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CAN CROATIAN URBAN HIERARCHY BE APPROXIMATED WITH THE FIBONACCI SEQUENCE? AN ANALYSIS ON HISTORICAL POPULATION DATA

UDC / UDK: 711.4(497.5)
JEL classification / JEL klasifikacija: P25, R11, R23
Original scientific paper / Izvorni znanstveni rad
Received / Primljeno: December 15, 2019 / 15. prosinca 2019.
Accepted for publishing / Prihvaćeno za tisak: June 8, 2020 / 8. lipnja 2020.

Abstract

Fibonacci numbers can be found in nature and have application in various fields of human activity. The Fibonacci sequence can also be used to predict population of settlements. The goal of this paper is to examine whether the hierarchy of the Croatian urban system can be approximated with the Fibonacci sequence on historical census data from 1857 to 2011. Actual values of urban rank according to settlements size were compared to the predicted values using two Fibonacci methods from the Fibonacci sequence. First method divides the population of the largest city by the golden ratio constant while the second method takes the population of each successive city and divides it by the golden ratio constant. The conducted analysis has shown that Croatian urban system conforms to the Fibonacci sequence with very good precision. Method 2 gives more precise overlaps of the actual number of inhabitants by settlements in the urban hierarchy of Croatia with the Fibonacci sequence than Method 1. If the largest city Zagreb is excluded from the analysis, the estimations are more precise with smaller mean MAPE.

Keywords: Croatia, urban system, Fibonacci sequence, rank-size rule

1. INTRODUCTION

Fibonacci numbers appear everywhere in nature and are applicable to growth of every living thing, from single cell to spiral galaxies. The most important application of Fibonacci sequence in financial economics is in stock price market prediction, (Talreja, 2014, Kumar, 2014, Lahutta, 2016, and Kandalgaonkar, 2015). Fibonacci retracements are popular tool for technical analysis in the stock market primarily related to strategies for trading using (Giryn and Kozubski, 2012). Fibonacci retracements can be used to predict trends on forex market using graphics such as Fibonacci arcs, channel, fan and expansion, (Gaucan and Maiorescu, 2011). Other applications of Fibonacci sequence in the economics are in the field of financial mathematics, Goetzmann (2004), microeconomic relations, Suleiman (2017), artificial intelligence, Tanackov, Tepić and Kostelac (2011), Kalman filter and optimal control, Benavoli, Chisci and Farina (2009) and even on betting on soccer draws (Lahvicka, 2013).

Fibonacci sequence can be also used to predict population of settlements. The use of Fibonacci numbers in analyzing urban rank-size hierarchy so far has been isolated and scarce. Fonseca (1988) was first to investigate the U.S. urban area population system in 1970s and 1980s using Fibonacci sequence. He came to the conclusion that Fibonacci sequence was a valuable tool for studying urban population systems. This paper is based on Fonseca's work and follows up on Žmuk and Jošić (2019) who investigated urban (ir)regularities such as Zipf's law and Fibonacci growth sequence for Eastern Croatia in the case of decreasing population in the period from 1991 to 2011. The goal of this paper is to explore whether Croatian urban rank-size system follows Fibonacci growth sequence. We conducted an analysis using Croatian historical census data, from the year 1857 to 2011.

Therefore, the research question of this paper which will be investigated is whether Croatian urban system can be approximated with the Fibonacci growth sequence. In the analysis two Fibonacci methods will be used. The first method takes into account population of the largest city and divides it by golden ratio constant. The second method takes the population of each successive city and divides it by the golden ratio constant. The precision of the two forecasting methods based on the Fibonacci series will be checked by following overall forecasting errors; mean squared error (MSE), root mean squared error (RMSE) and mean absolute percentage error (MAPE). The analysis will be firstly conducted with inclusion of the largest city Zagreb, and after that without Zagreb in order to get more precise results, because it has been noticed that capital city is disproportionately larger than any other city in the urban hierarchy.

This paper is structured in six chapters. After the brief introduction, the second chapter presents literature review displaying main theoretical postulates in urban economics. In section three the Fibonacci sequence, golden ratio, golden rectangles and the Fibonacci growth spiral are explained and elaborated while methodological issues and data are presented in section four. Main results of the analysis and discussion are displayed in chapter five. In the last chapter concluding remarks are given.

2. LITERATURE OVERVIEW

As the goal of this paper is using the Fibonacci sequence to determine Croatia's urban rank-size hierarchy on historical population data, in this section main theoretical postulates in urban economics will be examined and explained. The most important urban regularity in the literature in this field is Zipf's law or rank-size rule. Another important regularity in urban economics is Gibrat's law which states that proportionate growth of cities is independent of its size (Gibrat, 1931). Auerbach (1913) was the originator of Zipf's law in urban economics but got its name after George Kinsley Zipf (Zipf, 1949). It stated that city size is inversely proportional to its rank, respectively the second largest city is two times smaller than the largest city, the third largest city is three times smaller than the largest city and so on. Many studies have been conducted in this field in the last few decades with results pointing to the conclusion that the Zipf's law holds in the upper tail of the Pareto distribution (Rosen and Resnick, 1980, Krugman, 1996, Jiang and Jia, 2011). However, there was some scepticism about validity of Zipf's law for all cities (Gabaix, 1999, 1999a, Soo, 2005). Fonseca (1988) demonstrated a new interpretation of a rank-size regularity, data which conform to the rank-size pattern approximate the pattern of an equiangular curve. He calculated predicted population of U.S. urbanized areas using data for 1970 and 1980 census. Urbanised area data are used because they are more scientifically defined than city populations or standard metropolitan area data. It appeared that spiral constant may be able to predict urban distributions more accurately than rank-size.

Research works in the field of urban economics in Croatia have been made by Jošić and Nikić (2013) and Jošić and Bašić (2018) on the population data for Croatia using 2011 Census of Population Survey. In both cases results corroborated to the existence of Zipf's law in Croatian urban system. Furthermore, rank-size distribution for settlements in Croatia holds for the majority of the settlements' sizes but it did not hold for extremely small and large sizes of settlements. The Croatian rank-size distribution indicates that Zagreb is too large in comparison to other large Croatian cities, Bačić and Šišinački (2014), but according to authors there is a potential for Croatia's polycentric development. The similar statement can be said for Serbian urban system. The urban system is marked by Belgrade and its prominent domination in terms of the size, (Živanović et al, 2019).

Žmuk and Jošić (2019) conducted the analysis on Eastern Croatia urban rank-size system using 1991, 2001 and 2011 population data. Five counties form the area of Eastern Croatia; the County of Osijek-Baranja, County of Požega-Slavonia, County of Slavonski Brod-Posavina, County of Virovitica-Podravina and County of Vukovar-Sirmium. In the analysis, two forecasting Fibonacci methods were used. The precision of these two models was checked using MSE, RMSE and MAPE metrics. The analysis has shown that the structure of Eastern Croatia's urban population system comports with the Fibonacci sequence. Mean absolute percentage errors (MAPE) for both models were relatively small (8.32% and 13.08%) implying small estimation errors between actual data and those predicted with the Fibonacci growth sequence.

3. FIBONACCI SEQUENCE

The Fibonacci sequence is a series of numbers:

$$0,1,1,2,3,5,8,13,21,34,55,89,144,233,\ldots \quad (1)$$

developed by the famous Italian mathematician Leonardo of Pisa, known as Fibonacci, in his 1202 book *Liber Abaci* in which he introduced this sequence to Western European Mathematics. The Fibonacci sequence has its origins in Indian mathematics. According to Goonatilake (1998) the development of the Fibonacci sequence is attributed to Pingala (200 BC), Virahanka (700 AD), Gopāla (1135 AD) and Hemachandra (1150 AD). The expression „Fibonacci sequence" was coined by a 19th century French mathematician Édouard Lucas who studied the Fibonacci sequence and invented his own Lucas series similar to the Fibonacci numbers.

In the Fibonacci sequence each number in the series is found by adding the two numbers before it. This expression can be written as:

$$x_n = x_{n-1} + x_{n-2} \quad (2)$$

with seed values $x_0 = 0$ and $x_1 = 1$.

Although, Fibonacci rabbits are unrealistic condition, the Fibonacci numbers do appear in nature such as flower petals, seed heads, pinecones, fruit and vegetables, shells, spiral galaxies, hurricanes, human body, DNA molecules, etc., (Watson, 2017). Golden ratio and the Fibonacci numbers are specially connected. The ratio of any two successive Fibonacci numbers is very close to the golden ratio:

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi \quad (3)$$

where $a > b > 0$. Ratio of a to b is equal to the ratio of whole length $(a+b)$ to the larger section a as shown in Figure 1.

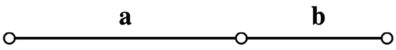


Figure 1: Golden ratio

The outcome of Equation 3 is an irrational number often called "golden number" φ (Greek letter phi) or sometimes τ (tau) which represents the golden ratio Debnath (2011). The method for finding the value of φ continues on Equation 3 and substitutes $b/a = 1/\varphi$:

$$\frac{a+b}{a}=\frac{a}{a}+\frac{b}{a}=1+\frac{b}{a}=1+\frac{1}{\varphi}, \tag{4}$$

implying that

$$1+\frac{1}{\varphi}=\varphi\,. \tag{5}$$

Multiplying with φ gives the quadratic equation:

$$\varphi^2-\varphi-1=0, \tag{6}$$

with two solutions

$$\varphi_1=\frac{1+\sqrt{5}}{2}=1.6180339887 \tag{7}$$

and

$$\varphi_2=\frac{1-\sqrt{5}}{2}=-0.6180339887\,. \tag{8}$$

In Figure 2 golden rectangles are illustrated whose sides are in the golden ratio.

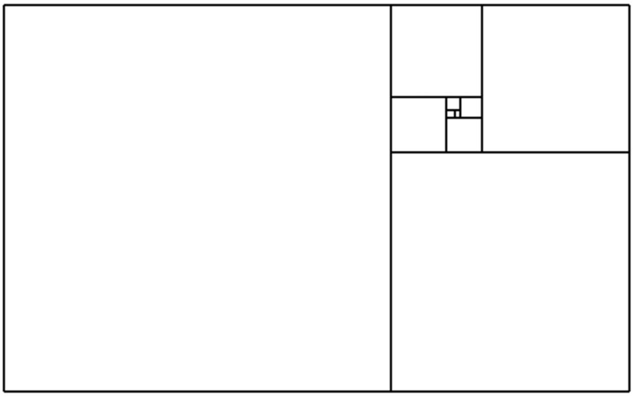


Figure 2 Golden rectangles

Source: Harvard Math Department, 2018

Inside the shorter side of the golden rectangle can be inserted a similar golden rectangle and so on. In Figure 3 is illustrated Fibonacci spiral.

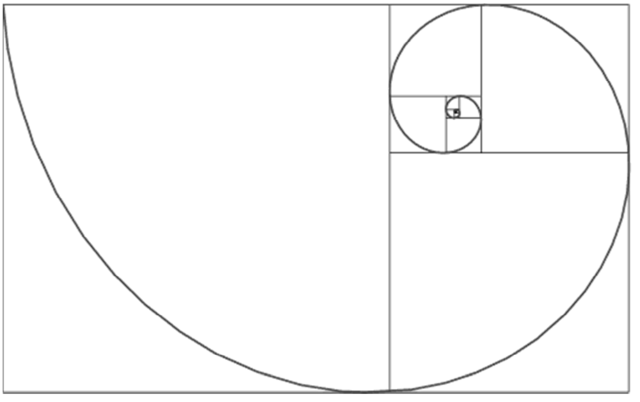


Figure 3 Fibonacci spiral

Source: Wolfram MathWorld (2018)

Logarithmic or equiangular spiral is a growth spiral whose polar equation is given by:

$$r = ae^{b\theta}, \tag{9}$$

where r is distance from origin, a and b are constants and θ is the angle from the x-axis. It is related to Fibonacci numbers, golden ratio and golden rectangle (Weisstein, 2018). The term equiangular refers to constant angle maintained between of extension of any radius of the curve and a line tangent to the curve at the point where the two met (Fonseca, 1988).

4. METHODOLOGY AND DATA

This paper uses rank-size distribution of cities which is the distribution of size by rank. This distribution usually follows the power law distribution. Size distribution of cities based on the Pareto distribution was firstly introduced by Felix Auerbach in 1913 (Auerbach, 1913):

$$R = CP^{-\alpha}, \tag{10}$$

where R is the city rank (largest city gets the rank one, second largest city gets the rank two and so on), C is a constant while P is the population of a city. α is a Pareto exponent which takes negative values but for the economic purposes is usually expressed in absolute value. This equation can also be written using natural logarithms. Right-skewed city size distribution provides a better fit for samples encompassing small cities (Gabaix, 1999a):

$$\ln(R)=\ln(C)-\alpha\ln(P). \tag{11}$$

Pareto exponent α is often employed as a measure of population concentration among cities of different sizes, (Zipf, 1949). When the α equals one, the values are centred around an average value and the rank-size rule is validated. Higher values of Pareto coefficient imply that the urban system is less concentrated. On the other hand, lower values of Pareto exponent imply more inequality between cities and more hierarchy between them. In order to predict population in settlements, two different methods which are based on Fibonacci series are going to be applied, (Fonseca, 1988). At the first step, the settlements are ranked according to their population size in decreasing order. After that the Fibonacci series is used to select settlements for which the population will be predicted. According to Method 1, the population from the largest settlement is taken and divided by φ_1 from Equation 7:

$$FP_{FR}=\frac{y_1}{\varphi_1^{FR-1}},\ FR=2,3,4,\ldots, \tag{12}$$

where FP is predicted population of a settlement, FR is the settlement rank according to the Fibonacci series (the largest settlement gets the rank one, the second largest settlement gets the rank two, the third largest settlement gets the rank three, the fifth largest settlement gets the rank four, the eight largest settlement gets the rank five and so on), y_1 is the actual population size in the largest settlement, φ_1 is the golden ratio (value of 1.6180339887). Contrary to Method 1, Method 2 takes into account the population of the previous ranked settlement and divides it by φ_1 from Equation 7:

$$FP_{FR}=\frac{y_{FR-1}}{\varphi_1},\ FR=2,3,4,\ldots \tag{13}$$

where FP is predicted population of a settlement, FR is the settlement rank according to the Fibonacci series (the largest settlement gets the rank one, the second largest settlement gets the rank two, the third largest settlement gets the rank three, the fifth largest settlement gets the rank four, the eight largest settlement gets the rank five and so on), y_{FR-1} is the actual population size at the previously Fibonacci ranked settlement, φ_1 is the golden ratio (value of 1.6180339887). The precision of the two prediction methods based on the Fibonacci series will be inspected by following overall prediction errors:

$$MSE=\frac{\sum_{FR=2}^{n}\left(y_{FR}-FP_{FR}\right)^2}{n}, \tag{14}$$

$$RMSE = \sqrt{\frac{\sum_{FR=2}^{n} \left(y_{FR} - FP_{FR} \right)^2}{n}}, \tag{15}$$

$$MAPE = \frac{\sum_{FR=2}^{n} \left| \frac{y_{FR} - FP_{FR}}{y_{FR}} \right| \cdot 100}{n}, \tag{16}$$

where MSE is mean squared error, FR is the settlement rank according to the Fibonacci series, y_{FR} is the actual population size at certain Fibonacci ranked settlement, FP_{FR} is predicted population at certain Fibonacci ranked settlement, n is the number of Fibonacci ranked settlements for which population is predicted, $RMSE$ is root mean squared error, $MAPE$ is a mean absolute percentage error. At all three overall prediction errors, the calculation begins from the second Fibonacci ranked settlement because there are no predicted values for the largest settlement. The lower the overall prediction errors are, the used method is considered to be better and more accurate than the other one. Data for analysis are provided from the Croatian Bureau of Statistics for years from 1857 to 2011, (Croatian Bureau of Statistics, 2018a and Croatian Bureau of Statistics, 2018b).

5. RESULTS AND DISCUSSION

In the analysis, population of Croatia according to the population censuses conducted in the period from 1857 to 2011 is observed. The population censuses include all population in a country. In this way full information about population and it characteristics can be obtained. However, population censuses are very expensive. Because of that, population censuses in Croatia are usually conducted only once per decade. The first population census conducted in Croatia was in 1857. Due to the Second World War there was unusually high gap of 17 years between two population censuses. Since 1961 population censuses are conducted every 10 years or, in other words, in year ending with number one. Table 1 shows basic descriptive statistics results for the 16 observed population censuses in Croatia so far, from year 1857 to 2011.

Table 1

Descriptive statistics results for population in Croatia, population censuses from 1857 to 2011

Statistics	Census year							
	1857	1869	1880	1890	1900	1910	1921	1931
No of settlements	5,382	5,128	5,910	6,238	6,347	6,345	5,665	5,829
Population (in mil.)	2.181	2.398	2.506	2.854	3.161	3.460	3.443	3.785
Mean	405	468	424	457	498	545	608	649
Mode	133	156	105	96	118	70	158	124
Standard deviation	778	927	1,040	1,220	1,567	1,984	2,327	3,375
Coeff. of variation	192	198	245	267	315	364	383	520
Kurtosis	626	600	923	1,215	1,717	1,734	2,434	3,571
Skewness	19	19	25	29	36	37	44	55
Range	32,200	36,998	48,135	60,908	87,236	109,028	140,813	227,836
Minimum	3	3	1	2	3	1	2	2
1st quartile	132	148	125	136	144	156	178	189
Median	237	270	239	264	282	306	343	360
3rd quartile	453	523	465	499	540	586	650	683
Maximum	32,203	37,001	48,136	60,910	87,239	109,029	140,815	227,838

Table 1

Descriptive statistics results for population in Croatia, population censuses from 1857 to 2011 – continued

Statistics	Census year							
	1948	1953	1961	1971	1981	1991	2001	2011
No of settlements	6,624	6,665	6,664	6,673	6,548	6,533	6,651	6,560
Population (in mil.)	3.779	3.935	4.159	4.426	4.601	4.784	4.437	4.276
Mean	571	590	624	663	703	732	667	652
Mode	138	206	93	102	79	69	4	2
Standard deviation	4,261	4,742	5,854	7,701	8,941	9,666	9,257	9,207
Coeff. of variation	747	803	938	1,161	1,273	1,320	1,388	1,413
Kurtosis	5,101	5,056	4,906	4,824	4,455	4,397	4,701	4,765
Skewness	68	67	66	66	63	62	65	65
Range	325,221	361,563	442,767	579,942	656,379	706,769	691,723	688,162
Minimum	2	1	1	1	1	1	1	1
1st quartile	150	150	138	117	97	87	57	51
Median	296	295	283	248	218	199	157	143
3rd quartile	568	567	556	510	471	447	382	358
Maximum	325,223	361,564	442,768	579,943	656,380	706,770	691,724	688,163

Source: authors' calculation.

According to the Table 1 the lowest number of settlements was in 1869, when there were 5,128 settlements, whereas the highest number of settlements, with value of 6,673 settlements, was registered in 1971. The population of

Croatia had a steady increase, except the population census right after the First World War in 1921, until 1991. In the last two population censuses, which were conducted in 2001 and in 2011, a quite large decrease of population was present. Namely, according to the population census conducted in 2011, Croatia had 4.276 million of citizens which is quite similar to the population size obtained with the population census in 1961. The average settlement size ranged from 405, according to population census in 1857, to 732 citizens, according to population census in 1991.

However, the obtained statistical results showed that there are huge differences in settlement sizes and that the distributions of settlements according to their size are highly positively skewed in all census years. Because of that median is here superior measure of central tendency than mean. Since 1948 population census, median values of settlement size are decreasing, which is mainly due to migration of population to the largest cities. In all population censuses capital Zagreb had the most citizens.

In Figure 4 the comparison of settlements, selected by the Fibonacci sequence, spiral constant and the ideal equiangular curve using data from population census conducted in 2011, is shown. In order to keep clarity of the figure only first eight settlements are shown. The rank of the settlements is determined by the Fibonacci sequence. Zagreb as the largest city got the rank 1 with the value of the angle from the x-axis, $\theta = 0^{\circ}$, Split as the second largest city got the rank 2 with $\theta = 90^{\circ}$, Rijeka got the rank 3 with $\theta = 180^{\circ}$, Zadar got the rank 5 with $\theta = 270^{\circ}$and so on.

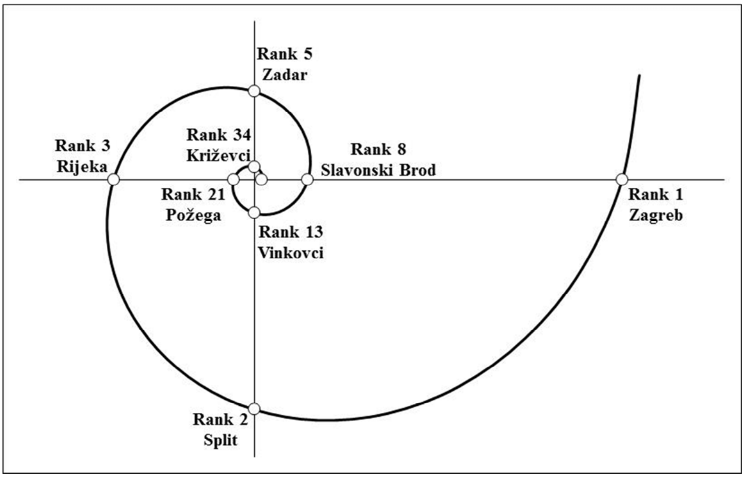


Figure 4 Comparison of settlements, selected by using Fibonacci series, by the spiral constant and the ideal equiangular curve, data from population census conducted in 2011

Source: authors

In Table 2 the results of predicted population in settlements according to the methods 1 and 2 (equations 12 and 13) for the population census from 2011 are provided.

Table 2

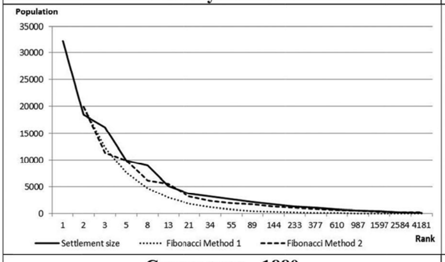
Predicted population in settlements by using Fibonacci series, applied Method 1 and Method 2, data from population census conducted in 2011

Fibonacci series / Settlement rank	Settlement	Settlement size	Method 1	Method 2
1	Zagreb	688,163	-----	-----
2	Split	167,121	425,308	425,308
3	Rijeka	128,384	262,855	103,286
5	Zadar	71,471	162,453	79,346
8	Slavonski Brod	53,531	100,402	44,172
13	Vinkovci	32,029	62,052	33,084
21	Požega	19,506	38,350	19,795
34	Križevci	11,231	23,702	12,055
55	Kaštel Sućurac	6,829	14,648	6,941
89	Prelog	4,324	9,053	4,221
144	Dugopolje	2,993	5,595	2,672
233	Mokošica	1,924	3,458	1,850
377	Novoselec	1,362	2,137	1,189
610	Funtana	907	1,321	842
987	Generalski Stol	589	816	561
1,597	Gornji Prnjarovec	369	505	364
2,584	Sedramić	205	312	228
4,181	Gostenje	87	193	127

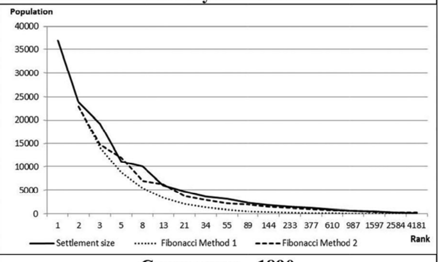
Source: authors' calculation

The full list of settlements, which have been selected by using Fibonacci series, along with their rank, name and population for all observed population censuses is given in Appendix in Table A1. According to the population census conducted in 2011, Croatia had 6,560 settlements with one or more citizens. Consequently, the last used number from the Fibonacci series is 4,181. It has to be emphasized that the first ranked settlement is the largest one. Actual settlement sizes are taken for chosen ranked settlements and Method 1 and Method 2 are applied to predict settlement size of the chosen settlements. In Table 2 predicted values are given just for the population census from 2011 as an example. Results of predictions by using Fibonacci series and Method 1 and Method 2 are graphically presented in Figure 5 for all census years.

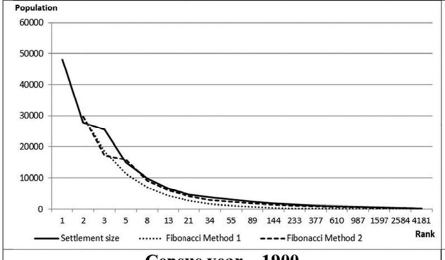
Census year – 1857



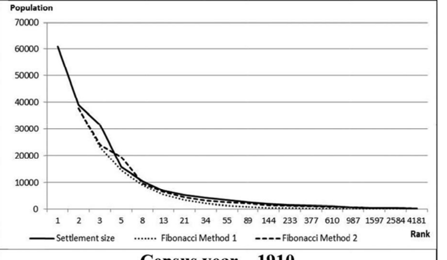
Census year – 1869



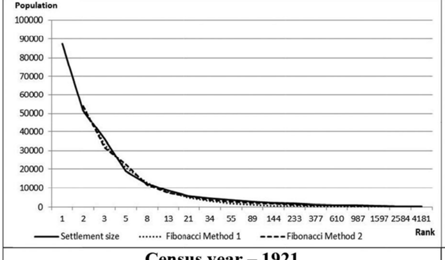
Census year – 1880



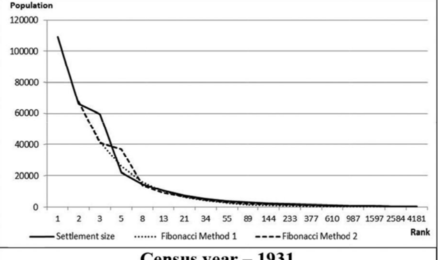
Census year – 1890



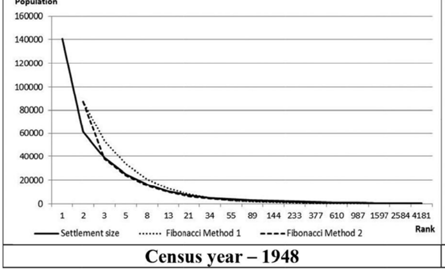
Census year – 1900



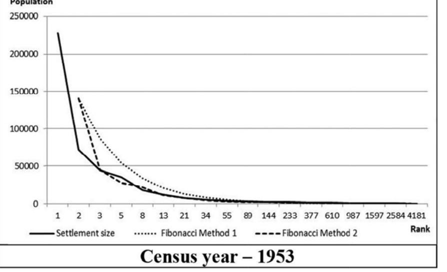
Census year – 1910



Census year – 1921



Census year – 1931



Census year – 1948

Census year – 1953

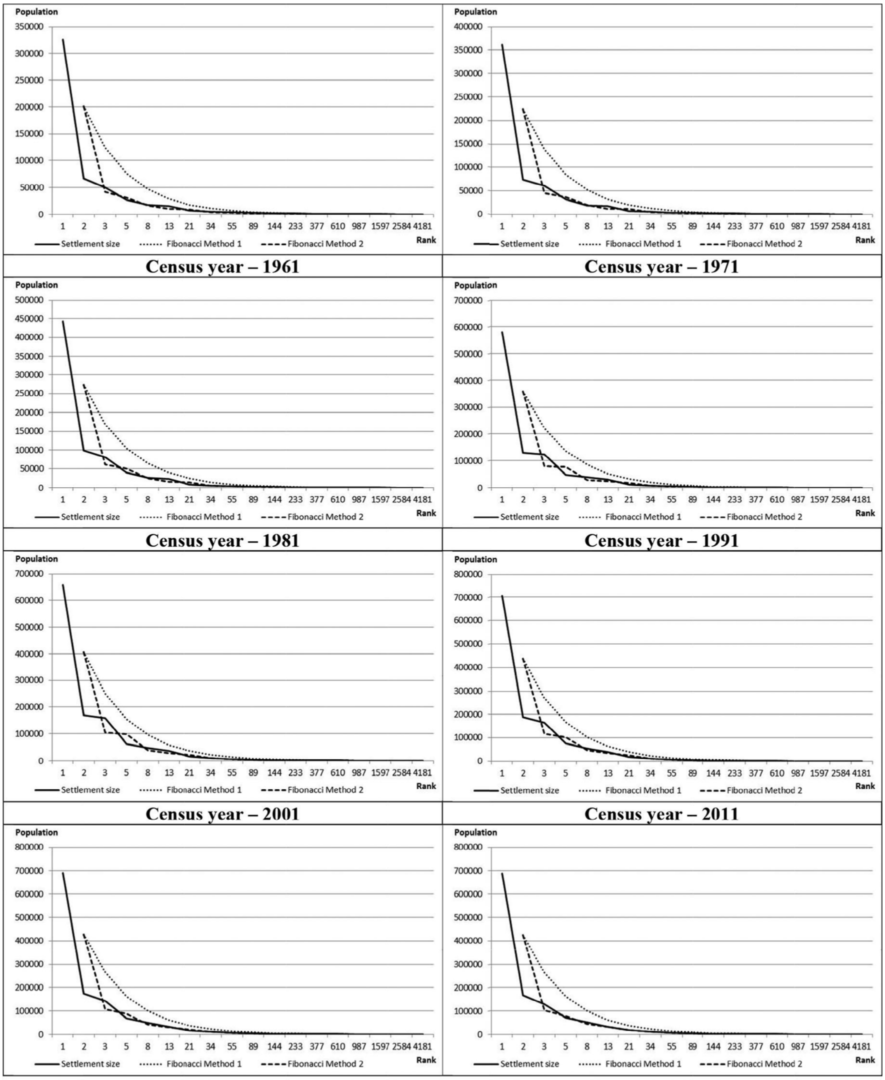


Figure 5 Predicted population in settlements by using Fibonacci series, applied Method 1 and Method 2, population censuses from 1857 to 2011

Source: authors' calculation

In Figure 5 the actual settlement sizes and predicted values calculated by Method 1 and 2 are provided. In that way the precision of the used methods can be observed. The closer the predicted line is to the line of actual values, the more precise predicted values are. Generally speaking, the used methods seem not to be so precise when settlements with lower ranks are observed (that would be the largest settlements). On the other hand, the used methods seem to be more precise in predicting settlement sizes when small settlements are considered. However, Figure 5 does not reveal which of the two used methods is better.

Table 3

Mean squared errors, root mean squared errors and mean absolute percentage errors of applied Method 1 and Method 2, population censuses from 1857 to 2011

Census year	Method 1			Method 2		
	Mean squared error	Root mean squared error	Mean absolute percentage error	Mean squared error	Root mean squared error	Mean absolute percentage error
1857	3,809,035	1,952	62	2,033,289	1,426	16
1869	5,179,341	2,276	62	2,046,204	1,430	16
1880	5,756,925	2,399	57	4,413,454	2,101	15
1890	5,565,635	2,359	52	4,422,466	2,103	16
1900	1,838,820	1,356	44	2,416,607	1,555	14
1910	20,648,624	4,544	43	34,236,067	5,851	18
1921	58,754,309	7,665	43	39,670,048	6,298	14
1931	427,891,223	20,686	57	283,758,937	16,845	19
1948	1,602,069,304	40,026	88	1,063,912,859	32,618	28
1953	1,932,738,013	43,963	86	1,339,540,986	36,600	29
1961	2,619,385,097	51,180	80	1,832,347,644	42,806	27
1971	4,312,754,912	65,672	83	3,264,225,551	57,133	29
1981	4,490,375,279	67,010	74	3,533,096,380	59,440	23
1991	4,925,832,978	70,184	72	3,789,539,327	61,559	19
2001	5,364,513,958	73,243	81	3,846,107,487	62,017	18
2011	5,689,594,172	75,429	82	3,967,184,288	62,986	16

Source: authors' calculation

Overall prediction errors mean squared error (MSE), root mean squared error (RMSE) and mean absolute percentage error (MAPE) are calculated and given in Table 3. In order to get better insight which method is more precise, RMSE or MAPE, the values of the two used methods are graphically compared in Figure 6 as well.

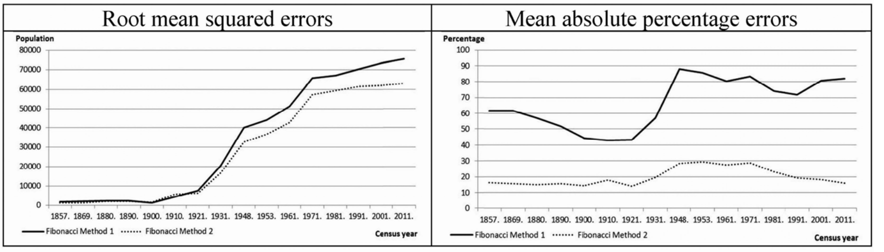


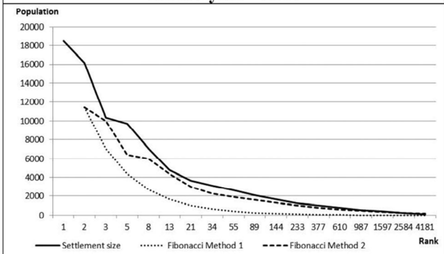
Figure 6: Root mean squared errors and mean absolute percentage errors of applied Method 1 and Method 2, population censuses from 1857 to 2011

Source: authors' calculation

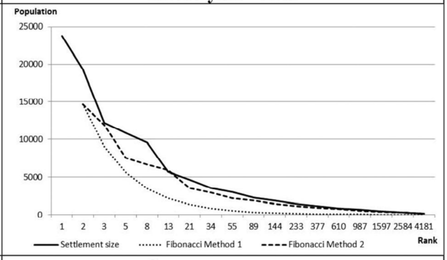
If values from Table 3 for Methods 1 and Method 2 are observed, it can be concluded that at the first couple of population censuses the MSE and RMSE for both used method were quite stable. After the population census conducted in 1900, the values of MSE and RMSE are continuously increasing up to the last population census in 2011. It can be concluded that the precision of both used methods is declining by each following census. According to the Figure 6, Methods 1 and 2 have about the same level of precision until population census conducted in 1921. After population census in 1921, Method 2 has shown to be more precise than Method 1. On the other side MAPE, according to results from Table 3, seems to be quite stable at Method 2 whereas MAPE value at Method 1 increased for about 20 percentage points in the observed period. Figure 6 shows that, according to MAPE, Method 2 seems to be more precise than Method 1 at all population censuses. The results from Table 3 and Figure 6 show that Method 1 did not achieve satisfactory level of precision. That is especially emphasized when MSE and RMSE values are observed. On the other hand, Method 2 seems to have good level of precision throughout of the observed period. According to Lewis (1982:40) the cut-off values for MAPE are as follows: <10 percent is highly accurate forecasting (in the paper it will be used as a prediction level), 10-20 percent is good forecasting, 20-50 percent is reasonable forecasting and >50 percent is inaccurate forecasting. It can be noticed that in the most of censuses the value of MAPE belongs to the good forecasting (prediction) level.

As previously noticed, the capital city Zagreb is too large in comparison with other large Croatian cities, therefore the largest city problem arises. In other word, it is suspected that there is too big a difference in size between Zagreb and the second largest settlement Split which is having a powerful impact on the results. Consequently, it has been decided to omit the largest settlement from the analysis and to repeat the analysis. Due to omission of the largest city Zagreb from the analysis, the rank one now has the second largest settlement Split, the rank two has the third largest settlement Rijeka and so on. The new predicted values with omitted capital city and by applying Methods 1 and Method 2 are given in Figure 7 for all census years.

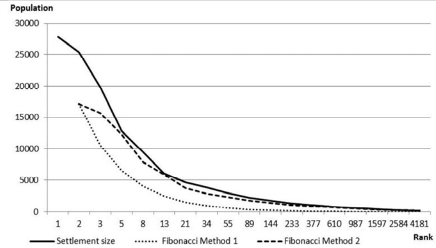
Census year – 1857



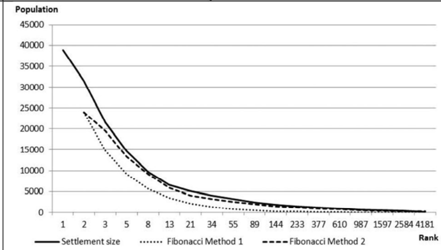
Census year – 1869



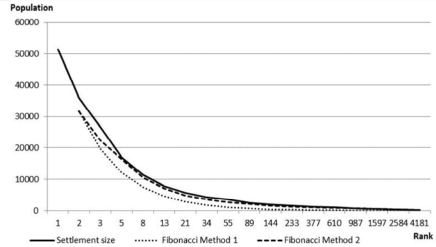
Census year – 1880



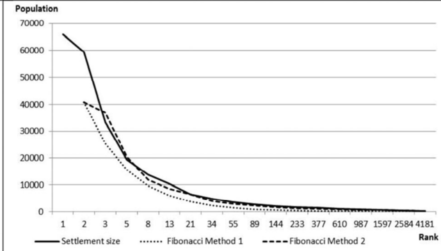
Census year – 1890



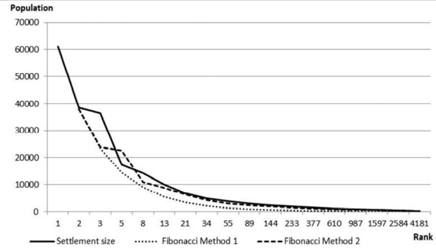
Census year – 1900



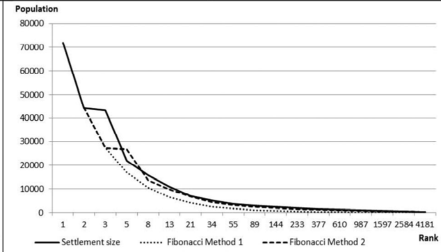
Census year – 1910



Census year – 1921



Census year – 1931



Census year – 1948

Census year – 1953

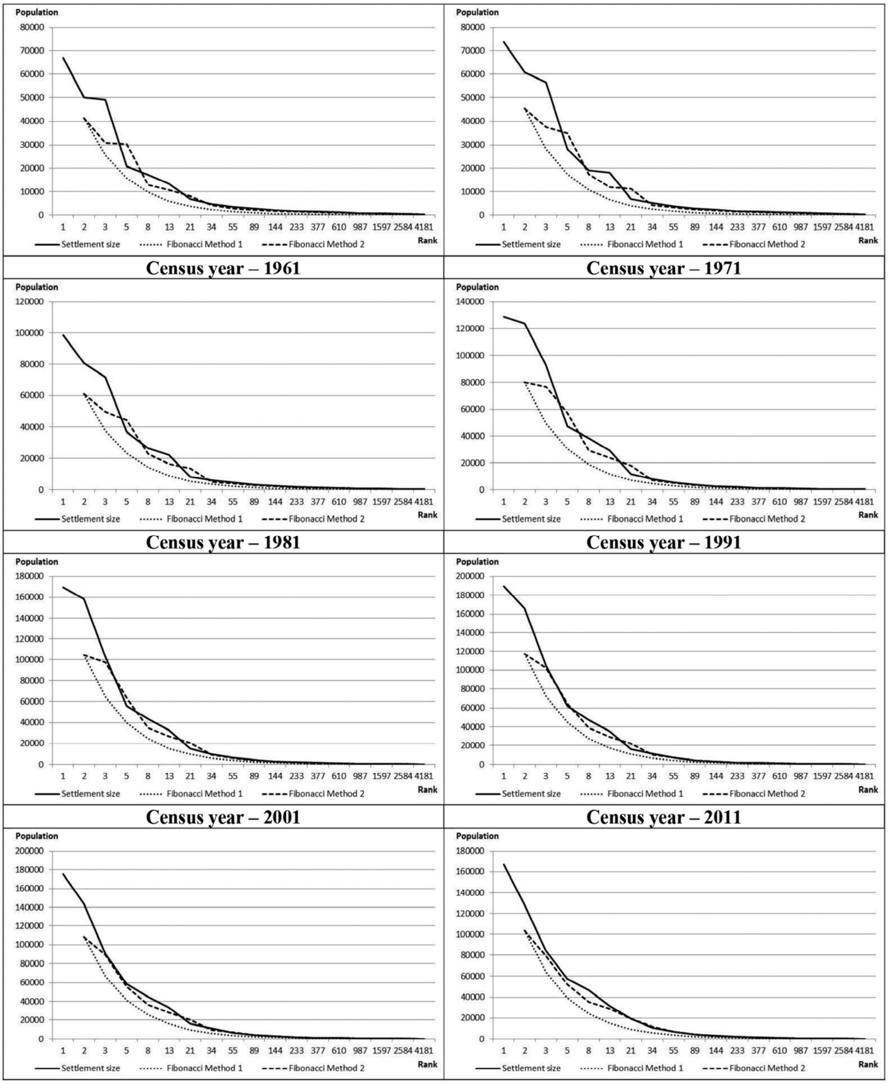


Figure 7 Predicted population in settlements by using Fibonacci series, applied Method 1 and Method 2, population censuses from 1857 to 2011, without the largest settlement

Source: authors' calculation

The precision of the new predictions is also observed by using the MSE, RMSE and MAPE metrics. The results from Table 4 show that, by omitting the largest settlement, the precision of both methods is significantly increased. The values of MAPE for both methods were lower in all census years than in the case of inclusion of the larget city Zagreb. Furthermore, the value of MAPE for Method 2 in the census year 2011 is 8 percent which can be interpreted as a highly accurate prediction.

Table 4

Mean squared errors, root mean squared errors and mean absolute percentage errors of applied Method 1 and Method 2, population censuses from 1857 to 2011, without the largest settlement

Census year	Method 1			Method 2		
	Mean squared error	Root mean squared error	Mean absolute percentage error	Mean squared error	Root mean squared error	Mean absolute percentage error
1857	6,912,072	2,629	72	2,172,040	1,474	18
1869	8,379,416	2,895	71	2,546,111	1,596	17
1880	15,719,785	3,965	70	5,289,464	2,300	16
1890	11,149,803	3,339	64	3,885,451	1,971	15
1900	8,829,393	2,971	60	2,532,111	1,591	15
1910	29,237,049	5,407	57	21,798,576	4,669	15
1921	15,673,153	3,959	58	11,869,377	3,445	17
1931	20,850,498	4,566	56	17,018,399	4,125	16
1948	46,466,205	6,817	60	30,674,842	5,538	20
1953	82,075,794	9,060	61	41,359,571	6,431	22
1961	123,691,422	11,122	57	59,017,126	7,682	20
1971	283,316,947	16,832	55	143,842,153	11,993	17
1981	312,336,022	17,673	48	181,145,898	13,459	12
1991	266,081,920	16,312	45	148,600,292	12,190	11
2001	169,216,688	13,008	46	82,563,546	9,086	10
2011	134,755,041	11,608	47	48,173,735	6,941	8

Source: authors' calculation

In addition, Figure 8 clearly shows that, by RMSE and MAPE criteria, Method 2 is more precise in predicting the settlement sizes than Method 1.

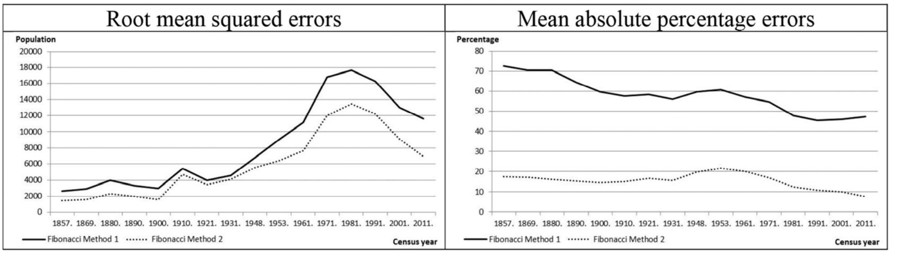


Figure 8 Root mean squared errors and mean absolute percentage errors of applied Method 1 and Method 2, population censuses from 1857 to 2011, without the largest settlement

Source: authors' calculation

It can be concluded that the hypothesis of the paper that Croatian urban system conforms to the Fibonacci growth sequence can be validated taking into account the results of the Fibonacci Method 2 and MAPE values. The value of MAPE with and without the inclusion of the capital city Zagreb belonged to the good prediction level. However, the values of MAPE for both methods were lower in all census years than in the case of inclusion of the largest city Zagreb. The capital city is usually the largest city in a country, disproportionately larger than any other in the urban hierarchy. This is also the case for Zagreb so the omittance of Zagreb from the analysis further improved the precision of the prediction methods. The value of MAPE for Method 2 in the census year 2011 was 8 percent which can be interpreted as a highly accurate prediction.

The aforementioned results shed a new light on this relatively unexplored field of urban economics. Contribution of the paper is twofold. The results point out to the conclusion that the Fibonacci sequence can be used as a convenient tool for analysing the structure of urban rank-size systems going hand in hand with other urban economics regularities such as Zipf's law and Gibrat's law. Second important contribution of this paper is the confirmation of previously obtained results from Žmuk and Jošić (2019). The value of mean absolute percentage errors (MAPE) obtained from Method 2 in which the capital city Zagreb was omitted from the analysis was 8%, which is very similar to previously obtained results of 8.32% and 13% in the case of Eastern Croatia urban hierarchy system which results were highly expected. Furthermore, better precision of the results has been achieved when the largest city Zagreb was excluded from the analysis. It is highly recommended to repeat the analysis with new data after a new population census in Croatia planned to be conducted in 2021. It will be interesting to see whether Fibonacci growth sequence can se used as a good approximation of Croatian urban system in the case of decreasing population.

6. CONCLUSIONS

The Fibonacci sequence can be found in various growth processes in nature but its application in estimating urban rank-size hierarchy was rare so far. Seminal paper in this field was work of Fonseca (1988) in the case of U.S. urbanized area in 1970 and 1980. Research question of this paper was to investigate if the structure of Croatian urban population system conforms to the Fibonacci growth sequence. For that purpose two Fibonacci methods have been used in prediction of Croatian urban rank-size hierarchy on historical population data, for 16 population censuses from 1857 to 2011. The precision of the two prediction methods was inspected with the help of overall forecasting (prediction) errors; mean squared error (MSE), root mean squared error (RMSE) and mean absolute percentage error (MAPE). The results have shown that the Croatian urban system conforms to the Fibonacci sequence. Method 2 has proven to be more precise than Method 1. Method 2 had a good level of precision throughout the observed period. Furthermore, the precision of results was higher when the largest city Zagreb was excluded from the analysis. Limitations of the paper are related to uneven number of years between the two population censuses and the problem of zeroes in the case when some settlements ceased to exist. There was also the problem of identifying the settlements with the same name. Recommendations for future research could be application of the Fibonacci growth sequence in the approximation of urban rank-size hierarchy for other countries and regions in the world in other time periods, inclusion of data for urbanized areas instead of settlement sizes, etc. Using urbanised areas' data instead of using the settlements' sizes data would be in the spirit of the Fibonacci sequence. Urbanised areas' data correspond more naturally to the distribution of population approximated with the Fibonacci sequence, instead of using data for administratively determined units or those defined by the political boundaries. In addition, usage of city data neglects the suburbanization and thus underestimates the growth of cities.

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Appendix

Table A1

List of settlements selected by using Fibonacci series, population censuses from 1857 to 2011

Fibonacci series / Settlement rank	Census year											
	1857		1869		1880		1890		1900		1910	
	Settlement	Size	Settlement	Size	Settlement	Size	Settlement	Size	Settlement	Size	Settlement	Size
1	Zagreb	32,203	Zagreb	37,001	Zagreb	48,136	Zagreb	60,910	Zagreb	87,239	Zagreb	109,029
2	Rijeka	18,466	Rijeka	23,747	Rijeka	27,780	Rijeka	38,841	Rijeka	51,419	Rijeka	66,042
3	Osijek	16,145	Osijek	19,281	Pula	25,390	Pula	31,498	Pula	36,143	Pula	59,498
5	Karlovac	9,968	Karlovac	11,175	Split	14,815	Split	15,697	Split	18,853	Split	21,738
8	Zadar	8,987	Varaždin	10,014	Varaždin	9,789	Varaždin	10,410	Varaždin	12,130	Šibenik	14,195
13	Virje	5,140	Virje	6,024	Virje	6,611	Vinkovci	6,954	Vinkovci	8,634	Vinkovci	10,455
21	Žminj	3,637	Cres	4,673	Bjelovar	4,671	Vodnjan	5,152	Blato	5,781	Đurđevac	6,987
34	Ilok	3,110	Vis	3,540	Novigrad Podravski	3,813	Požega	4,077	Drežnica	4,370	Poreč	4,854
55	Kotoriba	2,633	Tenja	3,038	Čepin	2,988	Samobor	3,228	Tenja	3,526	Pag	3,699
89	Križpolje	2,095	Milna	2,253	Rakovica	2,238	Milna	2,489	Molve	2,604	Vodice	2,771
144	Zlarin	1,643	Ivanić grad	1,829	Sigetec	1,772	Lopatinec	1,868	Slivno	1,971	Cerna	2,097
233	Koljane	1,275	Tinjan	1,438	Žrnovo	1,385	Otišić	1,497	Kutjevo	1,615	Gornja Močila	1,707
377	Lokvičići	1,003	Metković	1,143	Pješčanica	1,079	Brođanci	1,178	Posavski Podgajci	1,281	Pješčanica	1,360
610	Draškovec	764	Momjan	848	Gračišće	836	Veliki Bastaji	920	Župa	1,002	Nova Ploščica	1,076
987	Kupjak	560	Podstrana	634	Ilova	617	Križ	702	Otrovanec	757	Podmelnica	812
1,597	Donje Planjane	397	Komaji	441	Dabar	439	Kožljak	490	Borovci	537	Selnica Podravska	584
2,584	Brčino	248	Gornja Višnjica	269	Buzdohanj	276	Podkilavac	322	Velika Milešina	354	Orlec	381
4,181	Bogačevo Riječko	123	Bertelovci	121	Begovača	143	Kobilić	172	Zoretići	191	Benkovac	206

Source: authors

Table A1

List of settlements selected by using Fibonacci series, population censuses from 1857 to 2011 – continued

Fibonacci series / Settlement rank	Census year									
	1921		1931		1948		1953		1961	
	Settlement	Size	Settlement	Size	Settlement	Size	Settlement	Size	Settlement	Size
1	Zagreb	140,815	Zagreb	227,838	Zagreb	325,223	Zagreb	361,564	Zagreb	442,768
2	Rijeka	61,157	Rijeka	71,966	Rijeka	66,998	Rijeka	73,616	Rijeka	98,759
3	Pula	38,591	Pula	44,219	Split	50,075	Split	60,703	Split	80,902
5	Split	25,052	Split	35,332	Karlovac	26,690	Karlovac	31,842	Karlovac	40,180
8	Šibenik	16,294	Šibenik	17,669	Vukovar	17,223	Varaždin	19,341	Zadar	27,324
13	Vukovar	10,242	Sisak	11,860	Dubrovnik	15,875	Dubrovnik	18,286	Vinkovci	23,192
21	Požega	7,040	Požega	7,125	Čakovec	7,037	Nova Gradiška	7,548	Nova Gradiška	9,229
34	Pitomača	4,883	Nova Gradiška	4,905	Daruvar	4,812	Županja	5,391	Ilok	6,193
55	Primošten	3,700	Sinj	3,720	Darda	3,615	Delnice	3,840	Ivankovo	4,633
89	Senj	3,036	Branjin Vrh	2,942	Vodice	2,776	Glavice	2,914	Suhopolje	3,264
144	Suhopolje	2,240	Staro Petrovo Selo	2,292	Viljevo	2,190	Sveta Marija	2,300	Ervenik	2,397
233	Borovo	1,793	Sinac	1,867	Cista Velika	1,707	Sigetec	1,739	Koška	1,864
377	Gunja	1,414	Zlarin	1,480	Šarengrad	1,349	Negoslavci	1,384	Breznica	1,408
610	Podhum	1,117	Bibinje	1,158	Badljevina	1,052	Sedlarica	1,057	Stari Slatinik	1,072
987	Rovinjsko Selo	841	Lasinja	885	Zasadbreg	806	Gornji Miholjac	816	Omišalj	791
1,597	Vugrovec Donji	596	Bedenica	637	Sekirišće	587	Kaprije	588	Brčevec	576
2,584	Habjanovac	384	Zvečanje	417	Kupjak	386	Brajakovo Brdo	387	Dragotin	374
4,181	Rakovec Tomaševečki	183	Kraljevec Šemnički	208	Veliki Miletinac	214	Kruškovac	213	Lukarišće	202

Source: authors

Table A1

List of settlements selected by using Fibonacci series, population censuses from 1857 to 2011 – continued

Fibonacci series / Settlement rank	Census year									
	1971		1981		1991		2001		2011	
	Settlement	Size	Settlement	Size	Settlement	Size	Settlement	Size	Settlement	Size
1	Zagreb	579,943	Zagreb	656,380	Zagreb	706,770	Zagreb	691,724	Zagreb	688,163
2	Rijeka	129,173	Split	169,322	Split	189,388	Split	175,140	Split	167,121
3	Split	123,756	Rijeka	158,030	Rijeka	165,693	Rijeka	143,800	Rijeka	128,384
5	Karlovac	47,543	Zadar	63,364	Zadar	76,343	Zadar	69,556	Zadar	71,471
8	Slavonski Brod	38,705	Slavonski Brod	47,583	Slavonski Brod	55,683	Karlovac	49,082	Slavonski Brod	53,531
13	Dubrovnik	30,161	Šibenik	36,952	Šibenik	41,012	Velika Gorica	33,339	Vinkovci	32,029
21	Petrinja	12,155	Petrinja	15,778	Petrinja	18,706	Zaprešić	17,538	Požega	19,506
34	Velika Gorica	8,013	Ogulin	9,796	Slatina	11,416	Sinj	11,468	Križevci	11,231
55	Crikvenica	5,352	Darda	6,460	Ivanić0grad	7,104	Tenja	6,747	Kaštel Sućurac	6,829
89	Omiš	3,731	Orahovica	4,005	Orahovica	4,314	Nedelišće	4,430	Prelog	4,324
144	Rešetari	2,495	Donja Dubrava	2,719	Velika Mlaka	2,925	Vrbanja	2,952	Dugopolje	2,993
233	Oštarije	1,867	Donji Stupnik	1,917	Lučko	2,013	Banjol	1,971	Mokošica	1,924
377	Aržano	1,379	Medulin	1,362	Tisno	1,431	Preko	1,351	Novoselec	1,362
610	Orolik	1,024	Slavsko Polje	985	Sveti Petar u Šumi	999	Marijanci	935	Funtana	907
987	Kuševac	739	Breznica Našička	692	Cista Provo	687	Brezovica	632	Generalski Stol	589
1,597	Kuzmica	526	Šišan	481	Sekirišće	454	Lepajci	398	Gornji Prnjarovec	369
2,584	Bogomolje	337	Turkovići Ogulinski	293	Žaborić	273	Slani Dol	226	Sedramić	205
4,181	Podaca	176	Jasenik	146	Donji Bukovac Žakanjski	130	Parg	99	Gostenje	87

Source: authors

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MOŽE LI SE HIJERARHIJA HRVATSKOG URBANOG SUSTAVA APROKSIMIRATI FIBONACCIJEVIM NIZOM? ANALIZA NA POVIJESNIM PODACIMA O BROJU STANOVNIKA

Sažetak

Fibonaccijevi brojevi se mogu naći u prirodi i primjenjuju se u raznim područjima ljudske djelatnosti. Fibonaccijev niz se također može koristiti za predviđanje populacije naselja. Cilj ovog rada je ispitati može li se hijerarhija hrvatskog urbanog sustava aproksimirati Fibonaccijevim nizom na povijesnim podacima popisa stanovništva od 1857. do 2011. godine. Stvarne vrijednosti urbane hijerarhije s obzirom na veličinu naselja uspoređene su s predviđenim vrijednostima za dvije Fibonaccijeve metode dobivenima uz pomoć Fibonaccijevog niza. Prva metoda stavlja u omjer populaciju najvećeg grada i zlatnog omjera, dok druga metoda uzima stanovništvo svakog sljedećeg grada i dijeli ga sa zlatnim omjerom. Provedena analiza je pokazala da hrvatski urbani sustav odgovara Fibonaccijevom nizu uz vrlo dobru preciznost. Metoda 2 daje preciznija preklapanja stvarnog broja stanovnika po naseljima u urbanoj hijerarhiji Republike Hrvatske u odnosu na Metodu 1. Kada bi se najveći grad Zagreb izostavio iz analize, rezultati bi bili još precizniji s manjim MAPE.

Ključne riječi: Hrvatska, urbani sustav, Fibonaccijev niz, pravilo reda veličine.

JEL klasifikacija: P25, R11, R23.