Estimating the tail conditional expectation of Walmart stock data

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Abstract. Stable distribution, also known as Lévy stable distribution, which is a rich class of heavy–tailed distributions can capture asymmetry and heavy tails observed in financial data. In this paper, we fit an AR(1) process with α–stable innovations to the logarithms of volumes of Walmart stock traded daily on the New York Stock Exchange and estimate the TCE (Tail Conditional Expectation) risk measure.

1. Stable AR(1) process

The heavy–tailed auto–regressive models have various practical applications [7, 12, 18]. Stable distributions are often used to specify the innovations process of auto–regressive processes with infinite variance because of their interesting mathematical properties (heavy tails and asymmetry).

A random variable X has a stable distribution if and only if for every k and any family of independent and identically distributed variables $X_1, \ldots, X_k$, there exists $a_k > 0$ and b_k, two reals, such as:

$$X_1 + \ldots + X_k \stackrel{D}{=} a_k X + b_k,$$

where $\stackrel{D}{=}$ denotes equality in distribution. When $b_k = 0$, we speak of strictly stable distribution. It is shown in [5] that there exists a constant α, $0 < \alpha \leq 2$, such that $a_k = k^{1/\alpha}$ for $k \in \mathbf{N}$.

If X has a stable distribution, then we denoted by $X \sim S(\alpha, \mu, \beta, \sigma)$ and its characteristic function is written as:

$$\varphi_X(t) = \exp\left\{ i\mu t - \sigma^\alpha |t|^\alpha \left(1 + i\beta \frac{t}{|t|} w(t, \alpha) \right) \right\}, \tag{1}$$

where

$$w(t, \alpha) = \begin{cases} tg(\dfrac{\alpha\pi}{2}) \ if \ \alpha \neq 1 \\[2mm] \dfrac{2}{\pi} \ln|t| \ if \ \alpha = 1. \end{cases} \tag{2}$$

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A stable law is defined by four parameters:

- index of stability $0 < \alpha \leq 2$ is the main parameter. It characterizes the distribution tails. If α decreases, the tails are heavy. The case of $\alpha = 2$ corresponds to the normal distribution.

- position parameter $\mu \in \mathbf{R}$. It characterizes the law mean when $\alpha > 1$.

- asymmetry parameter $-1 \leq \beta \leq 1$. If $\beta = 0$, the law is symmetrical about the parameter μ. Moreover, when $\mu = 0$ the law is called symmetric α–stable law.

- scale parameter $\sigma > 0$.

According to [19], the stable laws have the following properties:

1. Let $X_1, X_2, \ldots$ i.i.d. $\sim S(\alpha, \mu, \beta, \sigma)$ and c_j a sequence of real values with $\sum_j |c_j|^\alpha < \infty$, then $\sum_{j=0}^{\infty} c_j X_j \sim S(\alpha^*, \mu^*, \beta^*, \sigma^*)$ with

$$
\begin{cases}
\alpha^* = \alpha \\
\mu^* = \mu \sum_{j=0}^{\infty} c_j \\
\beta^* = \beta \sum_{j=0}^{\infty} |c_j|^\alpha sign(c_j) \left(\sum_{j=0}^{\infty} |c_j|^\alpha \right)^{-1} \\
\sigma^* = \sigma \left(\sum_{j=0}^{\infty} |c_j|^\alpha \right)^{1/\alpha}
\end{cases}
\tag{3}
$$

2. If $0 < \alpha < 2$, the variance of a stable law is infinite and for $0 < \alpha < 1$, the mean becomes infinite.

3. Let $X \sim S(\alpha, \mu, \beta, \sigma)$, so as $x \to \infty$, we have

$$
x^\alpha \mathbf{P}(X > x) \longrightarrow C_\alpha \frac{1+\beta}{2} \sigma^\alpha
\tag{4}
$$

and

$$
x^\alpha \mathbf{P}(X < -x) \longrightarrow C_\alpha \frac{1-\beta}{2} \sigma^\alpha,
\tag{5}
$$

where $C_\alpha = \frac{2}{\pi} \Gamma(\alpha) sin \frac{\pi\alpha}{2}$.

If we denote by $G(x) := \mathbf{P}(|X| \leq x) = F_X(x) - F_X(-x), x > 0$, the d.f. of $Z = |X|$, then we have both following conditions:

- The regular variation condition

$$
\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\alpha}, \; x > 0
\tag{6}
$$

- The tails balance condition for $0 \leq a \leq 1$

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = a \quad , \quad \lim_{x \to \infty} \frac{F(-x)}{1 - G(x)} = 1 - a, \tag{7}$$

where $a = \dfrac{1 + \beta}{2}$.

The problem of estimating the parameters of a stable distribution is in general hampered by the lack of known closed–form density functions. However, there are numerical methods that have been found useful in practice like the quantile method proposed by [14], the method based on linear regression [11] and the maximum likelihood approach [16].

Let the AR(1) process:

$$\begin{aligned}
&X_t = \lambda X_{t-1} + \varepsilon_t, \ t = 1, \ldots, n \\
&-1 < \lambda < 1 \\
&\{\varepsilon_t\} \ i.i.d. \sim S(\alpha, \mu, \beta, \sigma) \\
&1 < \alpha < 2.
\end{aligned} \tag{8}$$

The AR(1) process defined in (8) is strictly stationary and it can be written as:

$$X_t = \sum_{j=0}^{\infty} \lambda^j \varepsilon_{t-j}, \ t = 1, \ldots, n. \tag{9}$$

Using (3) and (9) we have $X \sim S(\alpha^*, \mu^*, \beta^*, \sigma^*)$ with

$$\begin{cases}
\alpha^* = \alpha \\
\mu^* = \dfrac{\mu}{1 - \lambda} \\
\\
\beta^* = \begin{cases}
\beta & , \ 0 \leq \lambda < 1 \\
\dfrac{1 - |\lambda|^\alpha}{1 + |\lambda|^\alpha} \beta & , \ -1 < \lambda < 0
\end{cases} \\
\\
\sigma^* = \dfrac{\sigma}{[1 - |\lambda|^\alpha]^{1/\alpha}}
\end{cases} \tag{10}$$

The estimator for the auto-regressive coefficient λ is given by the following expression:

$$\widehat{\lambda}_n = \frac{\displaystyle\sum_{i=1}^{n-1}(X_{i+1} - \bar{X}_n)(X_i - \bar{X}_n)}{\displaystyle\sum_{i=1}^{n-1}(X_i - \bar{X}_n)^2}, \tag{11}$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. It has been proven in [2] that this estimator is consistent.

2. Tail conditional expectation

Distortion risk measure Π_g is a mapping from the set of losses random variables to R^+ defined by

$$\Pi_g(X) = \int_0^\infty g(1 - F_X(x))dx, \tag{12}$$

where $g : [0,1] \longrightarrow [0,1]$ is a non–decreasing and concave function with $g(0) = 0$ and $g(1) = 1$. $\Pi_g(X)$ was introduced by [3] in terms of Choquet integral and have been extensively been used in finance and insurance [21]. This class of risk measure fulfils all four axioms of a coherent risk measure [1].

The conception of tail distortion risk measure was introduced by [23] as:

$$\Pi_g(X|X > VaR_p(X)) = \int_0^\infty g_p(1 - F_X(x))dx, \tag{13}$$

with

$$VaR_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbf{R} : F_X(x) \geq p\}, \ 0 < p < 1 \tag{14}$$

and g_p is a distortion function defined by

$$g_p(s) = \begin{cases} g\left(\dfrac{s}{1-p}\right) & if \ 0 \leq s \leq 1-p \\\\ 1 & if \ 1-p \leq s \leq 1. \end{cases} \tag{15}$$

If $g = 1$, then we find the tail conditional expectation TCE

$$\Pi_g(X|X > VaR_p(X)) = TCE_p(X) = \frac{1}{1-p}\int_p^1 VaR_s(X)ds. \tag{16}$$

Let the order statistic $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ associated to the sample $(X_1, X_2, \ldots, X_n)$ of X. The empirical estimate of $TCE_p(X)$ is:

$$\widehat{TCE_p(X)}^{emp} = \frac{1}{1-p}\left(\frac{1}{n}\sum_{k=[np]+1}^n X_{k,n} + \left(\frac{[np]}{n} - p\right)X_{[np],n}\right), \tag{17}$$

where $[x]$ is the integer part of x [17].
If $X \frown S(\alpha, 0, \beta, 1)$, $1 < \alpha < 2$, [20] have represented TCE as follows:

$$TCE_p(X) = \frac{\alpha}{(1-\alpha)}\frac{|VaR_p|}{\pi(1-p)}\int_{-\bar{\theta}_0}^{\pi/2} g(\theta)\exp(-|VaR_p|^{\alpha/(\alpha-1)}v(\theta))d\theta, \tag{18}$$

where

$$g(\theta) = \frac{\sin(\alpha(\bar{\theta}_0 + \theta) - 2\theta)}{\sin(\alpha(\bar{\theta}_0 + \theta))} - \frac{\alpha\cos^2(\theta)}{\sin^2(\alpha(\bar{\theta}_0 + \theta))}$$

and

$$v(\theta) = (\cos(\alpha\bar{\theta}_0))^{1/(\alpha-1)}\left[\frac{\cos(\theta)}{\sin(\alpha(\bar{\theta}_0 + \theta))}\right]^{\alpha/(\alpha-1)}\frac{\cos(\alpha(\bar{\theta}_0 + \theta) - \theta)}{\cos(\theta)},$$

with

$$\bar{\theta}_0 = \frac{1}{\alpha}\arctan\left(\bar{\beta}\tan\left(\frac{\pi\alpha}{2}\right)\right), \ \bar{\beta} = -sign(VaR_p)\beta.$$

For $Y \backsim S(\alpha, \mu, \beta, \sigma)$, we have $\sigma X + \mu \backsim Y$, then

$$TCE_p(Y) = \sigma TCE_p(X) + \mu.$$

Limit (4) means that F_X is in Fréchet maximum domain of attraction [4]. More precisely, for a sample $X_1, \ldots, X_n$ from the random variable $X \sim S(\alpha, \mu, \beta, \sigma)$, we have

$$\frac{\max(X_1 \ldots, X_n)}{F_X^{-1}(1 - n^{-1})} \xrightarrow{D} \Phi_\alpha, \tag{19}$$

where $\xrightarrow{D}$ denotes convergence in distribution and

$$\Phi_\alpha(x) = \begin{cases} \exp(-x^{-\alpha}) \ , \ x > 0 \\ 0 \qquad\qquad , \ x \leq 0. \end{cases} \tag{20}$$

The tail index α can be estimated by the Hill estimator [10] defined by

$$\widehat{\alpha}^H = \left[\frac{1}{k} \sum_{i=1}^{k} \log X_{n-i,n} - \log X_{n-k+1,n} \right]^{-1}, \tag{21}$$

where $k = k_n$ is an intermediate sequence such that $k \to \infty, \ k/n \to 0, \ n \to \infty$.

The semi–parametric estimator of a high quantile $(p \to 1)$ proposed in [22] has the following form

$$\widehat{VaR}_p^H = X_{n-k,n} \left(\frac{k}{n(1-p)} \right)^{1/\widehat{\alpha}^H}. \tag{22}$$

It is known (e.g., [23]) that, for $p \to 1$ and $\alpha > 1$ we have

$$TCE_p = \frac{\alpha}{\alpha - 1} VaR_p. \tag{23}$$

Then we obtain the following estimator

$$\widehat{TCE}_p^H = \frac{\widehat{\alpha}^H}{\widehat{\alpha}^H - 1} X_{n-k,n} \left(\frac{k}{n(1-p)} \right)^{1/\widehat{\alpha}^H}. \tag{24}$$

3. Application

Our study is carried out on the natural logarithms of the volumes of Walmart stock traded daily on the New York Stock Exchange (time–series X_t) during the period from November 19, 2003 to January 4, 2005. The observations are shown on the Figure 1 and their empirical density is plotted on the Figure 2. The data, were taken from the public source https://finance.yahoo.com.
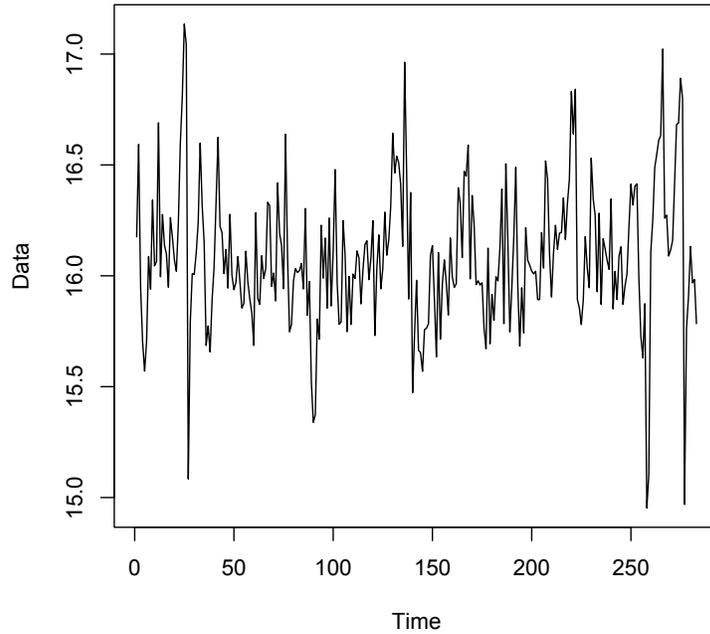
Figure 1: *Daily logarithms of the volumes of Walmart stock*
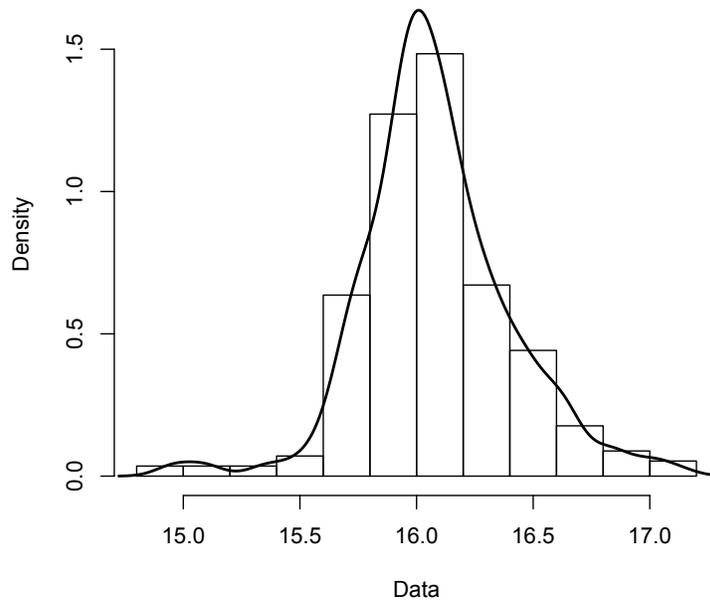
Figure 2: *Empirical density of time–series X_t*

To verify the stationarity of the data, we perform a Phillips-Perron test, we get a $p-value$ of 0.01, so the data are stationary. The data exhibit a high excess kurtosis $4.593 > 3$, indicating that the observations are not normally distributed. The $p-value$ of the Kolmogorov-Smirnov normality test is 4.7×10^{-6}, thus confirming the rejection of the assumption that the data would normally be distributed.

On the Figure 3, since the ACF decreases gradually and the partial ACF cut off after the first lag, it seems that the data follow an AR(1) process.
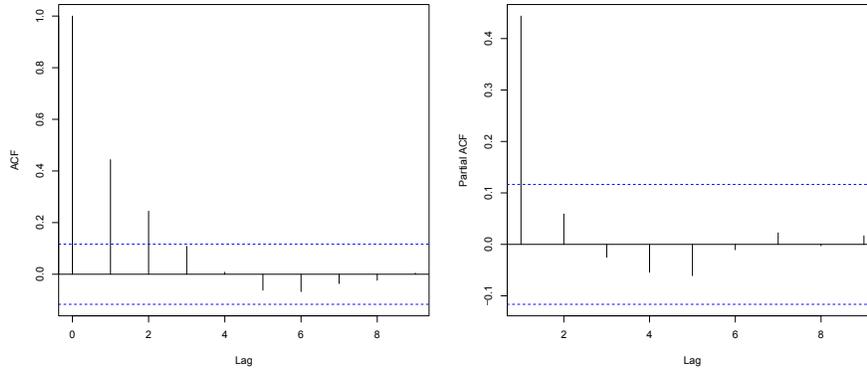
Figure 3: *ACF and PACF of time–series* X_t

Moreover, we use the AICC (Akaike's Information Criteria Corrected) and BIC (Bayesian Information Criterion) for all $ARMA(p,q)$ models with $p + q \leq 2$ as given in the Table 1.

$ARMA(p,q)$	$AICC$	BIC
AR(1)	96.67423	100.3054
AR(2)	97.68286	104.9309
MA(1)	110.6942	114.3254
MA(2)	101.2229	108.4709
ARMA(1,1)	97.82286	105.0709

Table 1: *AICC and BIC criteria's for different ARMA(p,q) models*

From this, it is obvious that the best model for X_t with respect to both criteria was an AR(1) defined by

$$X_t = \lambda X_{t-1} + \varepsilon_t, \ t = 1, \ldots, 283 \tag{25}$$

with $X_0 = 0$ and $\{\varepsilon_t\}$ *i.i.d.* whose distribution we will specify later.

The estimator provided by (11) of the coefficient λ is $\widehat{\lambda} = 0.4437255$. Statistical analysis of the residuals $\widehat{\varepsilon}_t = X_t - \widehat{\lambda} X_{t-1}$, $2 \leq t \leq 283$, leads to the following results:

1. The empirical ACF and PACF of the residuals on the Figure 4 shows the non–significance of auto–correlation coefficients.
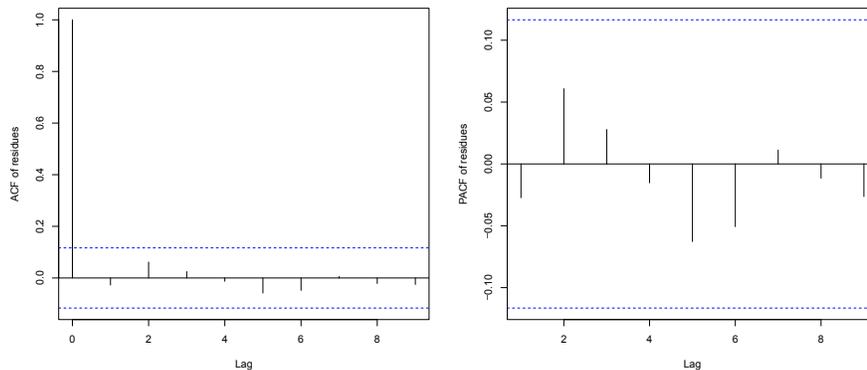
Figure 4: *The empirical ACF and PACF of the residuals* $\widehat{\varepsilon}_t$

2. Empirical density function of residuals on the left of the Figure 5 shows some asymmetry and the kurtosis of $6.941248 > 3$ indicates a heavy tail of distribution, which is confirmed by the deviation at the extremes of the normal Q-Q plot (Figure 5 on the right).
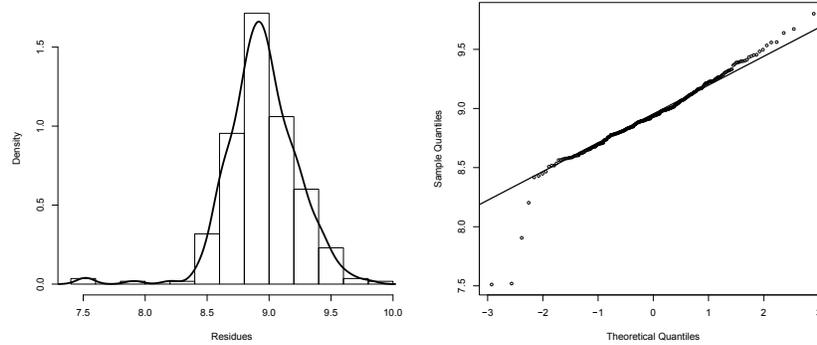
Figure 5: *Empirical density of $\widehat{\varepsilon}_t$ (left) and the normal Q-Q plot (right)*

The phenomena of heavy tails and excess kurtosis of the residues $\widehat{\varepsilon}_t$ suggest that a stable distribution $S(\alpha, \mu, \beta, \sigma)$ would be an appropriate model for $\widehat{\varepsilon}_t$. To estimate the parameters α, μ, β and σ we use the method of McCulloch [14]. The results are summarized in Table 2.

$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\beta}$	$\widehat{\sigma}$
1.834	8.9203869	0.95	0.1701674

Table 2: *Stable parameters of residuals*

To estimate the parameters of the AR(1) process defined in (25) we use the equations (10). The results are given in Table 3.

$\widehat{\alpha^*}$	$\widehat{\mu^*}$	$\widehat{\beta^*}$	$\widehat{\sigma^*}$
1.834	16.03594	0.95	0.1955845

Table 3: *Stable parameters of AR(1)*

In the Figure 6 the plots of the empirical density of the data and the estimated stable density $S(\widehat{\alpha^*}, \widehat{\mu^*}, \widehat{\beta^*}, \widehat{\sigma^*})$ show a good fit.
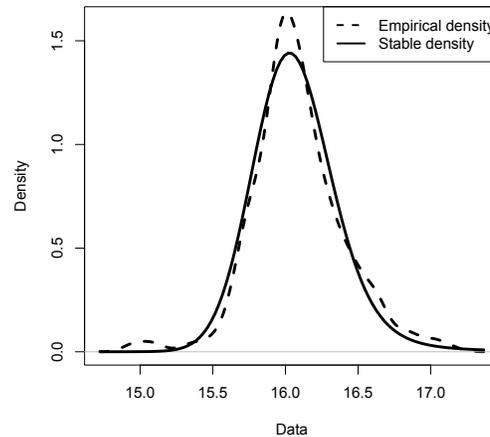
Figure 6: *The goodness of fit*

To confirm this, the $p - value = 0.6161$ of the Kolmogorov–Smirnov test implies that for our data there a better fit of the stable model around the center of the distribution while the $p - value = 0.3176$ of the Anderson–Darling test implies a better fit in the tails at the significance level 5%. The P-P plot on the Figure 7 is linear, also confirming the quality of fit for the stable distribution.
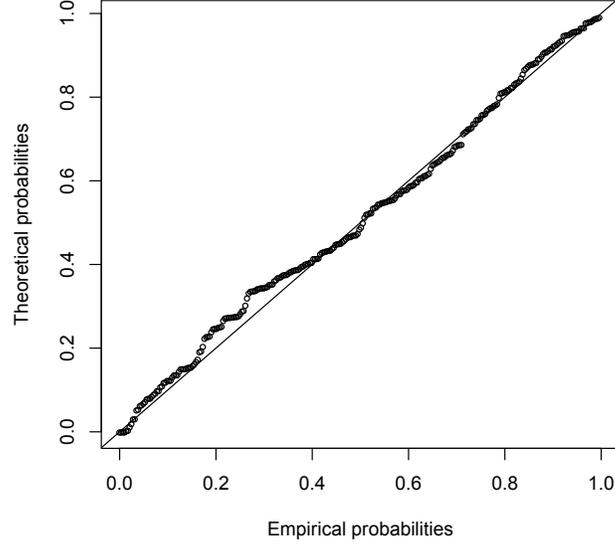
Figure 7: *The P-P plot*

From (18) we calculate the estimator $\widehat{TCE}^{1}_{X_t}$ using the parameters $\widehat{\alpha^*}, \widehat{\mu^*}, \widehat{\beta^*}$ and $\widehat{\sigma^*}$.

We adjust a stable law to the data X_t without passing by the residuals using the McCulloch estimators (Table 4).

$\widehat{\alpha}_{X_t}$	$\widehat{\mu}_{X_t}$	$\widehat{\beta}_{X_t}$	$\widehat{\sigma}_{X_t}$
1.678	16.0172717	0.821	0.1788372

Table 4: *Stable parameters obtained by McCulloch estimators*

Then, using stable parameters from Table 4, we calculate $\widehat{TCE}^{2}_{X_t}$ from (18). For comparison, we calculate the empirical $\widehat{TCE}^{emp}$ of the real data using (17) which are presented in Table 5.

p	0.90	0.95
$\widehat{TCE}^{emp}$	16.70186	16.83251
$\widehat{TCE}^{1}_{X_t}$	16.66029	16.81588
$\widehat{TCE}^{2}_{X_t}$	16.75967	17.00363

Table 5: *Estimation of TCE at 90% and 95% confidence levels*

To apply the Hill estimator for data coming from stable distribution, [6] proposed to center the data by subtracting the median. In the Figure 8 we have plotted $(k, \widehat{\alpha}^{H})$ for the data $Y_t = X_t - median(X_t)$.
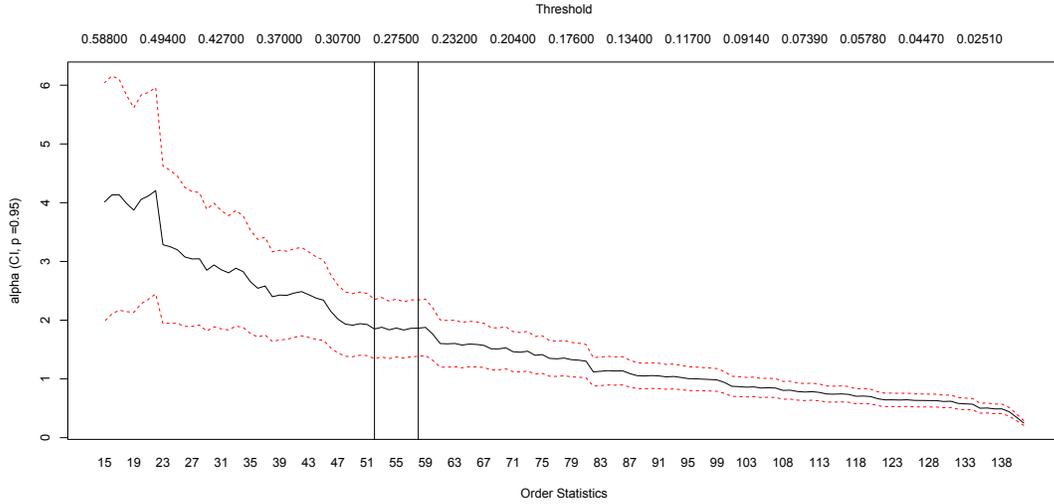
Figure 8: *Hill estimator for Y_t*

We see that there is a region of stability between $k = 52$ and $k = 58$, so we compute $\widehat{\alpha}^H$ for each value of this region and calculate $\widehat{TCE}_{Y_t}^H$ using (24), then we deduce $\widehat{TCE}_{X_t}^H$ from the following equation:

$$\widehat{TCE}_{X_t}^H = \widehat{TCE}_{Y_t}^H + median(X_t) \tag{26}$$

and obtain results in the Table 6.

		$\widehat{TCE}_{X_t}^H$	
k	$\widehat{\alpha}^H$	$p = 0.90$	$p = 0.95$
52	1.844519	16.91662	17.31423
53	1.801733	16.93763	17.35649
54	1.832314	16.92239	17.32583
55	1.799902	16.93892	17.35889
56	1.829332	16.92386	17.32878
57	1.832247	16.92248	17.32599
58	1.845455	16.91603	17.31314

Table 6: $\widehat{TCE}_{X_t}^H$ *for* $52 \leq k \leq 58$.

From the Tables 5-6 we remark following:

1. The estimators of a stable AR(1) via the residuals estimators give better results of $\widehat{TCE}$ than the estimators applied directly to the data. [13] have shown by simulations on synthetic samples that the estimates of the distribution parameters of the α–stable AR(1) process are better via the residuals.

2. The values of $\widehat{TCE}^H$ are farther from the values of $\widehat{TCE}^{emp}$ because [15] shows that the Hill estimator performs poorly on stable data when $1 < \alpha < 2$ leads to overestimates of α and thus overestimates the TCE^H.

4. Conclusion

In this paper, we adjusted a stable AR(1) model to the logarithms of the volumes of Walmart stock data and estimated the coherent risk measure TCE taking into account the dependence structure that exists in the model by estimating the parameters of the stable AR(1) via the residuals. Results are similar to [9] assuming the hypothesis of independence of the data involved in the modeling. The continuous evolution of risks in insurance and finance leads to reflections aimed at relaxing this hypothesis.

There are many possibilities for defining dependency, e.g. [8] estimated the TCE risk measure using the extremal index and the POT method.

Among the many possible mathematical tools to take into account such dependencies, we find the copulas which allow the introduction and characterization of a very flexible form of dependence between different random variables. It would be interesting to mix the concept of copula with the auto–regressive processes to describe another form of dependence in the multivariate case. These are topics for future research.

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