A Chaotic Jerk System with Different Types of Equilibria and its Application in Communication System

Karthikeyan RAJAGOPAL, Viet-Thanh PHAM, Serdar ÇİÇEK, Sajad JAFARI, Anitha KARTHIKEYAN, Sundaram ARUN

Abstract: In this paper, a new jerk system is designed. This system can display different characters of equilibrium points according to the value of its parameters. The proposed nonlinear oscillator can have both self-excited and hidden attractors. Dynamical properties of this system are investigated with the help of eigenvalues of equilibria, Lyapunov exponents' spectrum, and bifurcation diagrams. Also, an electronic circuit implementation is carried out to show the feasibility of this system. As an engineering application of this new chaotic jerk system, a chaotic communication system is realized by correlation delay shift keying. When the results of the communication system are examined, the transmitted information signal is successfully obtained in the receiving unit, and its performance efficiency is investigated in the presence of additive white Gaussian noise.

Keywords: chaotic; circuit; communication system; dynamics; equilibrium

1 INTRODUCTION

Most of the known chaotic systems are third-order differential equations with a finite number of equilibrium points [1-3]. Designing chaotic systems with particular features has been an exciting topic in the last years (for example chaotic systems with multi-wing [4, 5], with multi-stability [6], and fractional-order [7]). The numerical difficulties associated with the locating dynamical states whose basin of attraction does not overlap equilibrium points leads to the term "hidden attractor" [8]. Initially chaotic systems without equilibria were considered as "incomplete" or "misformulated" [9]. The first no-equilibrium chaotic system (Sprott A) was reported in 1994 [10]. Since then dynamical analysis of such systems became attractive and noticeable [11, 12]. Leonov & Kuznetsov studied [13, 14] and developed [15, 16] analytical and numerical methods to study the chaotic and hyper-chaotic hidden attractors.

The simplest of three-dimensional chaotic systems are the jerk systems [17]. Some chaotic jerk systems with two quadratic nonlinearities [18] and three quadratic nonlinearities [19] have been reported in the literature.

In [20] Piecewise-linear jerk function has been used to model simple chaotic systems. Sun and Sprott used a simple piecewise exponential nonlinear function and concluded that increasing nonlinearity will not end up with more chaos [21]. Vaidyanathan et al. announced a chaotic jerk system with two hyperbolic sinusoidal nonlinearities [22].

In chaos-based digital communication systems, each symbol to be transmitted is represented by a piece of the chaotic signal. Even if the same symbol information is repeatedly transmitted on this channel, the transmitted information will always be different because the chaotic signal is not periodic. Chaos-based communication systems have low predictability [23, 24]. Many chaos-based digital modulation and demodulation methods have been proposed such as Chaos Shift Keying (CSK), Chaotic On-Off Keying (COOK), Symmetric Chaos Shift Keying (SCSK), Differential Chaos Shift Keying (DCSK), Correlation Delay Shift Keying (CDSK), Frequency Modulated Differential Chaos Shift Keying (FM-DCSK)

for the field of communication [25-28]. In this study, chaos-based communication system by CDSK method with the new jerk chaotic system is designed for engineering application.

Motivated by the above discussions, we report a unique jerk system with different types of equilibrium points (depending on the parameters). Such systems are reported less in literature. We analyze the dynamical properties such as Lyapunov exponents' spectrum, eigenvalues, and bifurcation plots. We introduce the new chaotic jerk system in Section 2. In Section 3 the feasibility of this new system is verified through circuit implementation. Section 4 belongs to communication design, and finally, Section 5 is the conclusion of the paper.

The novelty of the system is that a simple 3D jerk system is showing multiple equilibrium types with self-excited and hidden oscillations. Also by just controlling the values of the parameters, we could change the type of the equilibrium.

2 JERK SYSTEM

Sprott proposed several cases of simple jerk systems [18, 19] of the form

$$\dddot{x} + a\ddot{x} + bx\dot{x}^2 + cx^3 = 0 \qquad (1)$$

which shows chaotic oscillations for e.g. $a = 3.6$, $b = -1$ and $c = 1$. We present a new chaotic jerk system by modifying Eq. (1)

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= a_1 z + a_2 x + a_3 y + a_4 x^3 + a_5 xy^2 + a_6
\end{aligned} \qquad (2)$$

The character of Eq. (2) can be discussed in two conditions $a_6 = 0$, $a_2 \neq 0$ and $a_6 \neq 0$, $a_2 = 0$. The equilibrium points of Eq. (2) for $a_6 = 0$, $a_2 \neq 0$ are $x = \sqrt{\dfrac{-a_2}{a_4}}$, $y = 0$, $z = 0$ and depend on the values of a_2. The system shows different types of equilibrium points.

Similarly, for the condition $a_6 \neq 0$, $a_2 = 0$, the equilibrium points are $x = \left(\dfrac{-a_6}{a_4}\right)^{\frac{1}{3}}$, $y = 0$, $z = 0$ and depending on the value of a_6 the systems have two types of equilibriums.

Tab. 1 shows the different types of equilibrium points,

the eigenvalues and the finite time Lyapunov exponents (LEs) of the systems for various choices of a_6 and a_2. The LEs are calculated using [29] for 20000 s.

The jerk Eq. (2) shows chaotic attractors (Fig. 1) for $a_1 = -3.6$, $a_3 = 0.1$, $a_4 = -1$, $a_5 = 1$ and the parameters a_6 and a_2 are taken as in Tab. 1.

Table 1 Types of equilibrium for Eq. (2)

Conditions		Eigen values	Equilibrium	LEs	Figure
$a_6 = 0$	$a_2 < 0$ $(a_2 = -1)$	-3.7 $0.0500 \pm 0.5175i$	Unstable spiral	0.225 0 -3.859	1a
	$a_2 > 0$ $(a_2 = 1.3)$	-	No equilibrium	0.153 0 -3.757	1b
$a_2 = 0$	$a_6 = 0$	0 0.0276 -3.6276	Non-hyperbolic	0.122 0 -3.724	1c
	$a_6 \neq 0$ $(a_6 = 0.01)$	$-3.6298 + 0.0000i$ $0.0149 \pm 0.0897i$	Unstable spiral	0.166 0 -3.769	1d
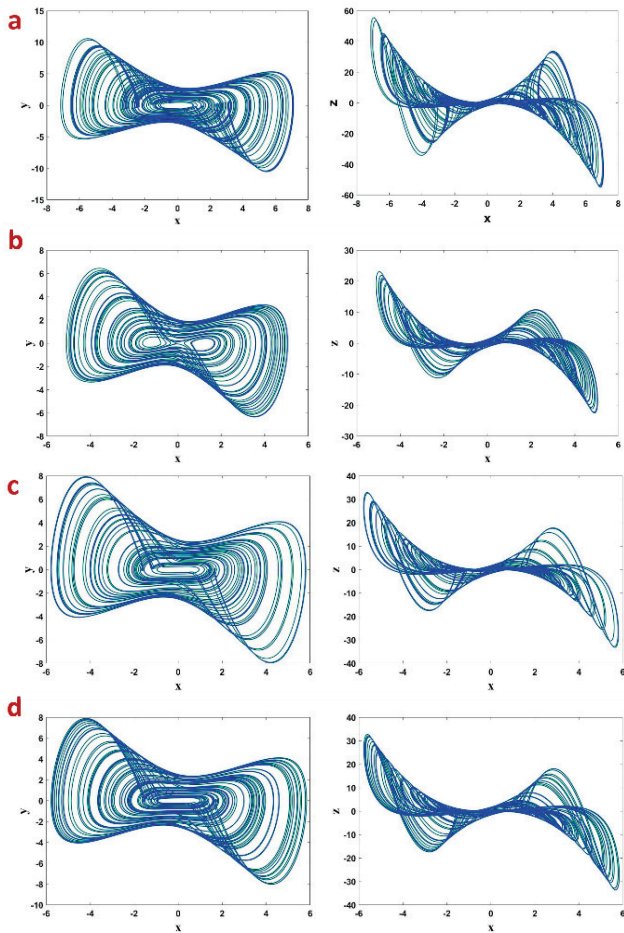
Figure 1 The 2D phase portraits of the system (1) with initial conditions [2, 2, −0.4]

Figure 2 Bifurcation of the jerk system for $a_6 = 0$ with (a) a_2 (b) a_3

Bifurcation plots are useful in investigating the characteristics of dynamical systems (see Fig. 2). We investigate the bifurcation of system (2) for the condition $a_6 = 0$ with a_2 and a_3 while the other parameters are fixed as $a_1 = -3.6$, $a_3 = 0.1$, $a_4 = -1$, $a_5 = 1$ and the initial condition for the first iteration is taken as [2, 2, −0.4] and is initialized to the end values of the state trajectories and the local maxima of the states plotted with the parameters. The jerk system exits chaos with period halving for the parameter a_2 and period doubling route to chaos with a_3.
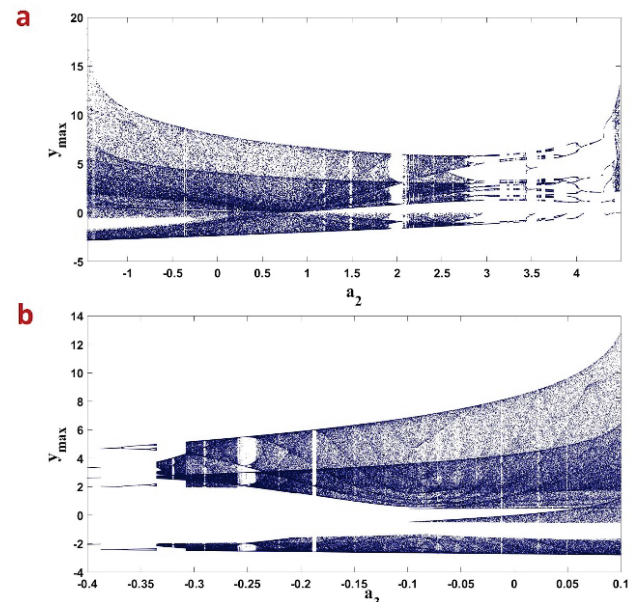
3 CIRCUIT DESIGN FOR THE JERK SYSTEM

In this section, we introduce a circuit design which provides a physical way to realize the theoretical jerk Eq. (2).

The circuit in Fig. 3 has been designed by using methods in [30-33], which are based on five operational amplifiers. As can be seen in Fig. 3, X, Y and Z are the voltages at the operational amplifiers U_1, U_2 and U_3.

From Fig. 3, the circuit equation is given by:

$$\dot{X} = \frac{1}{R_1 C_1} Y$$
$$\dot{Y} = \frac{1}{R_2 C_2} Z$$
$$\dot{Z} = -\frac{1}{R_3 C_3} Z - \frac{1}{R_4 C_3} X + \frac{1}{R_5 C_3} Y - \frac{1}{100 R_6 C_3} X^3 - \tag{3}$$
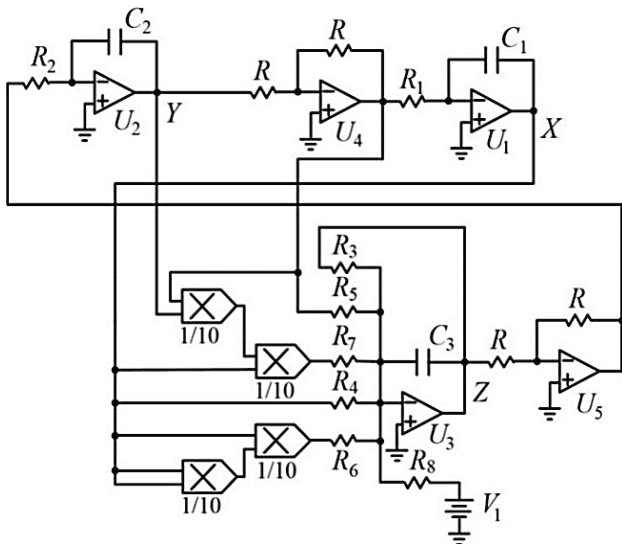$$- \frac{1}{100 R_7 C_3} XY^2 - \frac{1}{R_8 C_3} V_1$$

Figure 3 The electronic circuit designed for the new jerk system

It is simple to see that the circuit Eq. (3) agrees with the theoretical jerk Eq. (1). We have used the following electronics components to build the circuit: $R_1 = R_8 = R = 10$ kΩ, $R_3 = 2.778$ kΩ, $R_4 = 100$ kΩ, $R_5 = 1000$ kΩ, $R_2 = R_6 = R_7 = 1$ kΩ, $V_1 = 0V_{DC}$, $C_1 = C_2 = C_3 = C_4 = 10$ nF.

The electronic circuit has been implemented with PSpice. Fig. 4 presents PSpice chaotic attractors of the circuit for the case $a_2 = -1$, $a_6 = 0$.
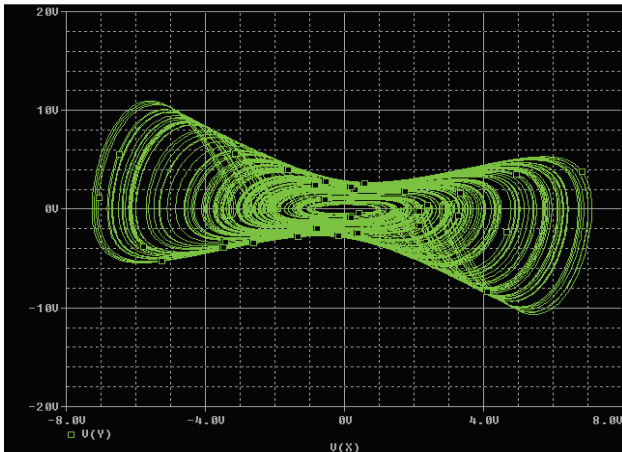
(a)
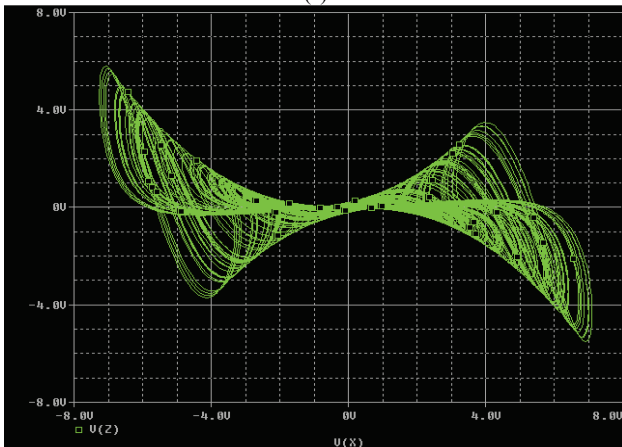
(b)

Figure 4 The 2D phase portraits of the electronic circuit obtained by using PSpice

4 COMMUNICATION DESIGN WITH THE NEW CHAOTIC JERK SYSTEM USING CDSK MODULATION

In this part, the new chaotic jerk system tested communication designs with the modulation method. Then, the BER performances of these communication systems were compared.

The CDSK modulation method can be considered as a type of DCSK method. The CDSK modulation method was proposed by Sushchic et al. in 2000. In the CDSK transmitter unit (TU), the chaotic signal $c(t)$ and delayed of this chaotic signal are multiplied by the information signal $m(t)$, then the chaotic signal is summed with the multiplication output signal and transmitted to the receiver unit. Block diagram of the CDSK transmitter unit is given in Fig. 5.
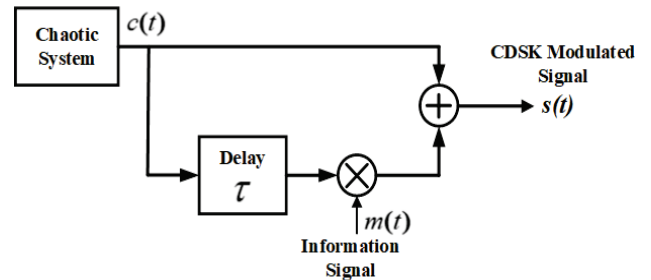

Figure 5 CDSK transmitter unit block diagram [28]

The modulated signal sent in the CDSK modulator is,

$$s(t) = \begin{cases} c(t) + c(t-\tau), & \text{"+1" information signal} \\ c(t) - c(t-\tau), & \text{"−1" information signal} \end{cases} \quad (4)$$

In Eq. (4), τ denotes the specified delay.

In the CDSK receiver unit (RU), the noisy incoming $n(t)$ signal $r(t)$ and its delayed version are multiplied and integrated into the correlator. The signal from the correlator is passed through the threshold detector to obtain the transmitted IS ~ $m(t)$.

The signal at the correlator output $o(iT_b)$ at the CDSK receiver unit,

$$o\left(iT_b\right) = \int\limits_{(i-1)T_b}^{iT_b} r_i\left(t\right) \cdot r_i\left(t-\tau\right) \mathrm{d}t =$$

$$= \int\limits_{(i-1)T_b}^{iT_b} \left[s_i\left(t\right) + n\left(t\right)\right] \cdot \left[s_i\left(t-\tau\right) + n\left(t-\tau\right)\right] \mathrm{d}t =$$

$$\int\limits_{(i-1)T_b}^{iT_b} s_i\left(t\right) \cdot s_i\left(t-\tau\right) \mathrm{d}t + \int\limits_{(i-1)T_b}^{iT_b} s_i\left(t\right) \cdot n\left(t-\tau\right) \mathrm{d}t +$$

$$+ \int\limits_{(i-1)T_b}^{iT_b} n\left(t\right) . s_i\left(t-\tau\right) \mathrm{d}t + \int\limits_{(i-1)T_b}^{iT_b} n\left(t\right) . n\left(t-\tau\right) \mathrm{d}t \quad (5)$$

In the Eq. (5), $n(t)$ denotes the noise. Eq. (5) becomes equal to Eq. (6) since $n(t) = 0$ in noiseless environment.

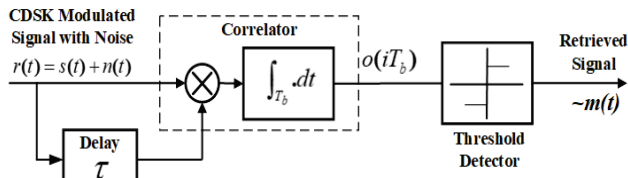The principle scheme of the CDSK RU is given in Fig. 6.

Figure 6 CDSK receiver unit block diagram [28]

$$o\left(iT_b\right) = \int\limits_{(i-1)T_b}^{iT_b} s_i\left(t\right) \cdot s_i\left(t - \tau\right) \mathrm{d}t \qquad (6)$$

The TU of the CDSK modulated communication system designed in the Matlab-Simulink® program using the new jerk CS is given in Fig. 7. In the TU, the x state variable of the system is used for the CS $c(t)$. The CS and the delayed of this chaotic signal by half the bit period ($\tau = T_b/2$) are multiplied, and this value is summed with the IS. The modulated signal thus obtained is sent to the RU.
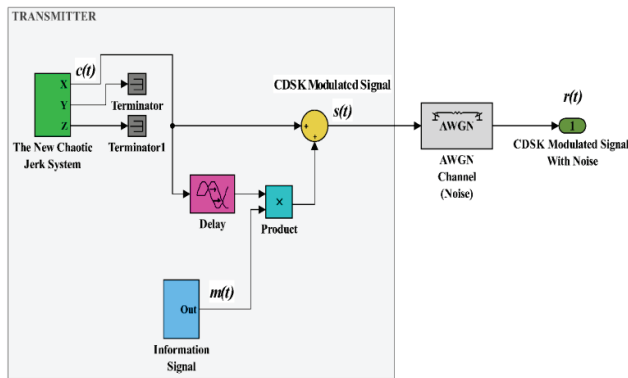

Figure 7 Matlab-Simulink® block diagram of the transmitter unit of the communication system with CDSK modulation method

The RU designed in the Matlab-Simulink® program using the new jerk chaotic system is given in Fig. 8. The modulation signal with noise $r(t)$ which is received by the RU is delayed by half the time of the bit period ($\tau = T_b/2$). Then these two signals are multiplied and integrated into the correlator unit. Thus the energy value of the signal is calculated. The correlator output signal is sent to the threshold detector and the IS $m(t)$ sent from the TU according to the threshold level is estimated, and the IS is obtained $\sim m(t)$.
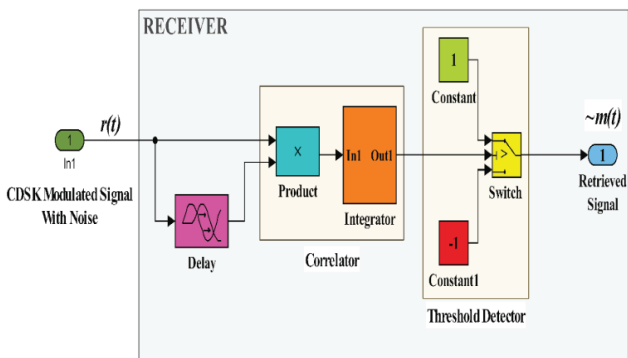

Figure 8 Matlab-Simulink® block diagram of the receiver unit of the communication system with CDSK modulation method

The system was tested under noise. Additive White Gaussian Noise (AWGN) was selected as a noise signal. The test was performed between 0 dB and −15 dB E_b/N_0

(Energy per bit to noise power spectral density ratio) values. In Fig. 9, the BER values graph is given.
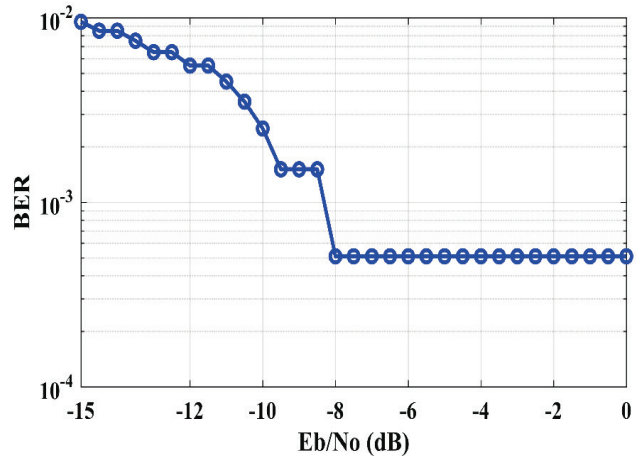

Figure 9 The BER performances of the CDSK modulated communication system using the new chaotic jerk system

The signal sent from the transmitter unit has been successfully received from the receiver unit up to a noise value of −8 dB E_b/N_0. In Fig. 10, the system simulated under the noise of −8 dB E_b/N_0 is given the IS, and the retrieved IS, respectively.
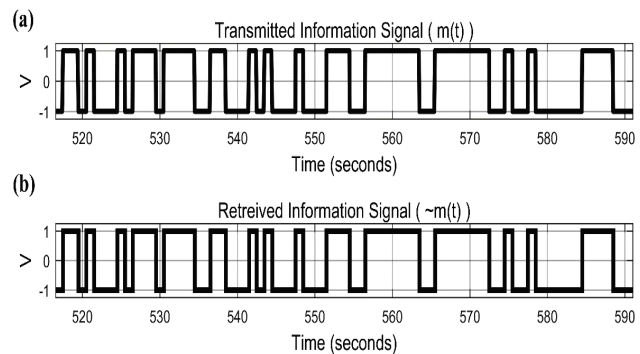

Figure 10 Signals of the CDSK modulated communication system using the new chaotic jerk system (a) transmitted information signal (b) retrieved information signal

5 CONCLUSION

A three-dimensional chaotic jerk system which shows different types of equilibrium points was reported in this research. Dynamical properties of this new system were analyzed by the help of Lyapunov exponents' spectrum and bifurcation plots. Analog circuit implementation was presented to show that the jerk system can be effectively realized in hardware. A correlation delay shift keying communication system was designed using the jerk system, and the performance efficiency of the communication model was analyzed in the presence of additive white Gaussian noise.

Acknowledgments

6 REFERENCES

[1] Lorenz, E. N. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, *20*, 130-41. https://doi.org/10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2

[2] Rössler, O. E. (1976). An equation for continuous chaos. *Physics Letters A*, *57*, 397-8. https://doi.org/10.1016/0375-9601(76)90101-8

[3] Chen, G. & Ueta, T. (1999). Yet another chaotic attractor. International *Journal of Bifurcation and Chaos*, *9*, 1465-6. https://doi.org/10.1142/S0218127499001024

[4] Shao-Bo, H., Ke-Hui, S., & Cong-Xu, Z. (2013). Complexity analyses of multi-wing chaotic systems. *Chinese Physics B*, *22*. https://doi.org/10.1088/1674-1056/22/5/050506

[5] Zhang, L., Sun, K., He, S., Wang, H., & Xu, Y. (2017). Solution and dynamics of a fractional-order 5-D hyperchaotic system with four wings. *The European Physical Journal Plus*, *132*, 31. https://doi.org/10.1140/epjp/i2017-11310-7

[6] Zhusubaliyev, Z. T. & Mosekilde, E. (2015). Multistability and hidden attractors in a multilevel DC/DC converter. *Mathematics and Computers in Simulation*, *109*, 32-45. https://doi.org/10.1016/j.matcom.2014.08.001

[7] Peng, D., Sun, K. H., & Alamodi, A. O. (2019). Dynamics analysis of fractional-order permanent magnet synchronous motor and its DSP implementation. *International Journal of Modern Physics B*. https://doi.org/10.1142/S0217979219500310

[8] Bragin, V., Vagaitsev, V., Kuznetsov, N., & Leonov, G. (2011). Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits. *Journal of Computer and Systems Sciences International*, *50*, 511-43. https://doi.org/10.1134/S106423071104006X

[9] Kuznetsov, A., Kuznetsov, S., Mosekilde, E., & Stankevich, N. (2015). Co-existing hidden attractors in a radio-physical oscillator system. *Journal of Physics A: Mathematical and Theoretical*, *48*. https://doi.org/10.1088/1751-8113/48/12/125101

[10] Sprott, J. C. (1994). Some simple chaotic flows. *Physical Review E*, *50*. https://doi.org/10.1103/PhysRevE.50.R647

[11] Wei, Z., Tang, Y., Chen, H., & Pehlivan, I. (2013). Adaptive reduced-order function projective synchronization and circuit design of hyperchaotic DLE with no equilibria. *Optoelectronics and Advanced Materials - Rapid Communications*, *7*, 984-90.

[12] Wei, Z., Wang, R., & Liu, A. (2014). A new finding of the existence of hidden hyperchaotic attractors with no equilibria. *Mathematics and Computers in Simulation*, *100*, 13-23. https://doi.org/10.1016/j.matcom.2014.01.001

[13] Leonov, G. & Kuznetsov, N. (2011). *Algorithms for searching for hidden oscillations in the Aizerman and Kalman problems*. Doklady Mathematics: Springer. https://doi.org/10.1134/S1064562411040120

[14] Leonov, G., Kuznetsov, N., & Vagaitsev, V. (2012). Hidden attractor in smooth Chua systems. *Physica D: Nonlinear Phenomena*, 241, 1482-6. https://doi.org/10.1016/j.physd.2012.05.016

[15] Leonov, G., Kuznetsov, N., Kiseleva, M., Solovyeva, E., & Zaretskiy, A. (2014). Hidden oscillations in mathematical model of drilling system actuated by induction motor with a wound rotor. *Nonlinear Dynamics*, *77*, 277-88. https://doi.org/10.1007/s11071-014-1292-6

[16] Leonov, G., Kuznetsov, N., & Mokaev, T. (2015). Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion. *The European Physical Journal Special Topics*, *224*, 1421-58. https://doi.org/10.1140/epjst/e2015-02470-3

[17] Schot, S. H. (1978). Jerk: the time rate of change of acceleration. *American Journal of Physics*, *46*, 1090-4. https://doi.org/10.1119/1.11504

[18] Sprott, J. (1997). Some simple chaotic jerk functions. *American Journal of Physics*, *65*, 537-43. https://doi.org/10.1119/1.18585

[19] Sprott, J. (1997). Simplest dissipative chaotic flow. *Physics Letters A*, *228*, 271-4. https://doi.org/10.1016/S0375-9601(97)00088-1

[20] Sprott, J. C. (2000). A new class of chaotic circuit. *Physics Letters A*, 266, 19-23. https://doi.org/10.1016/S0375-9601(00)00026-8

[21] Sun, K. & Sprott, J. (2009). A simple jerk system with piecewise exponential nonlinearity. *International Journal of Nonlinear Sciences and Numerical Simulation*, *10*, 1443-50. https://doi.org/10.1515/IJNSNS.2009.10.11-12.1443

[22] Vaidyanathan, S., Volos, C., Pham, V. T., Madhavan, K., & Idowu, B. A. (2014). Adaptive backstepping control, synchronization and circuit simulation of a 3-D novel jerk chaotic system with two hyperbolic sinusoidal nonlinearities. *Archives of Control Sciences*, *24*, 375-403. https://doi.org/10.2478/acsc-2014-0022

[23] Stavroulakis, P. (2005). *Chaos Applications in Telecommunications*: CRC press. https://doi.org/10.1201/9780203025314

[24] Yang, T. (2004). A survey of chaotic secure communication systems. *International Journal of Computational Cognition*, *2*, 81-130.

[25] Dedieu, H., Kennedy, M. P., & Hasler, M. (1993). Chaos shift keying: modulation and demodulation of a chaotic carrier using self-synchronizing Chua's circuits. *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, *40*, 634-42. https://doi.org/10.1109/82.246164

[26] Kolumban, G., Kennedy, M. P., Kis, G., & Jako, Z. (1998). FM-DCSK: A novel method for chaotic communications. *ISCAS'98 Proceedings of the 1998 IEEE International Symposium on Circuits and Systems (Cat No 98CH36187)*: IEEE, 477-80.

[27] Sushchik, M., Tsimring, L. S., & Volkovskii, A. R. (2000). Performance analysis of correlation-based communication schemes utilizing chaos. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *47*, 1684-91. https://doi.org/10.1109/81.899920

[28] Tam, W. M., Lau, F. C., & Chi, K. T. (2010). *Digital Communications with Chaos: Multiple Access Techniques and Performance*: Elsevier.

[29] Wolf, A., Swift, J. B., Swinney, H. L., & Vastano, J. A. (1985). Determining Lyapunov exponents from a time series. *Physica D: Nonlinear Phenomena*, *16*, 285-317. https://doi.org/10.1016/0167-2789(85)90011-9

[30] Buscarino, A., Fortuna, L., Frasca, M., & Sciuto, G. (2011). Design of time-delay chaotic electronic circuits. *IEEE Transactions on Circuits and Systems I: Regular Papers*, *58*, 1888-96. https://doi.org/10.1109/TCSI.2011.2107190

[31] Chen, M., Li, M., Yu, Q., Bao, B., Xu, Q., & Wang, J. (2015). Dynamics of self-excited attractors and hidden attractors in generalized memristor-based Chua's circuit. *Nonlinear Dynamics*, *81*, 215-26. https://doi.org/10.1007/s11071-015-1983-7

[32] Bao, B., Jiang, T., Xu, Q., Chen, M., Wu, H., & Hu, Y. (2016). Coexisting infinitely many attractors in active band-pass filter-based memristive circuit. *Nonlinear Dynamics*, *86*, 1711-23. https://doi.org/10.1007/s11071-016-2988-6

[33] Wu, H., Bao, B., Liu, Z., Xu, Q., & Jiang, P. (2016). Chaotic and periodic bursting phenomena in a memristive Wien-bridge oscillator. *Nonlinear Dynamics*, *83*, 893-903. https://doi.org/10.1007/s11071-015-2375-8

Contact information:

Karthikeyan RAJAGOPAL, PhD, Professor
Center for Nonlinear Dynamics, College of Engineering, Defence University,
Bishoftu, Ethiopia
E.mail: rkarthiekeyan@gmail.com

Viet-Thanh PHAM, PhD, Lecturer
(Corresponding author)
Nonlinear Systems and Applications, Faculty of Electrical and Electronics
Engineering, Ton Duc Thang University,
Ho Chi Minh City, Vietnam
E-mail: phamvietthanh@tdtu.edu.vn

Serdar ÇİÇEK, PhD, Assistant Professor
Department of Electronic & Automation, Vocational School of Hacibektas,
Nevsehir Haci Bektas Veli University,
50800 Hacibektaş, Nevşehir, Turkey
E-mail: serdarcicek@nevsehir.edu.tr

Sajad JAFARI, PhD, Assistant Professor
Department of Biomedical Engineering, Amirkabir University of Technology,
Tehran, 15875-4413, Iran
E-mail: sajadjafari@aut.ac.ir

Anitha KARTHIKEYAN, PhD, Assistant Professor
Center for Nonlinear Dynamics, College of Engineering, Defence University,
Bishoftu, Ethiopia
E-mail: mrs.anithakarthikeyan@gmail.com

Sundaram ARUN, PhD, Assistant Professor
Department of Electronics and Communication, Prathyusha Engineering College,
Chennai, India
E-mail: aruns@prathyusha.edu.in