. **71** (1951), 152–182.
[9] L. Mohler, *A characterization of smoothness in dendroids,* Fund. Math. **67** (1970), 369–376.
[10] S. B. Nadler, Jr., Hyperspaces of sets, Marcel Dekker, Inc., New York, 1978.
[11] S. B. Nadler, Jr., Continuum theory. An introduction, Marcel Dekker, Inc., New York, 1992.
[12] G. T. Whyburn, Analytic topology, Amer. Math. Soc. New York, 1942.

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