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An improved stability criterion for linear time-varying delay systems

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ABSTRACT

This paper considers the stability problem of linear systems with time-varying delays. A modified Lyapunov–Krasovskii functional (LKF) is constructed, which consists of delay-dependent matrices and double integral items under two time-varying subintervals. Based on the modified LKF, a less conservative stability criterion than some previous ones is derived. Furthermore, to obtain a tighter bound of the integral terms, the quadratic generalized free-weighting matrix inequality (QGFMI) is fully applied to different delay subintervals, which further reduces the conservatism of the stability criterion. Finally, three numerical examples are presented to show the effectiveness of the proposed approach.

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Delay-dependent stability; Lyapunov–Krasovskii functional; linear matrix inequalities; time-delayed system; time-varying delay

1. Introduction

The control theory of system is always a hot topic [1–8]. Time delays are of frequent occurrence in many practical systems, which often result in the major source of poor performance and instability. The stability problems of time-delayed systems have been a hot research topic. In this paper, the stability problems of linear systems with time-varying delays will been further analysed via the LKF method application. The linear systems with time-varying delays are described as:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t - h(t)), \\ x(s) = \psi(s), \quad s \in [-h, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system. A and A_1 are real constant matrices with appropriate dimensions. The time-varying delay $h(t)$ is continuous-time functional and satisfies the following conditions:

$$0 \le h(t) \le h, \quad |\dot{h}(t)| \le \mu < 1, \quad \forall t \ge 0, \tag{2}$$

where h and μ are positive constants.

To obtain stability criteria of system (1) by using the Lyapunov theorem, the main efforts are concentrated on the following several directions, one is finding an appropriate positive definite functional with a negative definite time derivative along the trajectory of system, e.g.LKF with delay partitioning approach [9,10], LKF with augmented terms [11–16], LKF with triple-integral and quadruple-integral terms [17–19], and so on. The other is reducing the upper bounds of the time derivative of LKF as much as possible by developing various inequality techniques, such as Jensen inequality [20], Wirtinger-based inequality [21], auxiliary function based inequality [22–25], Bessel-Legendre inequality [26], etc. Besides, further increasing the freedom of solving LMI, additional free-weighting-matrix technique is frequently introduced into the derivatives of LKF, for instance, the generalized zero equality [27,28], the one or second-order reciprocally convex combinations [29–31], the free-weighting-matrix approach [32], and so on. Based on those studies in [21,22,26,33], the tighter inequality seems to lead to less conservative results. Recently, Zhang [34] considered the effect of the LKFs and discussed the relationship between the tightness of inequalities and the conservatism of criteria. The results illustrate the integral inequality that makes the upper bound closer to the true value does not always lead to deduction of a less conservative stability condition if the LKF is not properly constructed. Thus, it is crucial to construct a proper LKF. Recently, to fully utilize the information of delay derivative, a novel LKF with delay-dependent matrix was constructed by Kwon et al. [35] as follows:

$$\begin{aligned} V(t) &= \zeta^T(t)P\zeta(t) + \int_{h-h(t)}^{t} \gamma^T(s)Q_1(t)\gamma(s)\,\mathrm{d}s \\ &\quad + \int_{t-h}^{h-h(t)} \gamma^T(s)Q_2(t)\gamma(s)\,\mathrm{d}s \\ &\quad + \int_{-h}^{0}\int_{t+\theta}^{t} \gamma^T(s)R\gamma(s)\,\mathrm{d}s\,\mathrm{d}\theta, \end{aligned} \tag{3}$$

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where $P > 0$, $R > 0$ and

$$Q_1(t) = Q_{01} - h(t)Q_{11} > 0,$$
$$Q_2(t) = Q_{20} + (h - h(t))Q_{21} > 0.$$
$$\zeta^T(t) = \Big[x^T(t)x^T(t-h(t))x^T(t-h) \times \int_{t-h(t)}^{t} x^T(s)\,ds \int_{t-h}^{t-h(t)} x^T(s)\,ds\Big],$$
$$\gamma^T(s) = [x^T(s)\ \dot{x}^T(s)].$$

Compared with the general delay-independent LKF, the Lyapunov matrices $Q_1(t)$, $Q_2(t)$ of the above LKF are delay-dependent, which can make further use of the information of time delay and derivative of delay. Kwon et al. have illustrated that the stability conditions based on the LKF (3) with delay-dependent matrices is less conservative than those based on the LKF without delay-dependent matrices. However, to bound the integral items $-\dot{h}(t)\int_{t-h(t)}^{t}\gamma^T\bar{Q}_a\gamma\,ds$ and $-\dot{h}(t)\int_{t-h}^{t-h(t)}\gamma^T\bar{Q}_b\gamma\,ds$ ($\bar{Q}_a > 0$, $\bar{Q}_b > 0$) via the QGFMI technique, two additional positive integral items $\mu\int_{t-h(t)}^{t}\gamma^T(s)\bar{Q}_a\gamma(s)\,ds$ and $\mu\int_{t-h}^{t-h(t)}\gamma^T(s)\bar{Q}_b\gamma(s)\,ds$ are introduced. Indeed, it is difficult to deal with the two integral terms due to their positive definiteness.

Moreover, the main inequalities of [35] were not incorrect due to the the following reasons:

the following terms in Theorem 1 of [35]

$$\begin{aligned} &H_2^TY_1G_3H_2, \quad H_3^TY_2G_4H_3,\\ &H_2^TY_3G_3H_2, \quad H_3^TY_4G_4H_3, \end{aligned} \tag{4}$$
$$\frac{\bar{d}-d(t)}{3}[H_3^TY_2\bar{Q}_b^{-1}Y_2^TH_3 + H_3^TY_4\bar{R}_b^{-1}Y_4^TH_3], \tag{5}$$
$$\frac{d(t)}{3}[H_2^TY_1\bar{Q}_a^{-1}Y_1^TH_2 + H_2^TY_3\bar{R}_a^{-1}Y_3^TH_2] \tag{6}$$

appeared in Ψ_2, Ψ_3 of inequalities (11)–(14) and (26) of [35], respectively, should be

$$\begin{aligned} &H_2^TW_1Y_1G_3H_2, \quad H_3^TW_2Y_2G_4H_3,\\ &H_2^TW_1Y_3G_3H_2, \quad H_3^TW_2Y_4G_4H_3, \end{aligned} \tag{7}$$
$$\frac{\bar{d}-d(t)}{3}[H_3^TW_2Y_2\bar{Q}_b^{-1}Y_2^TW_2H_3 + H_3^TW_2Y_4\bar{R}_b^{-1}Y_4^TW_2H_3], \tag{8}$$
$$\frac{d(t)}{3}[H_2^TW_1Y_1\bar{Q}_a^{-1}Y_1^TW_1H_2 + H_2^TW_1Y_3\bar{R}_a^{-1}Y_3^TW_1H_2] \tag{9}$$

with $W_1 = \mathrm{diag}\{d(t), 1, 1, 1,\}$ and $W_2 = \mathrm{diag}\{(\bar{d} - d(t)), 1, 1, 1,\}$.

Thus, all of the above terms (7)–(9) are nonlinear function of $d(t)$ and the inequalities (11)–(14) and (26) in Theorem 1 of [35] are not LMIs, although the Schur complement was used. Indeed, the first element of W_1, W_2, G_3, G_4 and $d(t)/3$, $(\bar{d} - d(t))/3$ lead to $d^2(t)$ and $d^3(t)$ contained in the terms (7)–(9). Although the terms (8)–(9) are transformed via the Schur complement application, the final forms $\Psi_{r[i,j]}$ still remain $d^2(t)$. Therefore, Theorem 1 of [35] is incorrect due to the above reasons.

Inspired by the above discussions, this paper constructs the following modified LKF (4) to analyse the stability of the linear time-delayed system (1).

$$V(t) = \sum_{i=1}^{4} V_i(t) \tag{10}$$

with

$$V_1(t) = \zeta^T(t)P\zeta(t),$$
$$V_2(t) = \int_{h-h(t)}^{t}\gamma^T(s)Q_1(t)\gamma(s)\,ds + \int_{t-h}^{h-h(t)}\gamma^T(s)Q_2(t)\gamma(s)\,ds,$$
$$V_3(t) = \int_{t-h(t)}^{t}\int_{\theta}^{t}\gamma^T(s)\bar{Q}_a\gamma(s)\,ds\,d\theta + \mu\int_{t-h}^{t-h(t)}\int_{\theta}^{t}\gamma^T(s)\bar{Q}_b\gamma(s)\,ds\,d\theta,$$
$$V_4(t) = \int_{t-h(t)}^{t}\int_{\theta}^{t}\gamma^T(s)R_1\gamma(s)\,ds\,d\theta + \int_{t-h}^{t-h(t)}\int_{\theta}^{t}\gamma^T(s)R_2\gamma(s)\,ds\,d\theta,$$

where, P, $Q_1(t)$, $Q_2(t)$, $\bar{Q}_a$, $\bar{Q}_b$, R_1 and R_2 are positive definite matrices.

The main contributions of this paper are summed up as follows:

- To avoid introducing the two additional positive integral items $\mu\int_{t-h(t)}^{t}\gamma^T(s)\bar{Q}_a\gamma(s)\,ds$ and $\mu\int_{t-h}^{t-h(t)}\gamma^T(s)\bar{Q}_b\gamma(s)\,ds$, the paper gives the modified LKF with two double integral items $V_3(t)$;
- All of the integral items $V_2(t)$, $V_3(t)$ and $V_4(t)$ are constructed under two subintervals ($[0, h(t)]$ and $[h(t), h]$) instead of being considered directly. It further makes full use of the information of time-varying delays $h(t)$, $h - h(t)$ and their derivative. The QGFMI technique can be utilized fully in each subinterval, which can further reduces the conservatism of the stability condition.
- To avoid introducing non-linear terms with $h^2(t)$ and $h^3(t)$, this paper introduces some necessary state variables into the LKF, which not only increases the freedom for solving LMIs, but also removes the non-linear terms with $h^2(t)$ and $h^3(t)$ in [35].

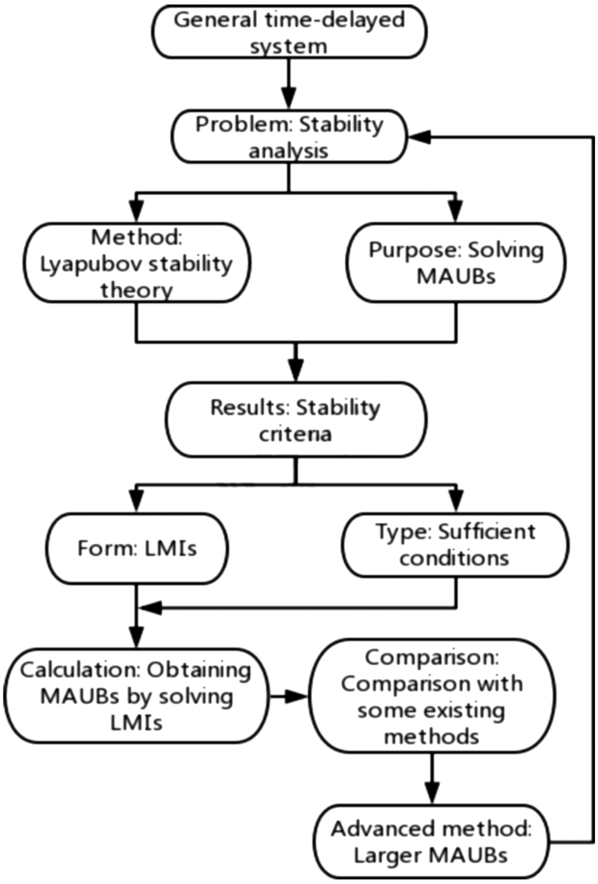

Figure 1. The research flow chart for this paper.

Finally, based on the LKF (10) proposed in this paper, a new less conservative stability criterion is obtained than some of the recent existing ones.

This paper is organized as follows. Section 2 gives the problem statement and provides some lemmas. Section 3 presents a new stability criterion for the linear time-varying delay system. Numerical examples and conclusions are drawn in Sections 4 and 5, respectively.

Notation: The notation $P > 0\,(< 0)$ mean that matrix P is positive (negative) definite. I represents an identity matrix with the corresponding dimension. $*$ denotes the symmetric terms in a block matrix and diag$\{\cdots\}$ denotes a block-diagonal matrix. e_i $(i = 1, \ldots, m)$ is block entry matrices. For example, $e_2 = [0\ I\ \underbrace{0 \cdots 0}_{m-2}]$. $F^{[h(t)]}$ denotes F is the function of $h(t)$. $\mathrm{Sym}\{B\} = B + B^T$.

2. Problem formulation

The main purpose of this paper is to derive the stability criterion for the system (1) satisfying the condition (2) by using a modifyed LKF. To further explain the topic of this paper, the following research flow chart is given in Figure 1. To achieve this purpose, the following lemma is important.

Lemma 2.1 (QGFMI [35]): *For given any matrices X, Y, a positive matrix R and a continuous differentiable function $\{\omega(s) \mid s \in [a, b]\}$, the following inequality holds*

$$
\begin{aligned}
&-\int_a^b \omega^T(s) R\omega(s)\,\mathrm{d}s \\
&\le \begin{bmatrix}\eta_0\\ \eta_1\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (b-a)XR^{-1}X^T & X[I\ 0] \\ * & \frac{b-a}{3}YR^{-1}Y^T \\ & +\mathrm{Sym}\{Y[-I\ 2I]\}\end{bmatrix} \\
&\times \begin{bmatrix}\eta_0\\ \eta_1\end{bmatrix},
\end{aligned}
$$

where η_0 is an any vector, and

$$
\eta_1^T = \left[\int_a^b \omega^T(s)\,\mathrm{d}s \quad \frac{1}{b-a}\int_a^b \int_\theta^b \omega^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\right].
$$

3. Main results

The main objective of this section is to describe the effect of the proposed LKF on reducing the conservatism of the stability criterion. A novel theorem is derived by employing the LKF proposed in previous section and a corollary for comparison is also given. For the sake of simplicity on matrix representation, the notations of several symbols and matrices are defined as:

$$
\begin{aligned}
h_d &= 1 - \dot{h}(t), \quad \bar{\mu} = \mu + \dot{h}(t), \\
\gamma^T(s) &= [x^T(s)\ \dot{x}^T(s)], \\
v_1(t) &= \int_{t-h(t)}^t \frac{x^T(s)}{h(t)}\,\mathrm{d}s, \\
v_2(t) &= \int_{t-h}^{t-h(t)} \frac{x^T(s)}{h-h(t)}\,\mathrm{d}s, \\
\zeta^T(t) &= [x^T(t)\, x^T(t-h(t))\, x^T(t-h) \\
&\quad \times h(t)v_1(t)\ (h-h(t))v_2(t)], \\
\xi^T(t) &= \Big[x^T(t)\ x^T(t-h(t))\, x^T(t-h)\ v_1(t)\ v_2(t) \\
&\quad \times \dot{x}^T(t)\ \dot{x}^T(t-h(t))\ \dot{x}^T(t-h)\frac{1}{h(t)} \\
&\quad \times \int_{t-h(t)}^t \int_u^t x^T(s)\,\mathrm{d}u\,\mathrm{d}s \frac{1}{h-h(t)} \\
&\quad \times \int_{t-h}^{t-h(t)} \int_u^{t-h(t)} x^T(s)\,\mathrm{d}u\,\mathrm{d}s \\
&\quad \times h(t)v_1(t)\ (h-h(t))v_2(t)\Big].
\end{aligned}
$$

Theorem 3.1: *The system (1) satisfying the conditions (2) is stable for given values of $h \ge 0$, $\mu < 1$, if there exist real positive definite matrices $P \in \mathbb{R}^{5n\times 5n}$, $(R_1, R_2, Q_1(t), Q_2(t), \bar{Q}_a, \bar{Q}_b \in \mathbb{R}^{2n\times 2n})$, symmetric matrices $(Q_a, Q_b, R_a, R_b \in \mathbb{R}^{n\times n})$ and any matrices $(\bar{U} \in \mathbb{R}^{3n\times n})$, $N \in \mathbb{R}^{4n\times 2n}$, $X_i \in \mathbb{R}^{7n\times 2n}$, $Y_i \in \mathbb{R}^{4n\times 2n}$,*

$(i = 1, 2, 3, 4)$ *such that* $\bar{R}_a > 0$, $\bar{R}_b > 0$ *and the following LMIs hold*

$$\begin{bmatrix} \Pi^{[0,\dot{h}(t)]} & \Omega_{a[1,4]} & \bar{\mu}\Omega_{a[1,2]} & h_d\Omega_{b[3,4]} & \Omega_{b[3,2]} \\ * & -h\bar{R}_b & 0 & 0 & 0 \\ * & * & -\bar{\mu}h\bar{Q}_b & 0 & 0 \\ * & * & * & -3h_dh\bar{R}_b & 0 \\ * & * & * & * & -3h\bar{Q}_a \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} \Pi^{[h,\dot{h}(t)]} & h_d\Omega_{a[1,3]} & \Omega_{a[1,1]} & h_d\Omega_{b[2,3]} & \Omega_{b[2,1]} \\ * & -h_dR_a & 0 & 0 & 0 \\ * & * & -h\bar{Q}_a & 0 & 0 \\ * & * & * & -3h_dh\bar{R}_a & 0 \\ * & * & * & * & -3h\bar{Q}_a \end{bmatrix} < 0, \tag{12}$$

where the definitions of some symbols are shown in Appendix 1.

Proof: Construct an **LKF** (10).

The time derivative of $V(t)$ with respect to time along the trajectory of the system (1) is as follows:

$$\dot{V}_1(t) = 2\zeta^T(t)P \begin{bmatrix} \dot{x}(t) \\ h_d\dot{x}(t-h(t)) \\ \dot{x}(t-h) \\ x(t) - h_dx(t-h(t)) \\ h_dx(t-h(t)) - x(t-h) \end{bmatrix},$$

$$\begin{aligned} \dot{V}_2(t) &= \gamma^T(t)Q_1(t)\gamma(t) + h_d\gamma^T(t-h(t)) \\ &\quad \times [Q_2(t) - Q_1(t)]\gamma(t-h(t)) \\ &\quad - \gamma^T(t-h)Q_2(t)\gamma(t-h) \\ &\quad - \dot{h}(t)\int_{t-h(t)}^{t} \gamma^T(s)Q_{11}\gamma(s)\,ds \\ &\quad - \dot{h}(t)\int_{t-h}^{t-h(t)} \gamma^T(s)Q_{21}\gamma(s)\,ds, \\ \dot{V}_3(t) &= h(t)\gamma^T(t)\bar{Q}_a\gamma(t) + \mu h_d(h-h(t))\gamma^T \\ &\quad \times (t-h(t))\bar{Q}_b\gamma(t-h(t)) \\ &\quad - h_d\int_{t-h(t)}^{t} \gamma^T(s)\bar{Q}_a\gamma(s)\,ds \\ &\quad - \mu\int_{t-h}^{t-h(t)} \gamma^T(s)\bar{Q}_b\gamma(s)\,ds, \\ \dot{V}_4(t) &= h(t)\gamma^T(t)R_1\gamma(t) + h_d(h-h(t))\gamma^T \\ &\quad \times (t-h(t))R_2\gamma(t-h(t)) \\ &\quad - h_d\int_{t-h(t)}^{t} \gamma^T(s)R_1\gamma(s)\,ds \\ &\quad - \int_{t-h}^{t-h(t)} \gamma^T(s)R_2\gamma(s)\,ds. \end{aligned}$$

For additional symmetric matrices Q_a, Q_b, R_a and R_b, the following zero equations are satisfied

$$\begin{aligned} 0 = \dot{h}(t)\Bigg[& x^T(t)Q_ax(t) - x^T(t-h(t))Q_ax(t-h(t)) \\ & - 2\int_{t-h(t)}^{t} x^T(s)Q_a\dot{x}(s)\,ds + x^T(t-h(t)) \\ & \times Q_bx(t-h(t)) - x^T(t-h)Q_bx(t-h) \\ & -2\int_{t-h}^{t-h(t)} x^T(s)Q_b\dot{x}(s)\,ds \Bigg], \end{aligned} \tag{13}$$

$$\begin{aligned} 0 = h_d\Bigg[& x^T(t)R_ax(t) - x^T(t-h(t))R_ax(t-h(t)) \\ & - 2\int_{t-h(t)}^{t} x^T(s)R_a\dot{x}(s)\,ds \Bigg] \\ & + x^T(t-h(t))R_bx(t-h(t)) - x^T(t-h)R_bx(t-h) \\ & - 2\int_{t-h}^{t-h(t)} x^T(s)R_b\dot{x}(s)\,ds. \end{aligned} \tag{14}$$

Taking the zero inequalities in $\dot{V}_2$ and $\dot{V}_4$, we have the following integral terms from $\dot{V}_2$ to $\dot{V}_4$

$$\begin{aligned} \varphi = & -\int_{t-h(t)}^{t} \gamma^T(s)\bar{Q}_a\gamma(s)\,ds \\ & - \bar{\mu}\int_{t-h}^{t-h(t)} \gamma^T(s)\bar{Q}_b\gamma(s)\,ds \\ & - h_d\int_{t-h(t)}^{t} \gamma^T(s)\bar{R}_a\gamma(s)\,ds \\ & - \int_{t-h}^{t-h(t)} \gamma^T(s)\bar{R}_b\gamma(s)\,ds. \end{aligned} \tag{15}$$

It follows from Lemma 1 with an augmented vector $\gamma(s)$ that

$$\begin{aligned} & -\int_{t-h(t)}^{t} \gamma^T(s)\bar{Q}_a\gamma(s)\,ds \\ & \leq \xi^T(t)\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} h(t)X_1\bar{Q}_a^{-1}X_1^T & X_1G_1 \\ * & \frac{d(t)}{3}Y_1\bar{Q}_a^{-1}Y_1^T + \mathrm{Sym}\{Y_1G_2\} \end{bmatrix} \\ & \quad \times \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}\xi(t), \end{aligned} \tag{16}$$

$$\begin{aligned} & -\bar{\mu}\int_{t-h}^{t-h(t)} \gamma^T(s)\bar{Q}_b\gamma(s)\,ds \\ & \leq \bar{\mu}\xi^T(t)\begin{bmatrix} H_1 \\ H_3 \end{bmatrix}^{\mathrm{T}} \\ & \quad \times \begin{bmatrix} (h-h(t))X_2\bar{Q}_b^{-1}X_2^T & X_2G_1 \\ * & \frac{h-h(t)}{3}Y_2\bar{Q}_b^{-1}Y_2^T + \mathrm{Sym}\{Y_2G_2\} \end{bmatrix} \\ & \quad \times \begin{bmatrix} H_1 \\ H_3 \end{bmatrix}\xi(t), \end{aligned} \tag{17}$$

$$- h_d\int_{t-h(t)}^{t} \gamma^T(s)\bar{R}_a\gamma(s)\,ds$$

$$\leq h_d\xi^T(t)\begin{bmatrix}H_1\\H_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}h(t)X_3\bar{R}_a^{-1}X_3^T & X_3G_1\\ * & \frac{h(t)}{3}Y_3\bar{R}_a^{-1}Y_3^T\\ & +\mathrm{Sym}\{Y_3H_3\}\end{bmatrix}$$

$$\times\begin{bmatrix}H_1\\H_2\end{bmatrix}\xi(t), \tag{18}$$

$$-\int_{t-h}^{t-h(t)}\gamma^T(s)\bar{R}_b\gamma(s)\,\mathrm{d}s$$

$$\leq \xi^T(t)\begin{bmatrix}H_1\\H_3\end{bmatrix}^{\mathrm{T}}$$

$$\times\begin{bmatrix}(h-h(t))X_4\bar{R}_b^{-1}X_4^T & X_4G_1\\ * & \frac{h-h(t)}{3}Y_4\bar{R}_b^{-1}Y_4^T\\ & +\mathrm{Sym}\{Y_4G_2\}\end{bmatrix}$$

$$\times\begin{bmatrix}H_1\\H_3\end{bmatrix}\xi(t). \tag{19}$$

For any appropriately dimensioned matrices $N\in\mathbb{R}^{4n\times 2n}$, $\bar{U}^T=[U_1^T\ U_2^T\ U_3^T]$, it is true that

$$0=2[v_1(t)\ v_2(t)\ h(t)v_1(t)\ (h-h(t))v_2(t)]N[h(t)e_4 - e_{11}\ (h-h(t))e_5-e_{12}]\xi(t), \tag{20}$$

$$0=2[x^T(t)\ x^T(t-h(t))\ \dot{x}^T(t)]\bar{U}[\bar{A}x(t) + \bar{A}_dx(t-h(t))-\dot{x}(t)]. \tag{21}$$

Finally, from the above derivation, we have

$$\begin{aligned}\dot{V}(t) &\leq \xi^T(t)\Big\{\Pi^{[h(t),\dot{h}(t)]}+h(t)[h_dH_1^TX_3\bar{R}_a^{-1}X_3^TH_1\\&+H_1^TX_1\bar{Q}_a^{-1}X_1^TH_1]+(h-h(t))[H_1^TX_4\bar{R}_b^{-1}X_4^TH_1\\&+\bar{\mu}H_1^TX_2\bar{Q}_b^{-1}X_2^TH_1]+h(t)\left[\frac{h_d}{3}H_2^TY_3\bar{R}_a^{-1}Y_3^TH_2\right.\\&\left.+\frac{1}{3}H_2^TY_1\bar{Q}_a^{-1}Y_1^TH_2\right]+(h-h(t))\\&\times\left[\frac{1}{3}H_3^TY_4\bar{R}_b^{-1}Y_4^TH_3+\frac{\bar{\mu}}{3}H_3^TY_2\bar{Q}_b^{-1}Y_2^TH_3\right]\Big\}\xi(t).\end{aligned} \tag{22}$$

Therefore, LMIs (11)–(12) hold for $[h(t),\dot{h}(t)]\in\{[0,h]\times[-\mu,\mu]\}$, which together with *Schur complement equivalence* imply that $\dot{V}(t)<0$. This shows that system (1) is stable from Lyapunov stability theory. This completes the proof. ■

Remark 3.1: It is important to reduce the conservativeness of stability conditions by introducing some delay-dependent matrices into the single-integral terms of LKF [35]. Unfortunately, the main result of [35] was incorrect due to the neglect of the terms with $h^2(t)$ even $h^3(t)$ discussed in the section 1. In this paper, all matrices inequations of Theorem 3.1 are LMI by introducing additional state variables $h(t)v_1(t)$, $(h-h(t))v_2(t)$ and a zero equation (20), which can remove the terms with $h^2(t)$ even $h^3(t)$. Moreover, the double integral items $V_4(t)$ of the LKF (10) divide the time delay interval $[0,h]$ into two subintervals ($[0,h(t)]$ and $[h(t),h]$) instead of using the item $\int_{t-h}^{t}\gamma^T(s)R\gamma(s)\,\mathrm{d}s$ directly. It further makes full use of the information of time-varying delays $h(t)$, $h-h(t)$ and their derivative. Thus, the QGFMI technique can be fully used in each subinterval, which can further reduce the conservatism of the stability conditions.

Remark 3.2: It is worth noting that in [35], to bound the integral item for $-\mu\leq\dot{h}(t)\leq\mu$, $(\mu>0)$

$$-\dot{h}(t)\int_{t-h(t)}^{t}\gamma^T\bar{Q}_a\gamma\,\mathrm{d}s-\dot{h}(t)\int_{t-h}^{t-h(t)}\gamma^T\bar{Q}_b\gamma\,\mathrm{d}s$$

via the QGFMI technique, the following addition zero equation is introduced

$$\begin{aligned}0&=\mu\int_{t-h(t)}^{t}\gamma^T(s)\bar{Q}_a\gamma(s)\,\mathrm{d}s\\&-\mu\int_{t-h(t)}^{t}\gamma^T(s)\bar{Q}_a\gamma(s)\,\mathrm{d}s\\&+\mu\int_{t-h}^{t-h(t)}\gamma^T(s)\bar{Q}_b\gamma(s)\,\mathrm{d}s\\&-\mu\int_{t-h}^{t-h(t)}\gamma^T(s)\bar{Q}_b\gamma(s)\,\mathrm{d}s.\end{aligned} \tag{23}$$

Then, the above integral item can be rewritten as the following form:

$$\begin{aligned}\tilde{\varphi}&=-(\mu+\dot{h}(t))\int_{t-h(t)}^{t}\gamma^T\bar{Q}_a\gamma\,\mathrm{d}s\\&-(\mu+\dot{h}(t))\int_{t-h}^{t-h(t)}\gamma^T\bar{Q}_b\gamma\,\mathrm{d}s\\&+\mu\int_{t-h(t)}^{t}\gamma^T\bar{Q}_a\gamma\,\mathrm{d}s+\mu\int_{t-h}^{t-h(t)}\gamma^T\bar{Q}_b\gamma\,\mathrm{d}s.\end{aligned} \tag{24}$$

The first two items on the right can be bounded via the QGFMI technique (for details see Equation (17) of this paper), however there are fewer proper techniques for obtaining a tight upper bound of the integral terms $\mu\int_{t-h(t)}^{t}\gamma^T(s)\bar{Q}_a\gamma(s)\,\mathrm{d}s$ and $\mu\int_{t-h}^{t-h(t)}\gamma^T(s)\bar{Q}_b\gamma(s)\,\mathrm{d}s$ due to their positive definiteness. Thus, to avoid introducing the two positive define integral terms, we give the modified LKF (10) with a double integral item $V_3(t)$.

To illustrate the effectiveness of delay-dependent matrices in reducing the conservativeness of stability conditions, we give the following corollary by choosing the following LKF:

$$\tilde{V}(t) = \sum_{i=1}^{3} \tilde{V}_i(t) \tag{25}$$

with

$$\tilde{V}_1(t) = \zeta^T(t)P\zeta(t),$$
$$\tilde{V}_2(t) = \int_{h-h(t)}^{t} \gamma^T(s)Q_1\gamma(s)\,\mathrm{d}s + \int_{t-h}^{h-h(t)} \gamma^T(s)Q_2\gamma(s)\,\mathrm{d}s,$$
$$\tilde{V}_3(t) = \int_{t-h(t)}^{t}\int_{\theta}^{t} \gamma^T(s)R_1\gamma(s)\,\mathrm{d}s\,\mathrm{d}\theta + \int_{t-h}^{t-h(t)}\int_{\theta}^{t} \gamma^T(s)R_2\gamma(s)\,\mathrm{d}s\,\mathrm{d}\theta,$$

where, P, Q_1, Q_2, R_1 and R_2 are positive definite matrices.

Corollary 3.1: *The system (1) satisfying the conditions (2) is stable for given values of $h \geq 0$, $\mu < 1$, if there exist real positive definite matrices $P \in \mathbb{R}^{5n\times 5n}$, $(R_1, R_2, Q_1, Q_2 \in \mathbb{R}^{2n\times 2n})$, symmetric matrices $(R_a, R_b \in \mathbb{R}^{n\times n})$ and any matrices $(\bar{U} \in \mathbb{R}^{3n\times n})$, $N \in \mathbb{R}^{4n\times 2n}$, $X_i \in \mathbb{R}^{7n\times 2n}$, $Y_i \in \mathbb{R}^{4n\times 2n}$, $(i = 1, 2)$ such that $\bar{R}_a > 0$, $\bar{R}_b > 0$ and the following LMIs hold*

$$\begin{bmatrix} \tilde{\Pi}^{[0,\dot{h}(t)]} & \tilde{\Omega}_{a[1,2]} & h_d\tilde{\Omega}_{b[3,2]} \\ * & -h\bar{R}_b & 0 \\ * & * & -3h_dh\bar{R}_b \end{bmatrix} < 0, \tag{26}$$

$$\begin{bmatrix} \tilde{\Pi}^{[h,\dot{h}(t)]} & h_d\tilde{\Omega}_{a[1,1]} & h_d\tilde{\Omega}_{b[2,1]} \\ * & -h_d\bar{R}_a & 0 \\ * & * & -3h_dh\bar{R}_a \end{bmatrix} < 0, \tag{27}$$

where the definitions of some symbols are shown in Appendix 2.

Proof: The steps of proof are similar to those of Theorem 3.1, so it is omitted here. ■

4. Numerical example

In this section, two numerical examples will be provided to illustrate the effectiveness and superiority of Theorem 3.1, in which the maximum allowable upper bounds (MAUBs) are carefully compared, including the numbers of decision variables (NoVs).

Remark 4.1: In this paper, the strict LMIs (11) and (12) can be solved easy by using the Matlab LMI-toolbox. The search steps are:

- Step 1. For given values of $h \geq 0$, $\mu < 1$, LMIs (11) and (12) can be solved to find the **MAUB** $h \triangleq h_1$;
- Step 2. Solve LMIs (11) and (12) by progressively increasing h from h_1 obtained in Step 1 until LMIs in (11) and (12) are infeasible. The obtained **MAUB** value is denoted as h_2;
- Step 3. Based on the search steps above, if there is no h_2 in Stept 2, the best result is h_1; otherwise, the best result is h_2.

Table 1. **MAUBs** h for different μ (Example 4.1).

Methods\\μ	0.05	0.1	0.3	0.5	NoVs
[36] (Th. 1)	2.613	2.424	2.131	1.793	$27n^2 + 4n$
[33] (Th. 2. C2)	2.598	2.397	2.128	1.787	$23n^2 + 4n$
[37] (Th. 1)	2.573	2.420	2.133	2.005	$142n^2 + 18n$
[38] (Th. 1)	2.575	2.425	2.230	2.019	$114n^2 + 18n$
[39] (Th. 3)	2.590	2.438	2.240	2.026	$70n^2 + 12n$
Corollary 3.1	2.571	2.419	2.131	2.001	$81.5n^2 + 11.5n$
Theorem 3.1	3.006	2.887	2.624	2.318	$105.5n^2 + 12.5n$

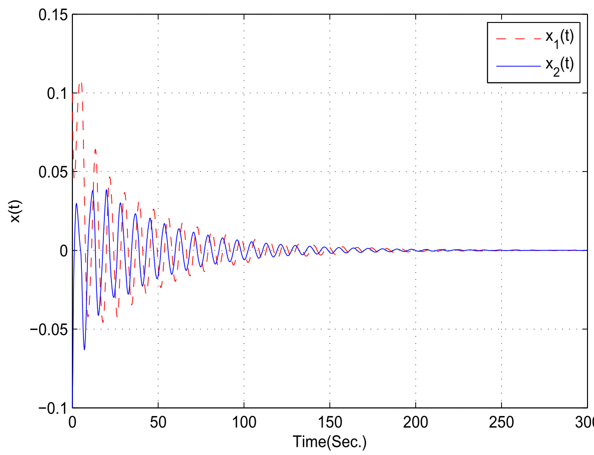

Figure 2. The state responses for Example 4.1.

Example 4.1: Consider the following systems:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x(t - h(t)). \tag{28}$$

The comparative results among some previous stability conditions, Theorem 3.1 and Corollary 3.1 are listed in Table 1. It is clear that the MAUBs obtained by Theorem 3.1 and Corollary 3.1 are all larger than those obtained by other conditions proposed in [33,35–39]. As expected, the MAUBs obtained by Theorem 3.1 are the largest for different values μ, which matches the explanation in Remark 3.1. On the other hand, the number of decision values in our criteria is larger than those in [33,35,36,39] but less than those in [37,38]. To confirm the obtained result ($h = 3.006$), the simulation result is shown in Figure 2, which shows that the state responses of the system (28) with $h(t) = \frac{3.006}{2} + \frac{3.006}{2}\sin(\frac{0.1t}{3.006})$ converge to zero under the random initial state.

Example 4.2: Consider the following systems:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - h(t)). \tag{29}$$

Table 2. **MAUBs** *h* for different μ (Example 4.2).

Methods\μ	0.1	0.2	0.5	0.8	NoVs
[33]	6.6103	4.0034	1.6875	1.0287	$23n^2+4n$
[37]	7.1672	4.5179	2.4158	1.8384	$142n^2+18n$
[38]	7.1765	4.5438	2.4963	1.9225	$114n^2+18n$
[40]	7.2030	4.5126	2.3860	1.8476	$203n^2+9n$
[39]	7.1905	4.5275	2.4473	1.8562	$70n^2+12n$
[11]	7.2734	4.6213	2.6505	2.0612	$78.5n^2+2.5n$
[12]	7.4001	4.7954	2.7175	2.0894	$108n^2+12n$
[41]	7.92	5.46	2.77	2.76	$91.5n^2+4.5n$
Corollary 3.1	7.1633	4.5143	2.4137	1.8349	$81.5n^2+11.5n$
Theorem 3.1	8.3352	5.6627	2.9275	2.8871	$104.5n^2+9.5n$

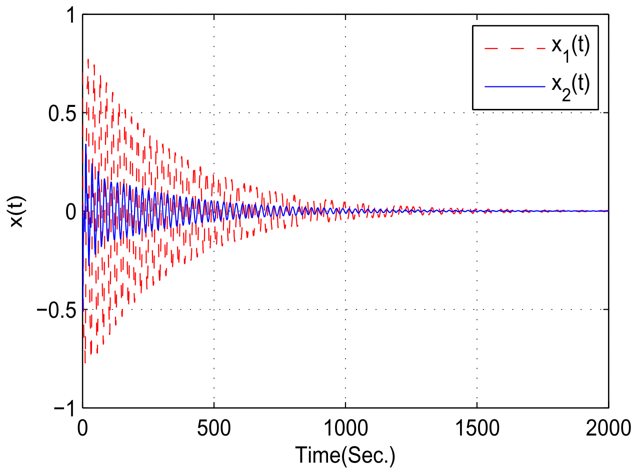

Figure 3. The state responses for Example 4.2.

For different lower bounds of delay derivative μ, the MAUBs are shown in Table 2 by utilizing Theorem 3.1, Corollary 3.1 and some methods in the literature. From Table 2, Theorem 3.1 gives larger MAUB than Corollary 3.1 and some existing ones. And the number of decision values in our criteria is larger than those in [11,33,35,39,41] and less than those in [12,37,38,40]. To confirm the obtained result ($h = 8.3352$), the simulation result is shown in Figure 3, which shows that the state responses of the system (29) with $h(t) = \frac{8.3352}{2} + \frac{8.3352}{2} sin(\frac{0.2t}{8.3352})$ converge to zero under the random initial state.

Example 4.3: Consider the following systems:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h(t)). \tag{30}$$

The MAUBs are shown in Table 3 for different μ by utilizing Theorem 3.1, Corollary 3.1 and some methods in the literature. From Table 3, Theorem 3.1 gives larger MAUB than Corollary 3.1 and some existing ones.

Remark 4.2: The final simulation comparison for the stability analysis of time-delayed systems is usual numerical example. The values of delay bound are the most convenient and direct comparison result. The maximum allowable upper bounds and the numbers of the decision variables can be found in the Tables 1 and 2, which can show our stability criterion is less conservative than some existing ones.

Table 3. **MAUBs** *h* for different μ (Example 4.3).

Methods\μ	0.1	0.2	0.5	0.8	NoVs
[36]	4.753	3.873	2.429	2.183	$27n^2+4n$
[33]	4.714	3.855	2.608	2.375	$23n^2+4n$
[32]	4.788	4.065	3.055	2.615	$65n^2+11n$
[34]	4.809	4.091	3.109	2.710	$25n^2+7n$
[37]	4.829	4.139	3.155	2.730	$142n^2+18n$
[38]	4.831	4.142	3.148	2.713	$114n^2+18n$
[39]	4.844	4.142	3.117	2.698	$70n^2+12n$
[42]	4.910	4.233	3.309	2.882	$54.5n^2+6.5n$
[40]	4.944	4.274	3.305	2.850	$203n^2+9n$
[12]	4.942	4.234	3.309	2.882	$108n^2+12n$
Corollary 3.1	4.835	4.149	3.192	2.836	$81.5n^2+11.5n$
Theorem 3.1	4.968	4.346	3.725	3.470	$104.5n^2+9.5n$

5. Conclusions

In this paper, the stability of time-varying delay systems is studied. A modified Lyapunov functional with delay-dependent matrices is proposed and used to derive the stability criterion. Based on delay decomposition method, the integral items of the LKF divide the time delay interval $[0, h]$ into two subintervals ($[0, h(t)]$ and $[h(t), h]$) instead of using the item $\int_{t-h}^{t} \gamma^T(s)R\gamma(s)\,ds$ directly. The Lemma 2.1 (QGFMI technique) can be fully used in each subinterval, which further reduces the conservatism of the stability condition. Three numerical examples illustrate the effectiveness of the proposed method.

The novel stability condition can also be generalized to some other time-delayed control systems, for example, time-delayed Lur'e systems, time-delayed neural networks, time-delayed neutral-type systems, and so on. However, there is still some distances from practical application because of the complexity of the control theory, which can be something that we will continue to study in the future.

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Appendices

Appendix 1

$$\Pi^{[h(t),\dot{h}(t)]} = \mathrm{Sym}\{\Pi_1^{[h(t),\dot{h}(t)]}\} + \Pi_2^{[h(t),\dot{h}(t)]} + \Pi_3^{[h(t),\dot{h}(t)]},$$

$$\begin{aligned}\Pi_1^{[h(t),\dot{h}(t)]} &= \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ h(t)e_4 \\ (h-h(t))e_5 \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} e_6 \\ h_d e_7 \\ e_8 \\ e_1 - h_d e_2 \\ h_d e_2 - e_3 \end{bmatrix} + \Pi_1 \bar{U} \Pi_2^T \\
&+ [e_4^T\ e_5^T\ e_{11}^T\ e_{12}^T] N [h(t)e_4^T \\
&- e_{11}^T\ (h-h(t))e_5^T - e_{12}^T]^T, + H_1^T X_1 G_1 H_2 \\
&+ \bar{\mu} H_1^T X_2 G_1 H_3 + h_d H_1^T X_3 G_1 H_2 + H_1^T X_4 G_1 H_3, \\
&+ h_d H_2^T Y_3 G_2 H_2 + H_3^T Y_4 G_2 H_3 \\
&+ H_2^T Y_1 G_2 H_2 + \bar{\mu} H_3^T Y_2 G_2 H_3,\end{aligned}$$

$$\begin{aligned}\Pi_2^{[h(t),\dot{h}(t)]} &= \begin{bmatrix} e_1 \\ e_6 \end{bmatrix}^{\mathrm{T}} Q_1(t) \begin{bmatrix} e_1 \\ e_6 \end{bmatrix} \\
&- h_d \begin{bmatrix} e_2 \\ e_7 \end{bmatrix}^{\mathrm{T}} (Q_1(t) - Q_2(t)) \begin{bmatrix} e_2 \\ e_7 \end{bmatrix} \\
&- \begin{bmatrix} e_3 \\ e_8 \end{bmatrix}^{\mathrm{T}} Q_2(t) \begin{bmatrix} e_3 \\ e_8 \end{bmatrix} + h(t) \begin{bmatrix} e_1 \\ e_6 \end{bmatrix}^{\mathrm{T}} \bar{Q}_a \begin{bmatrix} e_1 \\ e_6 \end{bmatrix} \\
&+ \mu h_d (h-h(t)) \begin{bmatrix} e_2 \\ e_7 \end{bmatrix}^{\mathrm{T}} \bar{Q}_b \begin{bmatrix} e_2 \\ e_7 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\Pi_3^{[h(t),\dot{h}(t)]} &= h(t) \begin{bmatrix} e_1 \\ e_6 \end{bmatrix}^{\mathrm{T}} R_1 \begin{bmatrix} e_1 \\ e_6 \end{bmatrix} \\
&+ h_d (h-h(t)) \begin{bmatrix} e_2 \\ e_7 \end{bmatrix}^{\mathrm{T}} R_2 \begin{bmatrix} e_2 \\ e_7 \end{bmatrix}, \\
&+ h_d [e_1^T R_a e_1 - e_2^T R_a e_2] + e_2^T R_b e_2 - e_3^T R_b e_3 \\
&+ \dot{h}(t) [e_1^T Q_a e_1 + e_2^T (-Q_a + Q_b) e_2 - e_3^T Q_b e_3],\end{aligned}$$

$$\begin{aligned}&\Omega_{a[i,j]} = hH_iX_j, \quad \Omega_{b[i,j]} = hH_iY_j, \\
&\quad i \in [1,2,3],\ j \in [1,2,3,4], \\
&H_1^T = [e_1^T\ e_2^T\ e_3^T\ e_4^T\ e_5^T\ e_{11}^T\ e_{12}^T], \\
&H_2^T = [e_{11}^T\ e_1^T - e_2^T\ e_9^T\ e_1^T - e_4^T], \\
&H_3^T = [e_{12}^T\ e_2^T - e_3^T\ e_{10}^T\ e_2^T - e_5^T], \\
&\Pi_1 = [e_1^T\ e_2^T\ e_6^T],\ \Pi_2^T = \bar{A}e_1 + \bar{A}_d e_2 - e_6, \\
&G_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},\ G_2 = \begin{bmatrix} -I & 0 & 2I & 0 \\ 0 & -I & 0 & 2I \end{bmatrix} \\
&\bar{Q}_a = Q_{11} + \begin{bmatrix} 0 & Q_a \\ Q_a & 0 \end{bmatrix},\ \bar{Q}_b = Q_{21} + \begin{bmatrix} 0 & Q_b \\ Q_b & 0 \end{bmatrix}, \\
&\bar{R}_a = R_1 + \begin{bmatrix} 0 & R_a \\ R_a & 0 \end{bmatrix},\quad \bar{R}_b = R_2 + \begin{bmatrix} 0 & R_b \\ R_b & 0 \end{bmatrix}.\end{aligned}$$

Appendix 2

$$\tilde{\Pi}^{[h(t),\dot{h}(t)]} = \mathrm{Sym}\{\Pi_1^{[h(t),\dot{h}(t)]}\} + \tilde{\Pi}_2^{[h(t),\dot{h}(t)]} + \tilde{\Pi}_3^{[h(t),\dot{h}(t)]},$$

$$\begin{aligned}\tilde{\Pi}_1^{[h(t),\dot{h}(t)]} &= \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ h(t)e_4 \\ (h-h(t))e_5 \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} e_6 \\ h_d e_7 \\ e_8 \\ e_1 - h_d e_2 \\ h_d e_2 - e_3 \end{bmatrix} + \Pi_1 \bar{U} \Pi_2^T \\
&+ [e_4^T\ e_5^T\ e_{11}^T\ e_{12}^T] N [h(t)e_4^T \\
&- e_{11}^T\ (h-h(t))e_5^T - e_{12}^T]^{\mathrm{T}} \\
&+ \mathrm{Sym}\{H_1^T X_1 G_1 H_2 + H_1^T X_2 G_2 H_3\} \\
&+ \mathrm{Sym}\{H_2^T Y_1 G_3 H_2 + H_3^T Y_2 G_4 H_3\},\end{aligned}$$

$$\begin{aligned}\tilde{\Pi}_2^{[h(t),\dot{h}(t)]} &= \begin{bmatrix} e_1 \\ e_6 \end{bmatrix}^{\mathrm{T}} Q_1 \begin{bmatrix} e_1 \\ e_6 \end{bmatrix} - h_d \begin{bmatrix} e_2 \\ e_7 \end{bmatrix} (Q_1 - Q_2) \begin{bmatrix} e_2 \\ e_7 \end{bmatrix} \\
&- \begin{bmatrix} e_3 \\ e_8 \end{bmatrix}^{\mathrm{T}} Q_2 \begin{bmatrix} e_3 \\ e_8 \end{bmatrix},\end{aligned}$$

$$\begin{aligned}\tilde{\Pi}_3^{[h(t),\dot{h}(t)]} &= h(t) \begin{bmatrix} e_1 \\ e_6 \end{bmatrix}^{\mathrm{T}} R_1 \begin{bmatrix} e_1 \\ e_6 \end{bmatrix} \\
&+ h_d (h-h(t)) \begin{bmatrix} e_2 \\ e_7 \end{bmatrix}^{\mathrm{T}} R_2 \begin{bmatrix} e_2 \\ e_7 \end{bmatrix}, \\
&+ h_d [e_1^T R_a e_1 - e_2^T R_a e_2] + e_2^T R_b e_2 - e_3^T R_b e_3,\end{aligned}$$

$$\tilde{\Omega}_{a[i,j]} = hH_iX_j, \quad \tilde{\Omega}_{b[i,j]} = hH_iY_j, \quad i,j \in [1,2].$$