Automatika

Journal for Control, Measurement, Electronics, Computing and Communications

ISSN: 0005-1144 (Print) 1848-3380 (Online) Journal homepage: https://www.tandfonline.com/loi/taut20

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To cite this article: Rawad Abdulghafor, Shahrum Shah Abdullah, Sherzod Turaev & Raini Hassan (2019) The nonlinear limit control of EDSQOs on finite dimensional simplex, Automatika, 60:4, 404-412, DOI: 10.1080/00051144.2019.1632063

To link to this article: https://doi.org/10.1080/00051144.2019.1632063

AUTOMATIKA
2019, VOL. 60, NO. 4, 404–412
https://doi.org/10.1080/00051144.2019.1632063

REGULAR PAPER

The nonlinear limit control of EDSQOs on finite dimensional simplex

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ABSTRACT

Consensus problems in multi agent systems (MAS) are theoretical aspect convergence of doubly stochastic quadratic operators. This work has presented the dynamic classifications of extreme doubly stochastic quadratic operators (EDSQOs) on finite-dimensional simplex (FDS) based on the limit behaviour of the trajectories. The limit behaviour of the trajectories of EDSQOs, on FDS is either in state of convergence, or fixed or periodic. This paper aimed at examining the behaviour of these states. The paper modelled the states and proves theoretically the characteristics of each state. The results indicate that convergence operators converge to the centre $(\frac{1}{m})$, and EDSQOs point are fixed with two or more points whereas periodic states exhibit sinusoidal behaviour. This work has contributed in understanding the limit of EDSQOs of the exterior initial points as fixed and periodic points developed spread attribute toward a fixed point.

ARTICLE HISTORY
Received 6 October 2018
Accepted 11 June 2019

KEYWORDS
Dynamic classifications; extreme doubly stochastic quadratic operators; convergence; fixed; periodic; finite-dimensional simplex

1. Introduction

Most research applications in biological implementations and physics use nonlinear mathematical models [1]. Crucial to this is feedback process, that is intended to reach an optimal result. One of the most important and complex systems of these applications is the MAS [2]. However, one of the central problems in MAS generated by feedback process is called consensus problem [3,4]. The nonlinear model of EDSQOs lies within the study of convergence of limit behaviour suitable for solving consensus problem in MAS [5–11]. EDSQOs refers to quadratic stochastic operators (QSOs) [12]. The history of QSOs can be traced back to Bernshtein [13]. In [14], an application of QSOs in population genetics has been modelled. QSQs are defined on dimensional simplex [15,16] where the simplex is a set of points. The higher dimension is more complicated to study and still an open problem for two-dimensional simplex [17,18]. Lyubich [19] has studied QSOs on one-dimensional simplex, where it was proven that it has a finite set for the $\omega - limit$ from any initial point. Vallander [20] has investigated results related to some of the cases associated to QSOs on two-dimensional simplex. These results have been extended on finite-dimensional simplex by Ganikhodzhaev in [21].

A theoretical publications of stability of QSOs have been studied in [22] and an incidence problem for the class in QSOs have been analysed in [23] for the steady topology. According to [24], four dimensional simplex has considered the set of the extreme of Volterra quadratic stochastic operators class. Similarly, a dynamic model has been defined using the class of QSOs on the dimensional simplex [25]. In another way, a weak convergence has been resulted from the iterations of kernel class of QSOs [26]. In [27], a conditional cubic stochastic operator has been introduced where it contains a unique fixed point. Besides, It has driven a cubic stochastic operators class for the genetic populations [28]. In addition, [29] has investigated a new subclass of QSOs on a finite-dimensional simplex. The trajectory behaviour of several classes of QSOs has been addressed in [30].

QSOs are developed through a majorization technique [31,32], where the majorization concept [33] of vectors has gained a reputation of becoming a beneficial method to classify QSOs into sub-classes. The QSOs are called DSQOs if the condition $Vx \prec x$ is satisfied (see the notations in the next section), where "$\prec$" is the notation of the majorization concept related to ordering of the respective vector or set elements [32,34,35]. Doubly stochastic quadratic operators (DSQOs) is defined in terms of the majorization concept, which was introduced in [19,20]. The ordering of the set follows the comparison of the coordinates' partial sums after a non-increasing rearrangement [36,37]. The class of DSQOs is very huge, therefor the study of limit behaviour is very difficult [23]. This work examines the extreme of DSQOs using the majorization concept. The definition

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of EDSQOs is derived from that of the DSQO's set points in space. These sets form a polyhedron that contains vertices known as extreme points. The limit behaviour of some EDSQOs has been studied in [38]. Thus, the limit behaviour of EDSQOs is characterized on 2DS, where the EDSQOs are defined using majorization techniques [25]. Up to the permutation of the components of EDSQOs on 2DS, 37 extreme points exist. Moreover, up to the permutation of the set points of each EDSQO on 2DS, 222 extreme points [12,32,35,37] exist as well. Therefore, the purpose of the present work is to classify EDSQOs on FDS based on the study of limit behaviour of trajectories.

2. Methodology

In this section, we present some definitions of majorization theory and a doubly stochastic operator, which are needed in our study.

Definition 2.1: The nonlinear discrete dynamic systems of QSOs are defined on the simplex [14,15,21]. In this case, A(m − 1)− dimensional simplex is defined as

$$S^{m-1} = \{x = (x_1, x_2, \ \ldots, x_m) \in R^m \ : \ x_i \geq 0,$$
$$\forall_i = \overline{1, m}, \ \sum_{i=1}^{m} x_i = 1\}. \tag{2.1}$$

Definition 2.2: The set $intS^{m-1} = \{x \in S^{m-1} : x_i \geq 0\}$ is called the interior of the simplex. The points $e_k = (0, 0, \ldots, \underbrace{1}_{k}, \ldots, 0)$, $(k = \overline{1, m})$ are the vertices of the simplex, and the scalar vector $\left(\frac{1}{m}, \frac{1}{m}, \ldots,\right)$ is the centre of the simplex.

Definition 2.3: A quadratic stochastic operator (QSO) V : $S^{m-1} \to S^{m-1}$ is defined as [15]:

$$(Vx)_k = \sum_{i=1}^{m} p_{ij,k} \ x_i x_j, \tag{2.2}$$

where coefficients $p_{ij,k}$ satisfy the following conditions [15,32]:

$$p_{ij,k} = p_{ji,k} \ \geq 0, \ \sum_{k=1}^{m} p_{ij,k} = 1, \tag{2.3}$$

The QSOs are related to population evolution. We consider a population consisting of m species. Let $x^0 = (x_1^0, x_2^0, \ldots, x_m^0)$ be the probability distribution of species in the initial generations, and $P_{ij,k}$ be the probability that individuals in the i_{th} and j_{th} species interbreed to produce an individual k. This probability is denoted as$P_{ij,k}$ (and referred to as "the heredity coefficient") and $\sum_{k=1}^{m} p_{ij,k} = 1$ for all i, j, that is, x_i and x_j are the fractions of species i and j in the population. In this case, parent pairs i and j arise from a fixed state $x = (x_1, x_2, \ldots, x_m) \in R^m$ with probability $x_i x_j$ [15].

If we denote $P_{ij,k}$ by A_k, $1 \leq i, j \leq m$, and $1 \leq k \leq m$, the operator can then be written equivalently in matrix form as follows:

$$V = (A_1 | A_2 | \ldots | A_m).$$

In this instance, matrices A_i- are non-negative and symmetric.

Definition 2.4: For any $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0)$, we define $x_\downarrow = (x_{[1]}, x_{[2]}, \ldots, x_{[m]})$ where $x_\downarrow = (x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[m]})$ – non-increasing rearrangement of x. Recall that for two elements x, y of the simplex S^{m-1}, the element x is majorized by y. We write $x \prec y$ or $x \succ y$ if the following holds:

$$\sum_{i=1}^{k} x_{[i]} \leq \ \sum_{i=1}^{k} y_{[i]} \ , \tag{2.4}$$

for any $k = \overline{1, m}$. This definition is called a weak majorization [33], and the definition of majorization requires $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$. However, we consider points only from the simplex, so we may drop this condition.

Definition 2.5: Matrix $P = (p_{ij})_{i,j=\overline{1,m}}$ is called doubly stochastic (sometimes bistochastic) if

$$p_{ij} \geq 0, \ for \ all \ i, j = \overline{1, m},$$

$$\sum_{i=1}^{m} p_j = 1, \ \text{and} \ \sum_{j=1}^{m} p_i = 1, \ \text{for all} \ i, j = \overline{1, m}. \tag{2.5}$$

Definition 2.6: The stochastic operator $V : S^{m-1} \to S^{m-1}$V : $S^{m-1} \to S^{m-1}$ is called doubly stochastic [15], if

$$Vx \prec x \text{ for all } x \in S^{m-1} \tag{2.6}$$

In the DSQOs, element $x \in S^{m-1}$ is the rearrangement of non-increasing $x_\downarrow = (x_{[1]}, \ldots, x_{[m]}) \in S^{m-1}$, where $x_{[1]} \geq \ldots \geq x_{[m]}$.

If we have two elements $x, y \in S^{m-1}$, and if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \ k = 1, \ \ldots, \ m. \tag{2.7}$$

Then, we can say that x is majorized by y, and write $x \prec y$.

It was defined in [33] that $x \prec y$ if the corresponding doubly stochastic matrix P is $x = Py$.

Hence, if P is a doubly stochastic matrix, then $Px \prec x$ for any element $x \in S^{m-1}$.

The operator is therefore called a DSQO because the operator $V(x)$ has a doubly stochastic matrix P, which satisfies the definition of majorization [33].

The works in [15] and [32] proved that the operator $V: S^{m-1} \to S^{m-1}$ is a DSQO if the coefficient $P_{ij,k}$ in QSO satisfies the condition $V(x) \prec x$. We call the conditions DSQOs a set of U_1 [32]

$$U_1 = \left\{ A = (a_{ij}) : a_{ij} = a_{ji} \geq 0, \times \sum_{i,j \in \alpha} a_{ij} \leq |\alpha|, \sum_{i,j \in I} a_{ij} = m \right\}. \quad (2.8)$$

where the sum of sub-block of size m by m is $\leq m, \sum_{i,j=1} a_{ij,k} \leq |\alpha|$ for $\alpha \subset \{1, 2, \ldots, m\}$.

Definition 2.7: Let $\boldsymbol{U}_1$ be a set of the DSQOs. It is a fact that the set of DSQO form a polyhedron, meaning that each DSQO is considered as a point in some dimensional space. A polyhedron has vertices that are defined as EDSQOs. These vertices are defined by adding two extra conditions to the set of $\boldsymbol{U}_1$ existence. Therefore, the extra conditions for EDSQOs are called a set of $\boldsymbol{ExtrU}_1$ and are given as follows,

The $V: S^{m-1} \to S^{m-1}$ belongs to $ExtrU_1$ [15], [15] if the following conditions hold

$$P_{ii,k} = 0 \;\; or \;\; 1; \quad (a)$$

$$P_{ij,k} = 0, \;\; \frac{1}{2} \;\; or \;\; 1, \;\; for \;\; i \neq j. \quad (b)$$

Therefore, the evolution operator of EDSQOs has $m * m$ edges with coefficients equal to1.

Definition 2.8: Let V be DSQO and$x^0 \in S^{m-1}$. The sequence $\{x^0, V(x^0), V^2(x^0), \ldots, V^n(x^0)\}$ is called the trajectory of the DSQO starting atx^0, where$V^2(x^0) = V(V(x^0))$. Usually, the initial state is defined by $V(x^0) = x^0$ and the set of limit points of the trajectory is denoted by $\omega(x^0)$, which are the ω- limit set of the trajectory. The set of points x^0 is considered to converge with (2.2), if $V^p(x^0)$ converges to the centre $C = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}\right)$ of the simplex S^{m-1} as $n \to \infty$.

Definition 2.9: The point x^0 is considered as having an undirected interconnection only if it has m interconnections, and at least one interconnection is with other points. It can also be referred to as a linear point, where it has linear function. The linear function means that it is characterized by an equation which has one of the points x_i^0 repeating in all its terms.

Definition 2.10: The point x^0 is considered to have a directed interconnection only if it has m interconnections, and all interconnections are between itself and others. It can also be called nonlinear point, where it does not contain a linear function. In turn, the nonlinear function is characterized by an equation with no repetitions of the points x^0 in its terms.

Definition 2.11: Points x_i^0 converges if the there exists a positive integer p for the limit behaviour, such that $V^p(x^0) = x^0 V^p(x^0) = x^0$, and $V^i(x^0) = x^0$. For all $i = \overline{1, p-1}$, if $p = 1 V^i(x^0) \neq x^o \forall i = \overline{1, p-1}$.p = 1,, then the point is said to be fixed.

Definition 2.12: Point x_i^0 is fixed if a positive integer p exists, such that $V^p(x_i^0) = x_i^0 V^p(x^0) = x^0$, and $V^i(x^0) = x_i^0$. For all $i = \overline{1, p-1}$, if $p = 1$, we say that the point is fixed.

Definition 2.13: Point x_i^0 is p-periodic if there is positive integer p, such that $V^p(x_i^0) = x_i^0 V^p(x^0) = x^0$, and $V^i(x^0) = x^0$ and $V^i(x^0) \neq x^0 \forall i = 1, p-1$. We usually say the point is periodic if the period is larger than one.

Definition 2.14: let the set of points $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0)$ be on the boundary of the simplex, then the boundary points are referred to as exterior points.

Definition 2.15: let the set of points $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0)$ be inside of the simplex, then these points are referred to as interior points.

Definition 2.16: let the set of points $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0)$ be on the vertices of the simplex, where one of x_i^0 has value equal to one while the others are equal to zero, then the set of points are referred to as extreme exterior points (see definition 2.2).

The DSQOs evaluate the next generation starting from the initial state x^0 of the probability distribution. They then continue to improve the probability distribution of the first generation, $x^{(1)} = V(x^0)$, followed by the second generation iteratively, $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})) = V^{(2)}(x^{(0)})$, and so on. Therefore, the iterative notation defining the EDSQOs is given by $V^{(t+1)}(x_i^0)$, where t is the number iterations (generations).

3. Results and discussion

In this section, we study the dynamic classification of limit behaviour of EDSQOs.

Theorem 3.1: *Let V be EDSQO defined on S^{m-1} and the initial values $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}, \sum_{i=1}^m x_i = 1\}$ are extreme exterior points, then the $\lim_{t \to \infty} V^{(t+1)}(x_i^0)$ never converges, for all $x_i^0 \in S^{m-1}$.*

Proof: Assuming that $V^{(t+1)}(x_i^0)$ is EDSQO on S^{m-1} and the initial values given by $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}, \sum_{i=1}^m x_i = 1\}$ are extreme exterior points, then by definition (2.16) we have$1 \leq j \leq m$, *if* $x_j^0 = 1$ *then for all* $1 \leq k \leq m, k \neq j, x_k^0 = 0$. ■

It is obvious from its definition that, each $V^{(t+1)}(x_i^0)$ on S^{m-1} has m point functions as $V^{(t+1)}(x_i^0) =$

$(V^{(t)}(x_1^0), V^{(t)}(x_2^0), \ldots, V^{(t)}(x_m^0))$ and each point function consists of the summation of m products. Each of the product terms in turn, constitute of either the same point $(x_i \cdot x_i)$ or different points $(x_i \cdot x_j$ with $i \neq j)$ and each of the points take the values of either zero or one $(x_i, x_j = 0, 1)$ where $i \leq i, j \leq m$. Thus, $V^{(t+1)}(x_i^0)$ has a total of m^2 product terms in all the summations, where there are m products of the form $x_i x_i, k \leq i \leq m$, and $(m^2 - m)$ products of the form $x_i x_j, i \leq i \neq j \leq m$, (with reference to Equation (2.8) (a) and (b)).

Therefore, the point function which has the product of $x_j x_j$ converges to one (provided $x_j = 1$), while the others converge to zero as they consist a multiple of zero ($x_i x_k$ *with* $x_i = 0$ *or* $x_k = 0$). Hence, the limit of any EDSQOs on S^{m-1} of any extreme exterior points converges to extreme exterior points (taking values of either one or zero) which are either fixed or periodic points.

Definition A: x_i^t is the (common) factor of function point $(V^{(t+1)}(x_i^t))$ if

$$V^{(t+1)}(x_i^t) = x_i^t.(x_1^t + x_2^t + \ldots + x_m^t) \in R^m : x_i \geq 0,$$

$$\forall_i = \overline{1, m}, \sum_{i=1}^{m} x_i = 1, t : 0 \to \infty\}$$

where should be $(x_1^t + x_2^t + \ldots + x_m^t) = 1$.

Definition C: $MAX(x_i^0)$ is the maximum initial value and $MIN(x_i^0)$ is the minimum initial value of $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}, \sum_{i=1}^m x_i = 1$. The difference between $MAX(x_i^0)$ and $MIN(x_i^0)$ is $d(x_i^0)$. We say that V is the EDSQO and defined on S^{m-1} converges if $d(x_i^t) = 0$, where $t : 0 \to \infty$.

Theorem 3.2: *Let V be EDSQO defined on S^{m-1} and $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}, \sum_{i=1}^m x_i = 1$ are initial values. If x_i^0 is the (common) factor of a function point $V^{(t+1)}(x_i^0)$ and is fixed, then the operator V does not converge.*

Proof: *Assuming that V is EDSQO and defined on S^{m-1} with set points of $V^{(t+1)}(x_i^0)$ (see theorem 3.4), and $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}, \sum_{i=1}^m x_i = 1$ be initial values.*

If

$$V^{(t+1)}(x_i^0) = x_i^0.(x_1^0 + x_2^0 + \ldots + x_m^0), \ \textit{and}$$

$$i = \overline{1, m}, \sum_{j=1}^{m} x_j = 1 \tag{3.1}$$

and

$$(x_1^0 + x_2^0 + \ldots + x_m^0) = 1, \tag{3.2}$$

Then it follows naturally that,

$$V^{(t+1)}(x_i^0) = x_i^0, \tag{3.3}$$

That is, $V^{(t+1)}(x_i^0)$ is fixed, and the operator does not converge.

Theorem 3.3: *Let V be EDSQO defined on S^{m-1} and $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}$, $\sum_{i=1}^m x_i = 1$ be initial values. If x_i^0 is the (common) factor of a function point $V^{(t+1)}(x_j^0)$ and x_j^0 is the (common) factor of function point $V^{(t+1)}(x_i^0)$, then these points are periodic and the operator V does not converge.*

Proof: Assuming that V is EDSQO and defined on S^{m-1} with set points of $V^{(t+1)}(x_i^0)$, and $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}, \sum_{i=1}^m x_i = 1$ be initial values.

If

$$V^{(t+1)}(x_i^0) = x_j^0.(x_i^0 + \ldots + x_m^0), \text{ and}$$

$$i = \overline{1, m}, \sum_{i=1}^{m} x_i = 1, \tag{3.4}$$

and

$$V^{(t+1)}(x_j^0) = x_i^0.(x_j^0 + \ldots + x_m^0), \text{ and}$$

$$j = \overline{1, m}, \sum_{j=1}^{m} x_j = 1, \tag{3.5}$$

then, starting from $t = 0$,

$$V^{(t+1)}(x_i^t) = x_j^t \cdot \ (x_i^t + \ldots + x_m^t) = x_j^t,$$

$$V^{(t+1)}(x_j^t) = x_i^t \cdot (x_j^t + \ldots + x_m^t) = x_i^t. \tag{3.6}$$

After the first iteration as shown, the resulting point is a different point. Now, continuing with the next iteration $(t = 1)$ then the

$$V^{(t+1)}(x_i^t) = x_j^t. \ (x_i^t + \ldots + x_m^t) = x_i^t,$$

$$V^{(t+1)}(x_j^t) = x_i^t. \ (x_i^t + \ldots + x_m^t) = \ x_j^t. \tag{3.7}$$

Thus, in every step $V^{(t+1)}(x_i^0)$ periodically repeats, and the operator V does not converge. ∎

Theorem 3.4: *Let V be EDSQO and defined on S^{m-1} and the initial values $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}, \sum_{i=1}^m x_i = 1$ be interior points, where $(0 < x_i^0 < 1)$. If the function point $V^{(t+1)}(x_i^0)$ does not have factor (common), then the $\lim_{t \to \infty} V^{(t+1)}(x_i^0)$ converges to the centre $\left(\frac{1}{m}\right)$.*

Proof: Assuming that V is EDSQO and defined on S^{m-1} with the initial values $x_i^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in R^m : x_i \geq 0, \forall_i = \overline{1, m}, \sum_{i=1}^m x_i = 1$ as interior points, where $(0 < x_i^0 < 1)$. ∎

If

$$V^{(t+1)}(x_i^0) \neq x_i^0.(x_i^0 + \ldots + x_m^0),$$

$$i = \overline{1, m}, \sum_{i=1}^{m} x_i = 1, \tag{3.8}$$

then by Theorem 3.1 $V^{(t+1)}(x_i^0)$ on S^{m-1} has m point functions $V^{(t+1)}(x_i^0) = (V^{(t)}(x_1^0), V^{(t)}(x_2^0), \ldots,$

$V^{(t)}(x_m^0))$ and by definition (C) we have the following three possible cases of products of values:

$$MAX(x_i^0)*MIN(x_i^0) \text{ or } MAX(x_i^0)*MAX(x_i^0) \text{ or } MIN(x_i^0)*MIN(x_i^0) \quad (3.9)$$

It follows that, with $\lim_{t\to\infty} V^{(t+1)}(x_i^0)$ and based on the distribution defined by EDSQO then the max and min points behave in the following manner

$$MAX(x_i^0) \gg MAX(x_i^1) \gg MAX(x_i^2) \gg \dots$$
$$MIN(x_i^0) \ll MIN(x_i^1) \ll MIN(x_i^2) \ll \dots \quad (3.10)$$

In turn, $MAX(x_i^0)$ decreases gradually while $MIN(x_i^0)$ increases gradually with each successive iteration and eventually reaching a common point of convergence as

$$MAX(x_i^0) = MIN(x_i^0) \quad (3.11)$$

which means that

$MAX(x_i^0)$ *and* $MIN(x_i^0)$ are bounded by the same point of convergence, and

$$d(x_i^t) = MAX(x_i^t) - MIN(x_i^t) = 0 \quad (3.12)$$

Finally, since the function point $V^{(t+1)}(x_i^0)$ has m summations of products of two points $x_i x_j$ and the sums' coefficients of each $V^{(t+1)}(x_i^0)$ equal to$1/m$, then $MAX(x_i^0)$ *and* $MIN(x_i^0)$ are bounded by $\left(\frac{1}{m}\right)$.

Consequently, the $V^{(t+1)}(x_i^0)$ can be expressed in terms of a common factor of $\left(\frac{1}{m}\right)$ and the x_i^0 as follows

$$V^{(t+1)}\left(\frac{1}{m}\right) = \frac{1}{m}.(x_1^0 + x_2^0 + \dots + x_m^0) \text{ for all } \times V^{(t+1)}(x_i^0) \text{ and } i = \overline{1, m}, \sum_{i=1}^{m} x_i = 1 \quad (3.13)$$

Then, the operator V of EDSQO on S^{m-1} converges to the centre $\left(\frac{1}{m}\right)$.

4. Simulation

In this section, we present the software simulation by Matlab for all operators of EDSQOs on S^{m-1}.

Let us to consider some examples of the EDSQOs on 2DS,3DS, 4DS:

(1) Some EDSQOs on 2DS ($V_{3.1}, V_{3.2}, V_{3.3}$):

$$V_{3.1.F}\begin{cases} V(x_1) = x_1^2 + x_1x_2 + x_1x_3, \\ V(x_2) = x_3^2 + x_1x_2 + x_2x_3, \\ V(x_3) = x_2^2 + x_1x_3 + x_2x_3, \end{cases}$$

$$V_{3.2.P}\begin{cases} V(x_1) = x_1^2 + x_1x_2 + x_1x_3, \\ V(x_2) = x_3^2 + x_1x_3 + x_2x_3, \\ V(x_3) = x_2^2 + x_1x_2 + x_2x_3, \end{cases}$$

$$V_{3.3.C}\begin{cases} V(x_1) = x_1x_2 + x_1x_3 + x_2x_3, \\ V(x_2) = x_1^2 + x_2^2 + x_3^2, \\ V(x_3) = x_1x_2 + x_1x_3 + x_2x_3, \end{cases}$$

(2) Some EDSQOs on 3DS ($V_{4.1}$, $V_{4.2}$, $V_{4.3}$):

$$V_{4.1.F}\begin{cases} V(x_1) = x_1^2 + x_2x_4 + x_1x_3 + x_1x_4, \\ V(x_2) = x_2^2 + x_1x_2 + x_2x_3 + x_2x_4, \\ V(x_3) = x_3^2 + x_1x_3 + x_2x_3 + x_3x_4, \\ V(x_4) = x_4^2 + x_1x_4 + x_1x_2 + x_3x_4, \end{cases}$$

$$V_{4.2.P}\begin{cases} V(x_1) = x_4^2 + x_1x_4 + x_2x_4 + x_3x_4, \\ V(x_2) = x_3^2 + x_1x_3 + x_2x_3 + x_3x_4, \\ V(x_3) = x_2^2 + x_1x_2 + x_2x_3 + x_2x_4, \\ V(x_4) = x_1^2 + x_1x_2 + x_1x_3 + x_1x_4, \end{cases}$$

$$V_{4.3.C}\begin{cases} V(x_1) = x_1x_2 + x_2x_3 + x_2x_4 + x_1\ x_4, \\ V(x_2) = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ V(x_3) = x_1x_3 + x_1x_3 + x_1x_4 + x_3x_4, \\ V(x_4) = x_1x_2 + x_2x_3 + x_2x_4 + x_3x_4, \end{cases}$$

(3) Some EDSQOs on 4DS ($V_{5.1}$, $V_{5.2}$, $V_{5.3}$):

$$V_{5.1.F} = \begin{cases} V(x_1) = x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5, \\ V(x_2) = x_2^2 + x_1x_2 + x_2x_3 + x_2x_4 + x_2x_5, \\ V(x_3) = x_3^2 + x_1x_3 + x_2x_3 + x_3x_4 + x_3x_5, \\ V(x_4) = x_4^2 + x_1x_4 + x_2x_4 + x_3x_4 + x_4x_5, \\ V(x_5) = x_5^2 + x_1x_5 + x_2x_5 + x_3x_5 + x_4x_5, \end{cases}$$

$$V_{5.2.P} = \begin{cases} V(x_1) = x_4^2 + x_1x_4 + x_2x_4 + x_3x_4 + x_4x_5, \\ V(x_2) = x_3^2 + x_1x_3 + x_2x_3 + x_3x_4 + x_3x_5, \\ V(x_3) = x_2^2 + x_1x_2 + x_2x_3 + x_2x_4 + x_2x_5, \\ V(x_4) = x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5, \\ V(x_5) = x_5^2 + x_1x_5 + x_2x_5 + x_3x_5 + x_4x_5, \end{cases}$$

$$V_{5.3.C} = \begin{cases} V(x_1) = x_1x_3 + x_1x_4 + x_1x_5 + x_2x_5 + x_3x_5, \\ V(x_2) = x_1x_2 + x_2x_3 + x_2x_4 + x_3x_4 + x_4x_5, \\ V(x_3) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2, \\ V(x_4) = x_1x_2 + x_2x_3 + x_2x_4 + x_3x_4 + x_4x_5, \\ V(x_5) = x_1x_3 + x_1x_4 + x_1x_5 + x_2x_5 + x_3x_5, \end{cases}$$

In the following section, we present the simulation of the limit behaviour of the trajectories for each EDSQO of ($V_{3.1}, V_{3.2}, V_{3.3}, V_{4.1}, V_{4.2}, V_{4.3}, V_{5.1}, V_{5.2}, V_{5.3}$).

4.1. Limit behaviour of the initial extreme exterior values

The initial extreme exterior values on 2DS of $(V_{3.1}, V_{3.2}, V_{3.3})$ are:

$$x_1^0 = 0, x_2^0 = 0, x_3^0 = 1.$$

The initial extreme exterior values on 3DS of $(V_{4.1}, V_{4.2}, V_{4.3})$ are:

$$x_1^0 = 1, x_2^0 = 0, x_3^0 = 0, x_4^0 = 0.$$

The initial extreme exterior values on 4DS of $(V_{5.1}, V_{5.2}, V_{5.3})$ are:

$$x_1^0 = 0, x_2^0 = 0, x_3^0 = 0, x_4^0 = 1, x_5^0 = 0.$$

It has been obtained in the simulation as depicted in Figure 1 that the limit behaviour of trajectories for EDSQOs of $V_{3.1.F}, V_{3.1.P}$, $V_{3.1.C}$ on 2DS of the initial extreme exterior values $(0, 0, 1)$, EDSQOs of $V_{4.1.F}, V_{4.1.P}$ and $V_{4.1.C}$ on 3DS of the initial extreme exterior values $(1, 0, 0, 0)$ and EDSQOs of $V_{5.1.F}, V_{5.1.P}$ and $V_{5.1.C}$ on 4DS of the initial extreme exterior values $(0, 0, 0, 1, 0)$ are fixed or periodic points. In fact, it confirms the theorem 3.1 that the limit of any EDSQOs on FDS of any extreme exterior points of initial values does not converge, it has either fixed or periodic points.

4.2. Limit behaviour of fixed points

It is depicted in Figure 2 that the limit behaviour of EDSQOs of $V_{3.1.F}$ on 2DS, $V_{4.1.F}$ on 3DS and $V_{5.1.F}$ on 4DS. It is shown that the operator of $V_{3.1.F}$ has one point of $V(x_1)$ as linear function which has a common factor of the same point as in Equation (3.1) and the limit behaviour of this point is a fixed point. In addition, the operator of $V_{4.1.F}$ has two points of $V(x_2)$ and $V(x_3)$ as linear functions and both having a common factor (as in Equation (3.1)) of their respective points and the limit behaviour of these points is fixed points. Moreover, the operator of $V_{5.1.F}$ has five points of $V(x_1)$, $V(x_2)$, $V(x_3)$, $V(x_4)$ and $V(x_5)$ that are linear functions and having a common factor (as in Equation (3.1)) of the respective points and the limit behaviour of these points is fixed points. Surely, that is what has

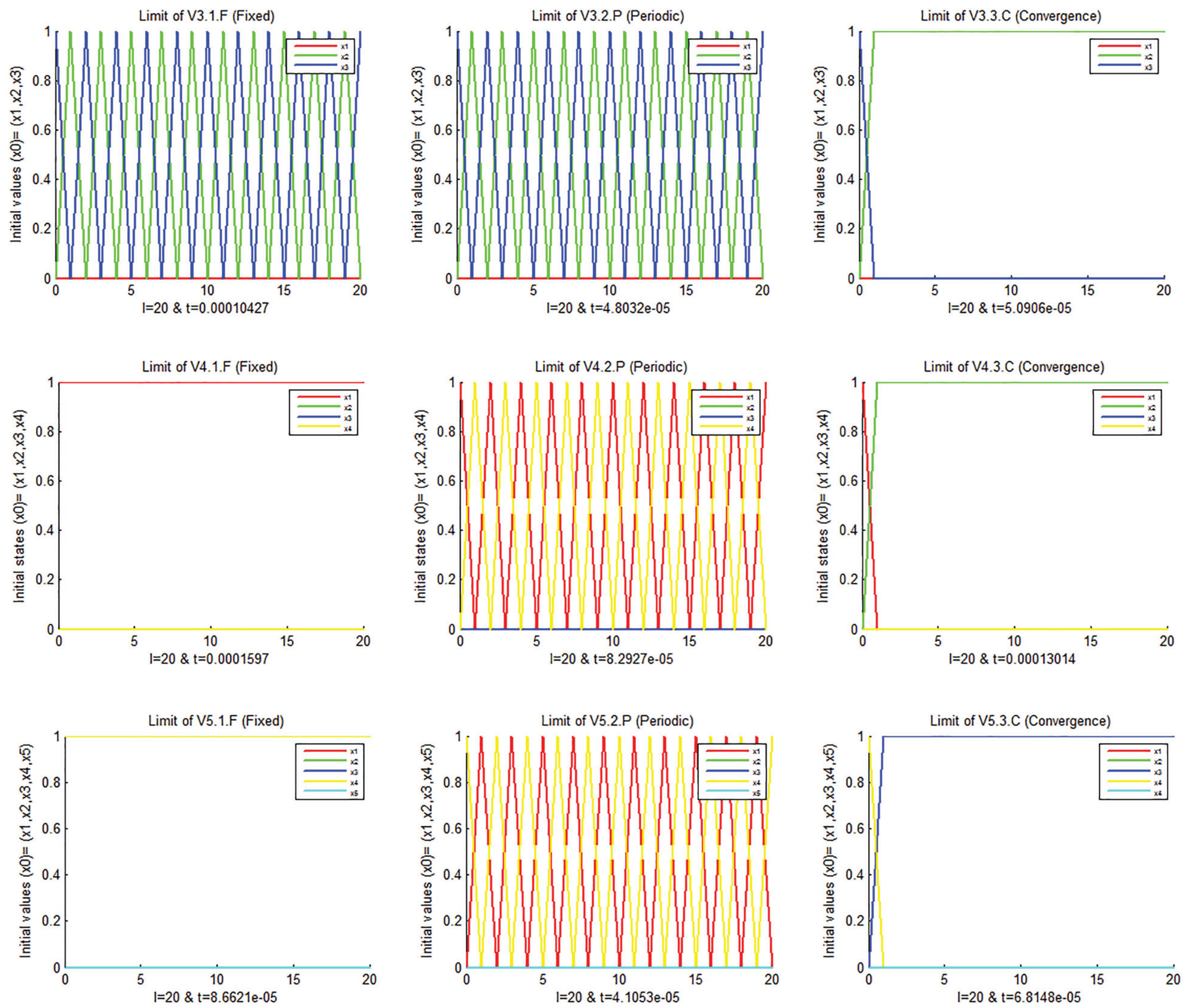

Figure 1. Limit behaviour of the trajectories for each EDSQO of V3.1.F, V3.1.P, V3.1.C, V4.1.P, V4.1.F, V4.1.C, V5.1.F, V5.1.P and V5.1.C of the initial extreme exterior values.

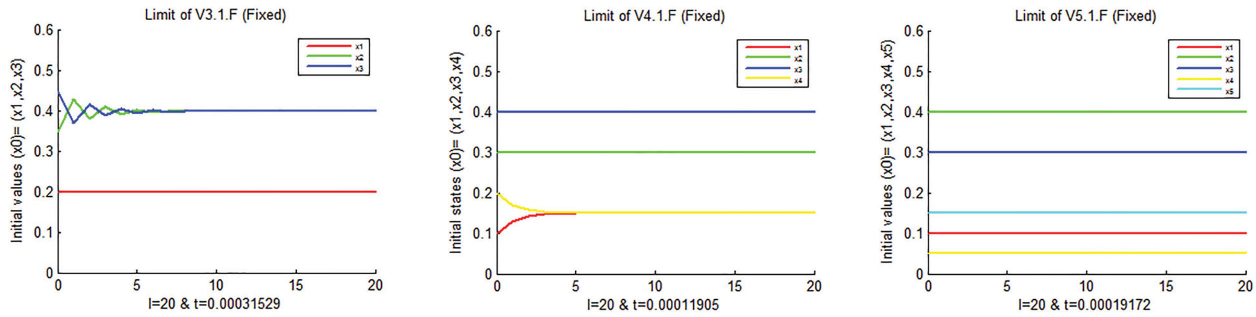

Figure 2. Limit behaviour of fixed points for EDSQOs of V3.1.F, V4.1.F and V5.1.F on 2DS, 3DS and 4DS, respectively.

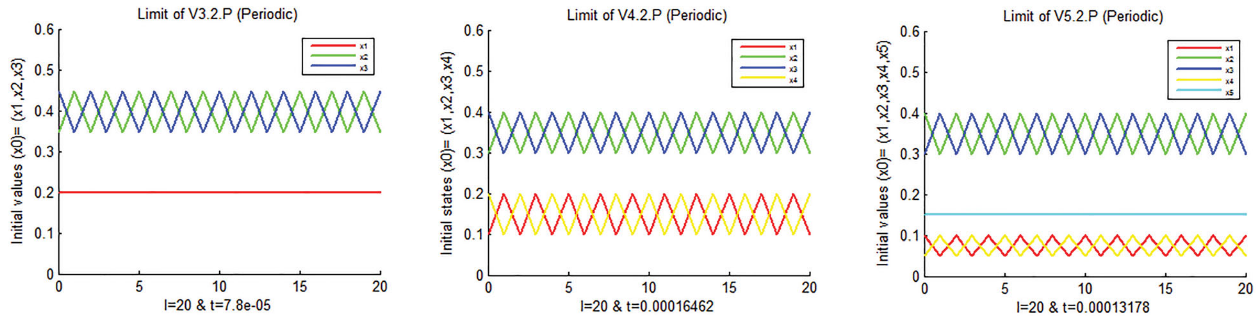

Figure 3. Limit behaviour of periodic points for EDSQOs V3.2.F, V4.2.F and V5.2.F on 2DS, 3DS and 4DS, respectively.

been demonstrated and proved in theorem 3.2 and if operator of EDSQOs on FDS has a point, which has a common factor as in Equation (3.1), then the limit of such points is fixed, consequently the operator does not converge.

4.3. *Limit behaviour of periodic points*

With reference to the fixed point in the theorem 3.2, the finding is that if there are two points or that can be expressed in terms of each other (as in Equation (3.6)) in the operator of EDSQOs, then these points are periodic as seen in Figure 3 in $V(x_2)$ with $V(x_3)$ of operator $V_{3.1.P}$, as well as $V(x_2)$ with $V(x_3)$ and $V(x_1)$ with $V(x_4)$ of operators $V_{4.1.P}$ and $V_{5.1.P}$ respectively. This is proven in theorem 3.3.

4.4. *Limit behaviour of convergence points*

Finally, we can investigate either the EDSQO on FDS is convergent to the centre $\left(\frac{1}{m}\right)$ point (as in Equation (3.8)) or that it contains no points which can be factored in terms of one another (see in Figure 4) in operators of $V_{3.1.C}$, $V_{4.1.C}$ and $V_{5.1.C}$. It is estimated that, the convergence for EDSQOs can be reached given that none of the initial values are equal to one. It is clear that the convergence of any EDSQOs is towards the centre $\left(\frac{1}{m}\right)$ as portrayed in Figure 4 that, the operators of $V_{3.1.C}$ on 2DS converge to $\left(\frac{1}{3}\right)$, while those of $V_{3.1.C}$ on 3DS converge to $\left(\frac{1}{4}\right)$ and of $V_{3.1.C}$ on 4DS converge to $\left(\frac{1}{5}\right)$. In essence, this is what demonstrated and proven in theorem 3.4.

We note that in Figures 1–4 the x-axis indicates initial values and y-axis indicates the number of iterations executed to reach the convergence. Moreover, I represents the number of iterations and t is the time spent on calculations of the operator.

5. Conclusion and future work

This work has studied the dynamic classifications of EDSQOs on FDS and investigated the limit behaviour

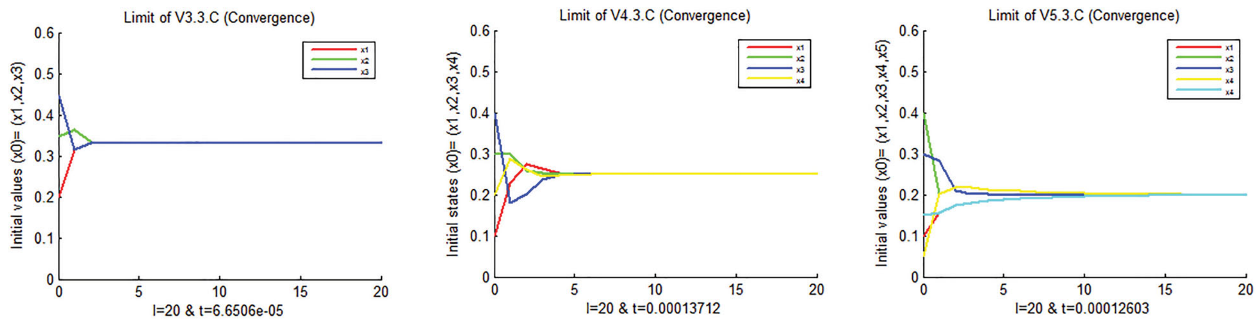

Figure 4. Limit behaviour of convergence points for EDSQOs V3.3.F, V4.3.F and V5.3.F on 2DS, 3DS and 4DS, respectively.

of trajectories of EDSQOs. It is obtained that the limit of EDSQOs converges to the central point if the operator satisfies the condition; it has no point with a common factor amongst its terms and it does not consist of two or more points that can be expressed in terms of one another. Empirically, the work has proven in theorem 3.1 that the EDSQO never converge from the extreme exterior initial points (where it must range from 0 to 1 and not exactly equal to one), it has fixed or periodic points. However, it has also been proven in theorem 3.2 that the operator of EDSQOs has a fixed point if this point has a (common) factor of its functional point. Meanwhile, it has been proven as well in theorem 3.3 that, if two points or more have a (common) factor amongst each other, then these points are periodic. In turn, it has been proved in theorem 3.4 that the EDSQOs on FDS is convergent to the centre $\left(\frac{1}{m}\right)$ if it has no fixed or periodic points and the initial values are not extremes. From this study, the EDSQOs have been classified on FDS to three classes: (i) fixed, (ii) periodic and (iii) convergence. Finally, the results of this study have been simulated by MatLab software and presented with the help of appropriate figures depicting the specific classes. The class of EDSQOs is very large on FDS. Therefore, the presented simulation has been considered for EDSQOs on 2DS, 3DS and 4DS, but the results are in turn generalized for FDS as it is proven in theorems 3.1, 3.2, 3.3 and 3.4. This work demonstrates that it is possible to achieve convergence in nonlinear-complexity protocol for consensus problem in MAS.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

We would like to thank Faculty of Information and Communication Technology, International Islamic University Malaysia for the support, and Malaysia-Japan International Institute of Technology, University Technology Malaysia KL Campus to fund this work by grant project [R.K430000.7743.4J009].

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