GLASNIK MATEMATIČKI
Vol. 55(75)(2020), 129 – 142

APPROXIMATE INVERSE LIMITS AND (m, n)–DIMENSIONS

Matthew Lynam and Leonard R. Rubin
East Central University, USA and University of Oklahoma, USA

Abstract. In 2012, V. Fedorchuk, using m-pairs and n-partitions, introduced the notion of the (m, n)-dimension of a space. It generalizes covering dimension. Here we are going to look at this concept in the setting of approximate inverse systems of compact metric spaces. We give a characterization of (m, n)-dim$X$, where X is the limit of an approximate inverse system, strictly in terms of the given system.

1. Introduction

In [2], V. Fedorchuk introduced a new generalization of covering dimension which he called (m, n)-dimension, written (m, n)-dim, and such that for each normal space X, $(2, 1)\text{-}\dim X = \dim X$. Fedorchuk's (m, n)-dim is defined using m-pairs and n-partitions; in Section 2 we will provide what is needed to define such pairs and partitions, and with that in hand, we shall give the definition of the (m, n)-dimension of a space. We shall also cite in that section a few fundamental facts from this theory that will be used in the sequel.

Since the introduction of (m, n)-dimension, the theory has been developed in parallel to that of the classical notions of dimension which one can find in [1]. For example, a strong inductive version was presented in [4], a transfinite type in [10], and for (m, n)-dimension, both a factorization theorem and one about the existence of universal spaces were given in [9] and [12], respectively. In [11], Martynchuk proved that for every strongly hereditarily normal space X, $(m, n)\text{-}\dim X = \left\lfloor \frac{\dim X}{n} \right\rfloor$; therefore Fedorchuk's notion of dimension deviates from that of covering dimension in infinitely many cases. One may also consult [3] and [5] for additional contributions of Fedorchuk.

2020 *Mathematics Subject Classification.* 54F45.
Key words and phrases. Dimension, (m, n)-dim, approximate inverse system.

Our main result gives a characterization of the (m,n)-dimension of a space X where X is the limit of an approximate (inverse) system, strictly in terms of the given system. One of the consequences of our result, Corollary 6.3, is that (m,n)-dim $\leq k$ is preserved by limits of approximate inverse systems of metric compacta. Most readers are familiar with inverse systems and their limits, but are perhaps not as well-versed with approximate systems. Approximate systems were introduced in [7] where it was shown that each compact Hausdorff space X can be written as the limit of an approximate system of compact polyhedra each having dimension less than or equal to $\dim X$. In general, this fact is not true of inverse systems.

In Section 3 we will provide the definition of an approximate system and its limit, as well as some basic results. The reader will see that the coordinate spaces of an approximate system are compact metric spaces each of which is assigned a positive number. The limit of such a system is always compact and Hausdorff. In Section 4 we shall prove several new facts dealing with finite covers of limits of approximate systems. These will be used in the proof of our main result, Theorem 5.2, which appears in Section 5. Section 6 gathers some corollaries to Theorem 5.2.

2. Introduction to (m,n)-dim

Throughout this paper, map will mean continuous function. We will denote the *order* of a nonempty finite family Φ of sets by $\mathrm{ord}(\Phi)$. By *order* we mean the largest $n \in \{0\} \cup \mathbb{N}$ such that Φ contains a subset Ψ with $\mathrm{card}(\Psi) = n$ and $\bigcap \Psi \neq \emptyset$. By this definition, $\mathrm{ord}(\Phi) = 0$ if and only if $\Phi = \{\emptyset\}$. On the other hand, $\mathrm{ord}(\Phi) = 1$ if and only if Φ is pairwise disjoint and there exists $F \in \Phi$ such that $F \neq \emptyset$. If B is a subspace of a metric space X and $\rho > 0$, then $N(B, \rho)$ will denote the ρ-neighborhood of B in X. In this section, X will always denote a normal space.

Definition 2.1 ([2, Definition 2.1]). *Let $u = (U_1, \dots, U_m)$ be a finite open cover of X and $\Phi = (F_1, \dots, F_m)$ be a family of closed subsets of X such that*

$$F_j \subset U_j, \quad j = 1, \dots, m;$$
$$\mathrm{ord}(\Phi) \leq 1.$$

*Then (u, Φ) is said to be an m-**pair in** X.*

Definition 2.2 ([2, Definition 2.5]). *Let (u, Φ) be an m-pair in X where $u = (U_1, \dots, U_m)$ and $\Phi = (F_1, \dots, F_m)$. A closed set $P \subseteq X$ is said to be an n-**partition** of (u, Φ) if there exists a family of open sets $v = (V_1, \dots, V_m)$ of X such that $F_j \subseteq V_j \subseteq U_j$, for $j = 1, \dots, m$; $\mathrm{ord}(v) \leq n$; and $X \setminus P = \bigcup v$.*

Definition 2.3 ([2, Definition 2.7]). *For each $i = 1, \dots, r$, let (u_i, Φ_i) be an m-pair in X. The sequence (u_i, Φ_i), $i = 1, \dots, r$, is called n-**inessential***

in *X if for each i, there exists an n-partition P_i of (u_i, Φ_i) such that $P_1 \cap \cdots \cap P_r = \emptyset$.*

Definition 2.4 ([2, Definition 2.8]). *Let $m, n \in \mathbb{N}$ with $n \leq m$. To every space X one assigns the (m,n)-**dimension** (m,n)-$\dim X$, which is an element of $\{-1\} \cup \{0\} \cup \mathbb{N} \cup \{\infty\}$ in the following way.*

(1) *(m,n)-$\dim X = -1$ if and only if $X = \emptyset$.*

In case $X \neq \emptyset$, then:

(2.1) *(m,n)-$\dim X = \infty$, if for each $k \in \{0\} \cup \mathbb{N}$, there is a sequence (u_i, Φ_i), $i = 1, \ldots, k+1$, of m-pairs in X, that is not n-inessential in X;*

(2.2) *(m,n)-$\dim X = r$, where $r \in \{0\} \cup \mathbb{N}$, if (m,n)-$\dim X \neq \infty$ and r is the minimum of those $k \in \{0\} \cup \mathbb{N}$ such that every sequence (u_i, Φ_i), $i = 1, \ldots, k+1$, of m-pairs in X, is n-inessential in X.*

Theorem 2.5 ([2, Theorem 2.9]). *One has that $(2,1)$-$\dim X = \dim X$.*

Of course Theorem 2.5 follows from Martynchuk's result mentioned in the Introduction. Moreover, from that very same expression, one can get values for (m,n)-$\dim X$ that are different from m. For example, if $X = [0,1]^3$, then $\dim X = 3$, but $(3,2)$-$\dim X = \left\lfloor \frac{3}{2} \right\rfloor = 1$.

Proposition 2.6 ([2, Proposition 2.19]). *Let Y be a space, $f : X \to Y$ be a map, and let a sequence (u_i, Φ_i), $i = 1, \ldots, r$, of m-pairs in Y be n-inessential in Y. Then $(f^{-1}(u_i), f^{-1}(\Phi_i))$, $i = 1, \ldots, r$, is an n-inessential sequence of m-pairs in X.*

Proposition 2.7 ([2, Proposition 2.20]). *Let (u_i, Φ_i) and (w_i, Ψ_i), $i = 1, \ldots, r$, be sequences of m-pairs in X where $u_i = (U_1^i, \ldots, U_m^i)$, $\Phi_i = (F_1^i, \ldots, F_m^i)$, $w_i = (W_1^i, \ldots, W_m^i)$, and $\Psi_i = (G_1^i, \ldots, G_m^i)$. Assume that*

$$F_j^i \subseteq G_j^i \subseteq W_j^i \subseteq U_j^i, \quad i = 1, \ldots, r, \quad j = 1, \ldots, m.$$

If the sequence (w_i, Ψ_i), $i = 1, \ldots, r$, is n-inessential in X, then the sequence (u_i, Φ_i), $i = 1, \ldots, r$, is n-inessential in X.

3. Approximate Inverse Systems

The following definition is from [7].

Definition 3.1. *An* **approximate inverse system**, $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$, *of metric compacta consists of the following: a partially ordered set $(A, \leq)$ which is directed and has no maximal element; for each $a \in A$, a compact metric space X_a with metric d and a real number $\varepsilon_a > 0$; for each pair $a \leq a'$ from A, a map $p_{aa'} : X_{a'} \to X_a$. Moreover, the following three conditions must be satisfied:*

(A1) $d(p_{a_1a_2}p_{a_2a_3}, p_{a_1a_3}) \leq \varepsilon_{a_1}$, $a_1 \leq a_2 \leq a_3$, $p_{aa} = \mathrm{id}$.

(A2) *For all $a \in A$ and $\eta > 0$ there exists an $a' \geq a$ such that for all $a_2 \geq a_1 \geq a'$ we have that $d(p_{aa_1}p_{a_1a_2}, p_{aa_2}) \leq \eta$.*

(A3) *For all $a \in A$ and $\eta > 0$ there exists an $a' \geq a$ such that for all $a'' \geq a'$ and $x, x' \in X_{a''}$ we have that if $d(x,x') \leq \varepsilon_{a''}$ then $d(p_{aa''}(x), p_{aa''}(x')) \leq \eta$.*

Definition 3.2. *A point $x = (p_a(x)) \in \prod_{a\in A} X_a$ belongs to $X = \lim \mathbf{X}$ provided the following condition is satisfied.*

(L) For all $a \in A$ and $\eta > 0$ there exists $a' \geq a$ such that for all $a'' \geq a'$ we have that $d(p_a(x), p_{aa''}p_{a''}(x)) \leq \eta$.

In the sequel we shall shorten the expression "approximate inverse system of metric compacta" to "approximate system."

The following theorem has several facts about approximate systems. The proofs can be found in [7].

Theorem 3.3. *Given an approximate system $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$ with $X = \lim \mathbf{X}$,*

1. *$X \neq \emptyset$ if and only if for all $a \in A$, $X_a \neq \emptyset$;*
2. *X is a compact Hausdorff space;*
3. *the collection of all sets of the form $p_a^{-1}(V_a)$, where $a \in A$ and $V_a \subseteq X_a$ is open, is a basis for the topology of X;*
4. *for every $a \in A$ and $\eta > 0$ there is an $a' \geq a$ such that for every $a'' \geq a'$ one has $d(p_{aa''}p_{a''}, p_a) \leq \eta$.*

Proposition 3.4 (Proposition 5.2 of [6]). *Given an approximate system $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$ with $X = \lim \mathbf{X}$, if F is closed in X, then for any neighborhood U of F in X, there exists an $a \in A$ such that for all $a' \geq a$, $p_{a'}^{-1}(p_{a'}(F)) \subseteq U$.*

4. New Results About Approximate Systems

We shall establish several facts concerning approximate systems. In this section, $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$ will denote an approximate system, and X will be its limit.

Definition 4.1. *In case $a \in A$ and $u = (U_1, \ldots, U_m)$ is a sequence of subsets of X, then by $p_a(u)$ we mean $(p_a(U_1), \ldots, p_a(U_m))$.*

The following is Theorem 3 of [7].

Theorem 4.2. *If $\mathcal{U}$ is an open cover of X, then there exist $a \in A$ and an open cover $\mathcal{V}$ of X_a such that $p_a^{-1}(\mathcal{V})$ refines $\mathcal{U}$.*

Corollary 4.3. *For each finite open cover $u = (U_1, \ldots, U_m)$ of X, there exist $a \in A$ and an open cover $(V_1, \ldots, V_m)$ of X_a such that $p_a^{-1}(V_j) \subseteq U_j$, for $j = 1, \ldots, m$.*

Proof. Use Theorem 4.2 to choose an index $a \in A$ and an open cover $\mathcal{V}$ of X_a such that $p_a^{-1}(\mathcal{V})$ refines u. For each $j = 1, \ldots, m$, set

$$V_j = \bigcup \{V \in \mathcal{V} \mid p_a^{-1}(V) \subseteq U_j\}.$$

Then the collection $(V_1, \ldots, V_m)$ is an open cover of X_a, and $p_a^{-1}(V_j) \subseteq U_j$ for all $j = 1, \ldots, m$. $\square$

Lemma 4.4. *For each finite open cover $u = (U_1, \ldots, U_m)$ of X and each index $a \in A$, there exists an index $a' \geq a$ such that for all $a'' \geq a'$, there is an open cover $(V_1, \ldots, V_m)$ of $X_{a''}$ with $p_{a''}^{-1}(V_j) \subseteq U_j$, for $j = 1, \ldots, m$.*

Proof. Using Corollary 4.3, choose an index $a_0 \in A$ and an open cover $(W_1, \ldots, W_m)$ of X_{a_0} such that $p_{a_0}^{-1}(W_j) \subseteq U_j$ for all $j = 1, \ldots, m$.

Choose a collection $(F_1, \ldots, F_m)$ of closed subsets of X_{a_0} that covers X_{a_0} and $F_j \subseteq W_i$ for all $j = 1, \ldots, m$. There exists an $\eta > 0$ such that for all j, $N(F_j, \eta) \subseteq W_j$.

Using Theorem 3.3(4.), choose an $a' \in A$ so that $a' \geq a$, $a' \geq a_0$, and for every $a'' \geq a'$ we have $d(p_{a_0 a''} p_{a''}, p_{a_0}) \leq \eta/2$. Now set $V_j = p_{a_0 a''}^{-1}(N(F_j, \eta/2))$. We claim that the collection $v = (V_1, \ldots, V_m)$ satisfies the conclusion.

It is clear that v is an open cover of $X_{a''}$. We now must show that for all $j = 1, \ldots, m$, $p_{a''}^{-1}(V_j) \subseteq U_j$. Let $x \in p_{a''}^{-1}(V_j)$. Then $p_{a''}(x) \in V_j = p_{a_0 a''}^{-1}(N(F_j, \eta/2))$. This gives us that $p_{a_0 a''} p_{a''}(x) \in N(F_j, \eta/2)$. Choose a $y \in F_j$ such that

$$d(p_{a_0 a''} p_{a''}(x), y) < \eta/2.$$

We also know that

$$d(p_{a_0 a''} p_{a''}(x), p_{a_0}(x)) \leq \eta/2.$$

And so by the triangle inequality,

$$d(p_{a_0}(x), y) < \eta.$$

This gives us that $p_{a_0}(x) \in N(F_j, \eta) \subseteq W_j$, and thus $x \in p_{a_0}^{-1}(W_j) \subseteq U_j$. We have shown that for all $j = 1, \ldots, m$, $p_{a''}^{-1}(V_j) \subseteq U_j$; hence the collection $v = (V_1, \ldots, V_m)$ satisfies the conclusion of the proposition. $\square$

Lemma 4.5. *If $\Phi = (F_1, \ldots, F_m)$ is a collection of closed subsets of X with* $\operatorname{ord}(\Phi) \leq 1$*, then there exists an $a \in A$ such that for all $b \geq a$,* $\operatorname{ord}(p_b(\Phi)) \leq 1$.

Proof. For each $j = 1, \ldots, m$, choose a neighborhood U_j of F_j so that for $i \neq j$, if $F_i = F_j$, then $U_i = U_j$, but if $F_i \neq F_j$, then $U_i \cap U_j = \emptyset$. Using Proposition 3.4, for each $j = 1, \ldots, m$, choose an $a_j \in A$ such that for all $a' \geq a_j$ we have $p_{a'}^{-1}(p_{a'}(F_j)) \subseteq U_j$.

Pick an $a \in A$ so that $a \geq a_j$ for all $j = 1, \ldots, m$, and let $b \geq a$. Then $p_b^{-1}(p_b(F_j)) \subseteq U_j$ for all $j = 1, \ldots, m$. This implies that if $i \neq j$ and $F_i \neq F_j$, then $p_b^{-1}(p_b(F_i)) \cap p_b^{-1}(p_b(F_j)) \subseteq U_i \cap U_j = \emptyset$. We claim that

$p_b(F_i) \cap p_b(F_j) = \emptyset$. For suppose the contrary, that $p_b(F_i) \cap p_b(F_j) \neq \emptyset$. One can find $q_i \in F_i$, $q_j \in F_j$, and $x \in p_b(F_i) \cap p_b(F_j)$ such that $p_b(q_i) = x = p_b(q_j)$. But then $\emptyset \neq \{q_i, q_j\} \subseteq p_b^{-1}(x) = p_b^{-1}(p_b(q_i)) = p_b^{-1}(p_b(q_j)) \subseteq p_b^{-1}(p_b(F_i)) \cap p_b^{-1}(p_b(F_j)) = \emptyset$, a contradiction. Thus $\operatorname{ord}(p_b(\Phi)) \leq 1$. ∎

LEMMA 4.6. *If (u, Φ) is an m-pair in X, where $u = (U_1, \ldots, U_m)$ and $\Phi = (F_1, \ldots, F_m)$, then there exist $a \in A$ and a finite open cover $v = (V_1, \ldots, V_m)$ of X_a such that $(v, p_a(\Phi))$ is an m-pair in X_a (and hence $\operatorname{ord}(p_a(\Phi)) \leq 1$), and for $j = 1, \ldots, m$, a closed neighborhood G_j of $p_a(F_j)$ in X_a and an open subset W_j of X_a such that,*

1. *$p_a^{-1}(V_j) \subseteq U_j$, and*
2. *$p_a(F_j) \subseteq \operatorname{int}(G_j) \subseteq G_j \subseteq W_j \subseteq \overline{W}_j \subseteq V_j$.*

Moreover, we may make the above choices such that if we define

$$w = (W_1, \ldots, W_m), \ g = (G_1, \ldots, G_m),$$

then w covers X_a, $\operatorname{ord}(g) \leq 1$ (so (w, g) is an m-pair in X_a), and so that for all $b \geq a$, $\operatorname{ord}(p_b(\Phi)) \leq 1$.

PROOF. Using Lemmas 4.4 and 4.5, choose an $a \in A$ and a finite open cover $t = (T_1, \ldots, T_m)$ of X_a such that $p_a^{-1}(T_j) \subseteq U_j$, $j = 1, \ldots, m$, and for all $b \geq a$, $\operatorname{ord}(p_b(\Phi)) \leq 1$. Since p_a is a closed map and $F_j \subseteq U_j$, there exists a neighborhood W_j of $p_a(F_j)$ such that $p_a^{-1}(W_j) \subseteq U_j$. Set $V_j = T_j \cup W_j$. Then, of course, $p_a(F_j) \subseteq V_j$, the open collection $v = (V_1, \ldots, V_m)$ covers X_a, and $p_a^{-1}(V_j) \subseteq U_j$. We leave the remaining details to the reader. ∎

LEMMA 4.7. *For every sequence (u_i, Φ_i), $i = 1, \ldots, k+1$, of m-pairs in X, where $u_i = (U_1^i, \ldots, U_m^i)$ and $\Phi_i = (F_1^i, \ldots, F_m^i)$, and every $a \in A$, there exists $b_0 \geq a$ such that for all $b \geq b_0$, there is a corresponding sequence $(y_i, p_b(\Phi_i))$, $i = 1, \ldots, k+1$, of m-pairs in X_b, where $y_i = (Y_1^i, \ldots, Y_m^i)$, and for all $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$,*

$$F_j^i \subseteq p_b^{-1}(p_b(F_j^i)) \subseteq p_b^{-1}(Y_j^i) \subseteq U_j^i.$$

PROOF. Let (u_i, Φ_i), $i = 1, \ldots, k+1$, be as above, and let $a \in A$. Using Lemma 4.6, for each i we can find an index $a_i \in A$ and a finite open cover $v_i = (V_i^i, \ldots, V_m^i)$ of X_{a_i} so that $(v_i, p_{a_i}(\Phi_i))$ is an m-pair of X_{a_i}, and for each $j = 1, \ldots, m$, a closed neighborhood G_j^i of $p_{a_i}(F_j^i)$ and an open subset W_j^i of X_{a_i} such that,

$$p_{a_i}^{-1}(V_j^i) \subseteq U_j^i, \text{ and} \tag{4.1}$$

$$p_{a_i}(F_j^i) \subseteq \operatorname{int}(G_j^i) \subseteq G_j^i \subseteq W_j^i \subseteq \overline{W_j^i} \subseteq V_j^i. \tag{4.2}$$

Moreover, if we define $w_i = (W_1^i, \ldots, W_m^i)$ and $g_i = \left(G_1^i, \ldots, G_m^i\right)$, then (w_i, g_i) is an m-pair in X_{a_i}, and for all $b \geq a_i$, $\operatorname{ord}(p_b(\Phi_i)) \leq 1$. There exists

$\delta > 0$ such that for all i and j,

$$N(p_{a_i}(F^i_j),\delta) \subseteq G^i_j, \tag{4.3}$$

and

$$N(\overline{W^i_j},\delta) \subseteq V^i_j. \tag{4.4}$$

Using Theorem 3.3(4.), pick $b_0 \in A$ so that $b_0 \geq a$ and $b_0 \geq a_i$ for all $i = 1,\ldots,k+1$, and for all $b \geq b_0$,

$$d(p_{a_i}, p_{a_i b}p_b) < \delta. \tag{4.5}$$

Fix $b \geq b_0$. For each $i = 1,\ldots,k+1$ and $j = 1,\ldots,m$, set $Y^i_j = p^{-1}_{a_i b}(W^i_j)$. Put $y_i = \left(Y^i_1,\ldots,Y^i_m\right)$. Then y_i is an open cover of X_b.

We claim that for each i, $(y_i, p_b(\Phi_i))$ is an m-pair in X_b. Since we have chosen a_i as in Lemma 4.6, and $b \geq a_i$, then $\mathrm{ord}(p_b(\Phi_i)) \leq 1$. We will now show that for each j, $p_b(F^i_j) \subseteq Y^i_j$. Let $x \in p_b(F^i_j)$. Choose $z \in F^i_j$ such that $p_b(z) = x$. We have that $p_{a_i}(z) \in p_{a_i}(F^i_j)$. By (4.5), $d\left(p_{a_i}(z), p_{a_i b}p_b(z)\right) < \delta$. So (4.3) and (4.2) show that $p_{a_i b}p_b(z) \in G^i_j \subseteq W^i_j$. Finally we have, $x = p_b(z) \in p^{-1}_{a_i b}(W^i_j) = Y^i_j$. Thus, $p_b(F^i_j) \subseteq Y^i_j$. Therefore, $(y_i, p_b(\Phi_i))$ is an m-pair in X_b.

We next demonstrate that for each i and j,

$$F^i_j \subseteq p_b^{-1}(p_b(F^i_j)) \subseteq p_b^{-1}(Y^i_j) \subseteq U^i_j.$$

The left inclusion is obvious, and the middle inclusion follows from the fact that $(y_i, p_b(\Phi_i))$ is an m-pair in X_b. To show the right inclusion, let $x \in p_b^{-1}(Y^i_j) = p_b^{-1}(p^{-1}_{a_i b}(W^i_j))$. Then $p_{a_i b}p_b(x) \in W^i_j$. By (4.5), $d(p_{a_i}(x), p_{a_i b}p_b(x)) < \delta$, so by (4.4), $p_{a_i}(x) \in V^i_j$. Using (4.1), $x \in p^{-1}_{a_i}(V^i_j) \subseteq U^i_j$, as needed. □

5. Characterization

Definition 5.1. *Let X be a space, $B \subseteq X$, and (u,Φ) an m-pair in X with $u = (U_1,\ldots,U_m)$ and $\Phi = (F_1,\ldots,F_m)$. Then by $(u \cap B, \Phi \cap B)$ or $(u,\Phi)\cap B$, we shall mean the m-pair in B given by $u \cap B = (U_1 \cap B,\ldots,U_m \cap B)$ and $\Phi \cap B = (F_1 \cap B,\ldots,F_m \cap B)$.*

Here is our main result.

Theorem 5.2. *Let $\mathbf{X} = \{X_a, \epsilon_a, p_{aa'}, A\}$ be an approximate system, $X = \lim \mathbf{X}$, $\{m,n\} \subset \mathbb{N}$, and $k \geq 0$. Then (m,n)-$\mathrm{dim}X \leq k$ if and only if for each $a \in A$ and sequence (w_i,Φ_i), $i = 1,\ldots,k+1$, of m-pairs in X_a, there exists $b_0 \geq a$ such that for all $b \geq b_0$, the sequence $\left(p^{-1}_{ab}(w_i), p^{-1}_{ab}(\Phi_i)\right) \cap p_b(X)$, $i = 1,\ldots,k+1$, of m-pairs in $p_b(X)$ is n-inessential in $p_b(X)$.*

Proof. ($\Leftarrow$) Let (u_i, Φ_i), $i = 1, \dots, k+1$, be a sequence of m-pairs in X. We wish to show that this sequence is n-inessential in X. For each i, let $u_i = (U_1^i, \dots, U_m^i)$ and $\Phi_i = (F_1^i, \dots, F_m^i)$. Using Lemma 4.6, we can find an index $a_i \in A$ and a finite open cover $v_i = (V_1^i, \dots, V_m^i)$ of X_{a_i} such that $(v_i, p_{a_i}(\Phi_i))$ is an m-pair in X_{a_i} along with sequences $w_i = (W_1^i, \dots, W_m^i)$ and $g_i = (G_1^i, \dots, G_m^i)$ where for each $j = 1, \dots, m$, G_j^i is a closed neighborhood of $p_{a_i}(F_j^i)$ in X_{a_i} and W_j^i is an open subset of X_{a_i}, so that,

$$p_{a_i}^{-1}(V_j^i) \subseteq U_j^i, \tag{5.1}$$

$$p_{a_i}(F_j^i) \subseteq \operatorname{int} G_j^i \subseteq G_j^i \subseteq W_j^i \subseteq \overline{W_j^i} \subseteq V_j^i, \tag{5.2}$$

$$(w_i, g_i) \text{ is an } m\text{-pair of } X_{a_i}. \tag{5.3}$$

There exists $\delta > 0$ such that for all $i = 1, \dots, k{+}1$ and for all $j = 1, \dots, m$,

$$N(p_{a_i}(F_j^i), \delta) \subseteq G_j^i, \text{ and} \tag{5.4}$$

$$N(\overline{W_j^i}, \delta) \subseteq V_j^i. \tag{5.5}$$

Using Theorem 3.3(4.) and Definition 3.1(A2), pick $a \in A$ so that $a \geq a_i$ for all $i = 1, \dots, k+1$, and for all $a' \geq a$ we have,

$$d(p_{a_i}, p_{a_i a'} p_{a'}) < \delta/2, \text{ and} \tag{5.6}$$

$$d(p_{a_i a'}, p_{a_i a} p_{a a'}) < \delta/2. \tag{5.7}$$

For each $i = 1, \dots, k+1$, set

$$\begin{aligned} w_i^0 &= p_{a_i a}^{-1}(w_i) = (p_{a_i a}^{-1}(W_1^i), \dots, p_{a_i a}^{-1}(W_m^i)), \text{ and} \\ g_i^0 &= p_{a_i a}^{-1}(g_i) = (p_{a_i a}^{-1}(G_1^i), \dots, p_{a_i a}^{-1}(G_m^i)). \end{aligned}$$

It readily follows from (5.3), that for each $i = 1, \dots, k+1$, (w_i^0, g_i^0) is an m-pair in X_a. Now use the assumption in ($\Leftarrow$) to choose $b \geq a$ such that the sequence of m-pairs, $\left(p_{ab}^{-1}\left(w_i^0\right), p_{ab}^{-1}\left(g_i^0\right)\right) \cap p_b(X)$, $i = 1, \dots, k+1$, in $p_b(X)$ is n-inessential in $p_b(X)$. It follows from Proposition 2.6, that the sequence of m-pairs in X,

$$\left(p_b^{-1}\left(p_{ab}^{-1}\left(w_i^0\right) \cap p_b(X)\right), p_b^{-1}\left(p_{ab}^{-1}\left(g_i^0\right) \cap p_b(X)\right)\right), \; i = 1, \dots, k+1$$

which equals $\left(p_b^{-1}\left(p_{ab}^{-1}\left(w_i^0\right)\right), p_b^{-1}\left(p_{ab}^{-1}\left(g_i^0\right)\right)\right)$, $i = 1, \dots, k+1$, is n-inessential in X.

We will now show that for each $i = 1, \dots, k+1$ and $j = 1, \dots, m$, $F_j^i \subseteq p_b^{-1}(p_{ab}^{-1}(p_{a_i a}^{-1}(G_j^i))) \subseteq p_b^{-1}(p_{ab}^{-1}(p_{a_i a}^{-1}(W_j^i))) \subseteq U_j^i$. Fix i and j.

To show the left inclusion, let $x \in F_j^i$. Then $p_{a_i}(x) \in p_{a_i}(F_j^i)$. By (5.6) we have

$$d(p_{a_i}(x), p_{a_i b} p_b(x)) < \delta/2,$$

and by (5.7) we have

$$d(p_{a_ib}p_b(x), p_{a_ia}p_{ab}p_b(x)) < \delta/2.$$

And so by the triangle inequality,

$$d(p_{a_i}(x), p_{a_ia}p_{ab}p_b(x)) < \delta.$$

By (5.4) we have $p_{a_ia}p_{ab}p_b(x) \in G^i_j$. Thus, $x \in p_b^{-1}(p_{ab}^{-1}(p_{a_ia}^{-1}(G^i_j)))$ which implies that,

$$F^i_j \subseteq p_b^{-1}(p_{ab}^{-1}(p_{a_ia}^{-1}(G^i_j))).$$

The middle inclusion follows from (5.2).

To show the right inclusion, let $x \in p_b^{-1}(p_{ab}^{-1}(p_{a_ia}^{-1}(W^i_j)))$. Then

$$p_{a_ia}p_{ab}p_b(x) \in W^i_j.$$

Using (5.7) and (5.6), one has that

$$d(p_{a_ib}p_b(x), p_{a_ia}p_{ab}p_b(x)) < \delta/2$$

and

$$d(p_{a_i}(x), p_{a_ib}p_b(x)) < \delta/2.$$

By the triangle inequality,

$$d(p_{a_i}(x), p_{a_ia}p_{ab}p_b(x)) < \delta.$$

So by (5.5) we have $p_{a_i}(x) \in V^i_j$. Thus, using (5.1), $x \in p_{a_i}^{-1}(V^i_j) \subseteq U^i_j$, and so

$$p_b^{-1}(p_{ab}^{-1}(p_{a_ia}^{-1}(W^i_j))) \subseteq U^i_j.$$

By Proposition 2.7, the sequence (u_i, Φ_i), $i = 1, \ldots, k+1$, is n-inessential in X, and so (m,n)-dim$X \le k$.

($\Rightarrow$) We will now assume that (m,n)-dim$X \le k$. Let $a \in A$, and (w_i, Φ_i), $i = 1, \ldots, k+1$, be a sequence of m-pairs in X_a, where we denote $w_i = (W^i_1, \ldots, W^i_m)$ and $\Phi_i = (F^i_1, \ldots, F^i_m)$. Recall that for each $i = 1, \ldots, k+1$, w_i is an open cover of X_a, $\operatorname{ord}(\Phi_i) \le 1$, and for each $j = 1, \ldots, m$, $F^i_j \subseteq W^i_j$.

For each $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$, choose a closed neighborhood $\widehat{F}^i_j$ of F^i_j in X_a and an open set $\widehat{W}^i_j$ in X_a such that

$$F^i_j \subseteq \operatorname{int} \widehat{F}^i_j \subseteq \widehat{F}^i_j \subseteq \widehat{W}^i_j \subseteq \overline{\widehat{W}^i_j} \subseteq W^i_j,$$

$$\hat{w}_i = (\widehat{W}^i_1, \ldots, \widehat{W}^i_m) \text{ covers } X_a \text{ , and}$$

$$\operatorname{ord}(\widehat{\Phi}_i) \le 1 \text{ where } \widehat{\Phi}_i = (\widehat{F}^i_1, \ldots, \widehat{F}^i_m).$$

Then $(\hat{w}_i, \widehat{\Phi}_i)$ is an m-pair in X_a.

There exists $\delta > 0$ such that for all $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$, we have that in X_a,

$$N(F^i_j, \delta) \subseteq \widehat{F}^i_j, \text{ and}$$

$$N(\overline{\widehat{W}^i_j}, \delta) \subseteq W^i_j.$$

Since (m,n)-dim$X \leq k$, the sequence $(p_a^{-1}(\hat{w}_i), p_a^{-1}(\widehat{\Phi}_i))$, $i = 1, \ldots, k+1$, of m-pairs in X is n-inessential in X. So, for each $i = 1, \ldots, k+1$, there exists an n-partition P_i of the m-pair $(p_a^{-1}(\hat{w}_i), p_a^{-1}(\widehat{\Phi}_i))$ such that

$$P_1 \cap \cdots \cap P_{k+1} = \emptyset.$$

By the definition of n-partition, for each $i = 1, \ldots, k+1$, we have a collection of open sets in X, $v_i = (V_1^i, \ldots, V_m^i)$, such that

$$p_a^{-1}\left(\widehat{F}_j^i\right) \subseteq V_j^i \subseteq p_a^{-1}\left(\widehat{W}_j^i\right) \text{ for } j = 1, \ldots, m; \tag{5.8}$$

$$\operatorname{ord}(v_i) \leq n; \text{ and}$$

$$X \setminus P_i = \bigcup v_i. \tag{5.9}$$

By (5.8) and (5.9) we have for each $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$,

$$p_a^{-1}\left(\widehat{F}_j^i\right) \subseteq V_j^i \subseteq \bigcup v_i = X \setminus P_i.$$

Thus, for each $i = 1, \ldots, k+1$,

$$\bigcup p_a^{-1}\left(\widehat{\Phi}_i\right) \subseteq X \setminus P_i,$$

and so,

$$P_i \subseteq X \setminus \bigcup p_a^{-1}\left(\widehat{\Phi}_i\right).$$

For each $i = 1, \ldots, k+1$, we choose an open set Q_i in X with the following properties:

$$P_i \subseteq Q_i, \tag{5.10}$$

$$Q_1 \cap \cdots \cap Q_{k+1} = \emptyset, \text{ and}$$

$$Q_i \subseteq X \setminus \bigcup p_a^{-1}\left(\widehat{\Phi}_i\right). \tag{5.11}$$

Consider the closed set $X \setminus Q_i$. Then by (5.9) and (5.10) the collection of open sets in $X \setminus Q_i$,

$$\left(V_1^i \cap (X \setminus Q_i), \ldots, V_m^i \cap (X \setminus Q_i)\right)$$

covers $X \setminus Q_i$. By (5.8) we have that $p_a^{-1}\left(\widehat{F}_j^i\right) \subseteq V_j^i$, and by definition, $p_a^{-1}(\widehat{F}_j^i) \subseteq \bigcup p_a^{-1}(\widehat{\Phi}_i)$. This and (5.11) imply that $p_a^{-1}\left(\widehat{F}_j^i\right) \subseteq \bigcup p_a^{-1}\left(\widehat{\Phi}_i\right) \subseteq X \setminus Q_i$. And so, $p_a^{-1}\left(\widehat{F}_j^i\right) \subseteq V_j^i \cap (X \setminus Q_i)$. This shows that there exist closed sets $G_1^i, \ldots, G_m^i$ in X such that for $j = 1, \ldots, m$ and $i = 1, \ldots, k+1$, we have,

$$p_a^{-1}\left(\widehat{F}_j^i\right) \subseteq G_j^i \subseteq V_j^i \cap (X \setminus Q_i), \text{ and} \tag{5.12}$$

$$\text{the collection } (G_1^i, \ldots, G_m^i) \text{ covers } X \setminus Q_i. \tag{5.13}$$

We will now choose $b_0 \geq a$ using Proposition 3.4, Theorem 3.3(4.), and Lemma 4.7. First choose $b_1 \geq a$ so that by Proposition 3.4 we have that for all $b \geq b_1$,

$$p_b^{-1}(p_b(G_j^i)) \subseteq V_j^i \text{ for all } j = 1,\ldots,m \text{ and } i = 1,\ldots,k+1. \tag{5.14}$$

Next, using Theorem 3.3(4.), choose $b_2 \geq b_1$ such that for all $b \geq b_2$ we have that

$$d(p_{ab}p_b, p_a) < \delta.$$

Apply Lemma 4.7 to the sequence $(p_a^{-1}(\hat{w}_i), p_a^{-1}(\widehat{\Phi}_i))$, $i = 1,\ldots,k+1$, of m-pairs in X (note that here we substitute $p_a^{-1}\left(\widehat{F}_j^i\right)$ for the F_j^i and $p_a^{-1}\left(\widehat{W}_j^i\right)$ for the U_j^i of the lemma) to get an index $b_0 \geq b_2$ so that for all $b \geq b_0$, there exists a corresponding sequence

$$(y_i, p_b(p_a^{-1}(\widehat{\Phi}_i))),\ i = 1,\ldots,k+1$$

of m-pairs in X_b. Moreover, for all $j = 1,\ldots,m$ and $i = 1,\ldots,k+1$, we have,

$$y_i = (Y_1^i,\ldots,Y_m^i),$$

$$p_b\left(p_a^{-1}\left(\widehat{\Phi}_i\right)\right) = \left(p_b\left(p_a^{-1}\left(\widehat{F}_1^i\right)\right),\ldots,p_b\left(p_a^{-1}\left(\widehat{F}_m^i\right)\right)\right),$$

and

$$p_a^{-1}\left(\widehat{F}_j^i\right) \subseteq p_b^{-1}\left(p_b\left(p_a^{-1}\left(\widehat{F}_j^i\right)\right)\right) \subseteq p_b^{-1}(Y_j^i) \subseteq p_a^{-1}\left(\widehat{W}_j^i\right). \tag{5.15}$$

We now use (5.8), (5.14) and the fact that p_b is a closed map to choose, for each $j = 1,\ldots,m$ and $i = 1,\ldots,k+1$, an open neighborhood T_j^i in $p_b(X)$ of $p_b(G_j^i)$ such that

$$p_b^{-1}(p_b(G_j^i)) \subseteq p_b^{-1}(T_j^i) \subseteq V_j^i \subseteq p_a^{-1}\left(\widehat{W}_j^i\right). \tag{5.16}$$

Let $t_i = (T_1^i,\ldots,T_m^i)$. Then since $\mathrm{ord}(v_i) \leq n$ and $T_j^i \subseteq p_b(X)$ we have that $\mathrm{ord}(t_i) \leq n$.

For each $j = 1,\ldots,m$ and $i = 1,\ldots,k+1$, let $\widehat{Y}_j^i = \left(Y_j^i \cup T_j^i\right) \cap p_b(X)$, and $\hat{y}_i = (\widehat{Y}_1^i,\ldots,\widehat{Y}_m^i)$. Then $(\hat{y}_i, p_b(p_a^{-1}(\widehat{\Phi}_i)))$, $i = 1,\ldots,k+1$, is a sequence of m-pairs in $p_b(X)$, and using (5.15) and (5.16) we have for all $i = 1,\ldots,m$ and $j = 1,\ldots,k+1$,

$$p_a^{-1}\left(\widehat{F}_j^i\right) \subseteq p_b^{-1}\left(p_b\left(p_a^{-1}\left(\widehat{F}_j^i\right)\right)\right) \subseteq p_b^{-1}\left(\widehat{Y}_j^i\right) \subseteq p_a^{-1}\left(\widehat{W}_j^i\right). \tag{5.17}$$

We claim that the sequence $(\hat{y}_i, p_b(p_a^{-1}(\widehat{\Phi}_i))) \cap p_b(X)$, $i = 1,\ldots,k+1$, which is the same as $(\hat{y}_i, p_b(p_a^{-1}(\widehat{\Phi}_i)))$, $i = 1,\ldots,k+1$, is n-inessential in $p_b(X)$.

For each $i = 1,\ldots,k+1$, let $R_i = p_b(X) \setminus \bigcup_{j=1}^m T_j^i$. Using (5.12) and the fact that T_j^i is an open neighborhood of $p_b(G_j^i)$ in $p_b(X)$, we have that $p_b\left(p_a^{-1}\left(\widehat{F}_j^i\right)\right) \subseteq T_j^i \subseteq \widehat{Y}_j^i$ for $j = 1,\ldots,m$. Since t_i is a family of open sets in $p_b(X)$ such that $p_b\left(p_a^{-1}\left(\widehat{F}_j^i\right)\right) \subseteq T_j^i \subseteq \widehat{Y}_j^i$, and $\operatorname{ord}(t_i) \leq n$, then R_i is an n-partition of $\left(\hat{y}_i, p_b\left(p_a^{-1}\left(\widehat{\Phi}_i\right)\right)\right)$ in $p_b(X)$. We will now show that $R_1 \cap \cdots \cap R_{k+1} = \emptyset$.

We first note that by (5.13) and (5.16) we have for each $i = 1, \ldots, k+1$,

$$X \setminus Q_i \subseteq \bigcup_{j=1}^{m} G_j^i \subseteq \bigcup_{j=1}^{m} p_b^{-1}(p_b(G_j^i)) \subseteq \bigcup_{j=1}^{m} p_b^{-1}(T_j^i).$$

It follows that

$$p_b^{-1}(R_i) = p_b^{-1}\left(p_b(X) \setminus \bigcup_{j=1}^{m} T_j^i\right) = X \setminus \left(\bigcup_{j=1}^{m} p_b^{-1}(T_j^i)\right) \subseteq Q_i.$$

Since

$$Q_1 \cap \cdots \cap Q_{k+1} = \emptyset,$$

we have that

$$p_b^{-1}(R_1) \cap \cdots \cap p_b^{-1}(R_{k+1}) = \emptyset,$$

and so, since $R_i \subseteq p_b(X)$ for each $i = 1, \ldots, k+1$,

$$R_1 \cap \cdots \cap R_{k+1} = \emptyset.$$

Thus, as stated above, the sequence $(\hat{y}_i, p_b(p_a^{-1}(\widehat{\Phi}_i))) \cap p_b(X)$, $i = 1, \ldots, k+1$, is n-inessential in $p_b(X)$.

To conclude the proof we will show that for all $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$, we have

$$p_{ab}^{-1}(F_j^i) \cap p_b(X) \subseteq p_b(p_a^{-1}(\widehat{F}_j^i)) \subseteq \widehat{Y}_j^i \subseteq p_{ab}^{-1}(W_j^i) \cap p_b(X),$$

and apply Proposition 2.7 (this means that in terms of Proposition 2.7, F_j^i corresponds to $p_{ab}^{-1}(F_j^i) \cap p_b(X)$, G_j^i to $p_b(p_a^{-1}(\widehat{F}_j^i))$, W_j^i to $\widehat{Y}_j^i$, and U_j^i to $p_{ab}^{-1}(W_j^i) \cap p_b(X)$). Fix i and j.

To show the left inclusion, let $x \in p_{ab}^{-1}(F_j^i) \cap p_b(X)$, and choose $y \in p_b^{-1}(x)$. Then we have that,

$$d(p_{ab}(x), p_a(y)) = d(p_{ab}(p_b(y)), p_a(y)) < \delta.$$

Thus, $p_a(y) \in N(F_j^i, \delta)$. It follows that $p_a(y) \in \widehat{F}_j^i$, and so $y \in p_a^{-1}(\widehat{F}_j^i)$. Finally,

$$x = p_b(y) \in p_b(p_a^{-1}(\widehat{F}_j^i)),$$

proving the left inclusion. The middle inclusion follows from (5.17).

To show the right inclusion, we first note that by (5.17), $\widehat{Y}_j^i \subseteq p_b(p_a^{-1}(\widehat{W}_j^i))$. We now let $x \in \widehat{Y}_j^i \subseteq p_b(p_a^{-1}(\widehat{W}_j^i))$. We have that $p_a(p_b^{-1}(x)) \subseteq \widehat{W}_j^i$. Choose $y \in p_b^{-1}(x)$. Then $p_a(y) \in \widehat{W}_j^i$. So,

$$d(p_{ab}(x), p_a(y)) = d(p_{ab}(p_b(y)), p_a(y)) < \delta.$$

Thus, $p_{ab}(x) \in N(\widehat{W}^i_j, \delta)$, so $p_{ab}(x) \in W^i_j$. It follows that $x \in p_{ab}^{-1}(W^i_j)$. Since $x \in p_b(p_a^{-1}(\widehat{W}^i_j)) \subseteq p_b(X)$, we have that $x \in p_{ab}^{-1}(W^i_j) \cap p_b(X)$, proving the right inclusion. □

6. Corollaries

It is shown in [2] that (m,n)-dim $\leq k$ is preserved by inverse limits of inverse systems of compact Hausdorff spaces whose coordinate spaces have (m,n)-dim $\leq k$.

Theorem 6.1 ([2, Theorem 2.21]). *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact Hausdorff spaces X_a with (m,n)-dim$X_a \leq k$ for all $a \in A$, and let $X = \lim \mathbf{X}$. Then (m,n)-dim$X \leq k$.*

The following corollary of Theorem 5.2 is parallel to Theorem 6.1. It shows that (m,n)-dim $\leq k$ is preserved by limits of approximate systems whose coordinate spaces have (m,n)-dim $\leq k$. First we need to recall Proposition 2.10 from [2] which shows that (m,n)-dimension is weakly hereditary.

Proposition 6.2. *Suppose that X is a space with (m,n)-dim$X \leq k$. Then for each closed subspace A of X, (m,n)-dim$A \leq k$.*

Corollary 6.3. *Let $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$ be an approximate system such that for all $a \in A$, (m,n)-dim$X_a \leq k$, and let $X = \lim \mathbf{X}$. Then (m,n)-dim$X \leq k$.* □

Using the next fact, which is Proposition 2 of [8], we can strengthen Corollary 6.3.

Proposition 6.4. *Let $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$ be an approximate system and $X = \lim \mathbf{X}$. Suppose that $B \subseteq A$ is a cofinal subset of A. Then $\mathbf{Y} = \{X_a, \varepsilon_a, p_{aa'}, B\}$ is an approximate system. Let Y be the limit of $\mathbf{Y}$. Then the restriction $p = \pi | X$ of the projection $\pi : \prod\{X_a \mid a \in A\} \to \prod\{X_a \mid a \in B\}$ is a homeomorphism $p : X \to Y$.*

Corollary 6.5. *Let $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$ be an approximate system and $X = \lim \mathbf{X}$. If there exists a cofinal subset $B \subseteq A$ such that for all $a \in B$, (m,n)-dim$X_a \leq k$, then (m,n)-dim$X \leq k$.* □

Corollary 6.6. *Let $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$ be an approximate system and $X = \lim \mathbf{X}$. If there exists an $a \in A$ such that (m,n)-dim$X_{a'} \leq k$ for all $a' \geq a$, Then (m,n)-dim$X \leq k$.* □

References

[1] R. Engelking, Theory of dimensions finite and infinite, Heldermann Verlag, Lemgo, 1995.

[2] V. V. Fedorchuk, *Finite dimensions defined by means of m-coverings*, Mat. Vesnik **64** (2012), 347–360.

[3] V. V. Fedorchuk, *Estimates of the dimension* (m, n)*-dim*, Mosc. Univ. Math. Bull. **68** (2013), 177–181.

[4] V. V. Fedorchuk, *Inductive* (m, n)*-dimensions*, Topology Appl. **169** (2014), 120–135.

[5] V. Fedorchuk, *Valuations for dimension* (m, n)*-dim*, Matem. Vestnik, MUS (2013) (in Russian), in press.

[6] M. Lynam and L. R. Rubin, *Extensional maps and approximate inverse limits*, Topology Appl. **239** (2018), 324–336.

[7] S. Mardešić and L. R. Rubin, *Approximate inverse systems of compacta and covering dimension*, Pacific J. Math. **138** (1989), no. 1, 129–144.

[8] S. Mardešić and J. Segal, $\mathcal{P}$*-like continua and approximate inverse limits*, Math. Japon. **33** (1988), no. 6, 895–908.

[9] N. Martynchuk, *Factorization theorem for the dimension* (m, n)*-dim*, Mosc. Univ. Math. Bull. **68** (2013), 188–191.

[10] N. Martynchuk, *Transfinite extension of dimension function* (m, n)*-dim*, Topology Appl. **160** (2013), 2514–2522.

[11] N. Martynchuk, *Several remarks concerning* (m, n)*-dimensions*, Topology Appl. **178** (2014), 219–229.

[12] N. Martynchuk, *Theorem on universal spaces for dimension function* (m, n)*-dim*, Mat. Zametki (2013), in press.

M. Lynam
Department of Mathematics
East Central University
Ada, Oklahoma 74820
USA
E-mail: mlynam@ecok.edu

L. R. Rubin
Department of Mathematics
University of Oklahoma
Norman, Oklahoma 73019
USA
E-mail: lrubin@ou.edu

Received: 8.3.2019.
Revised: 22.4.2019. & 13.11.2019.