Šest načina dokazivanja jednog poučka za pravilni petnaesterokut

Dragoljub Milošević,[1] *Aleksandar Sredojević*[2]

Zasigurno vam je poznato da za pravilan sedmerokut $A_0A_1\ldots A_6$ vrijedi jednakost

$$\frac{1}{|A_0A_1|}=\frac{1}{|A_0A_2|}+\frac{1}{|A_0A_3|}.$$

U [1] je dan sljedeći poučak (bez dokaza): *U pravilnom petnaesterokutu* $A_0A_1\ldots A_{14}$ *vrijedi jednakost*:

$$\frac{1}{|A_0A_1|}=\frac{1}{|A_0A_2|}+\frac{1}{|A_0A_4|}+\frac{1}{|A_0A_7|}.$$

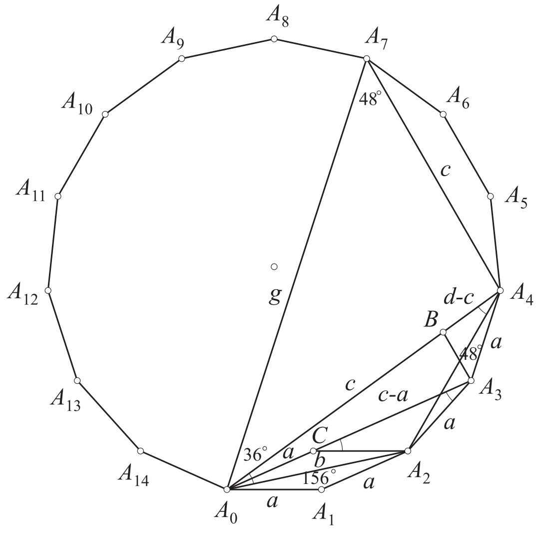
Prikazat ćemo šest načina dokazivanja ovog poučka.

Prvi način. Uvedimo sljedeće oznake: $|A_0A_1|=a$, $|A_0A_2|=b$, $|A_0A_3|=c$, $|A_0A_4|=d$, $|A_0A_5|=e$, $|A_0A_6|=f$ i $|A_0A_7|=g$. Tada navedena jednakost postaje

$$\frac{1}{a}=\frac{1}{b}+\frac{1}{d}+\frac{1}{g}. \tag{1}$$

Središnji kut nad stranicom pravilnog petnaesterokuta je $360^\circ : 15 = 24^\circ$, a odgovarajući obodni kut iznosi $24^\circ : 2 = 12^\circ$. S obzirom da je vanjski kut jednak središnjem kutu, tj. 24°, unutarnji kut pravilnog 15-erokuta jednak $180^\circ - 24^\circ = 156^\circ$.

Na dijagonali $|A_0A_4|=d$ odredimo točku B tako da $|A_0B|=|A_0A_3|=c$ (slika 1).



Slika 1.

[1] Profesor je u miru u srednjoj školi u Gornjem Milanovcu.

[2] Autor je profesor gimnazije u Gornjem Milanovcu.

Tada je $|BA_4| = d - c$. Trokut A_0A_3B je jednakokračni, pa je

$$\sphericalangle A_0A_3B = \sphericalangle A_0BA_3 = (180^\circ - 12^\circ) : 2 = 84^\circ.$$

U trokutu A_3A_4B imamo

$$\sphericalangle BA_3A_4 = \sphericalangle A_2A_3A_4 - (\sphericalangle A_2A_3A_0 + \sphericalangle A_0A_3B) = 156^\circ - (2 \cdot 12^\circ + 84^\circ) = 48^\circ$$

i $\sphericalangle A_3A_4B = 3 \cdot 12^\circ = 36^\circ$, a i u trokutu $A_0A_4A_7$ je

$$\sphericalangle A_4A_0A_7 = 3 \cdot 12^\circ = 36^\circ \quad \text{i} \quad \sphericalangle A_0A_7A_4 = 4 \cdot 12^\circ = 48^\circ.$$

Zaključujemo da su trokuti A_3A_4B i $A_7A_0A_4$ slični, jer imaju jednake kutove. Zbog toga je

$$|A_3A_4| : |A_7A_0| = |BA_4| : |A_4A_0|,$$

odnosno $a : g = (d - c) : d$.

Odavde slijedi $ad = g\,(d - c)$, tj.

$$dg - ad = cg. \tag{2}$$

Na dijagonali $\overline{A_0A_3}$ odredimo točku C tako da $|A_0C| = |A_0A_1| = a$, pa je $|CA_3| = c - a$. Kako je $A_0A_3 \| A_1A_2$ i $|A_0A_1| = |A_1A_2| = |A_0C| = a$, zaključujemo da je četverokut $A_0A_1A_2C$ romb, što znači da je $|A_2C| = a$. Dakle, trokut A_2A_3C je jednakokračni $(|A_2C| = |A_2A_3| = a)$. U njemu je

$$\sphericalangle CA_2A_3 = \sphericalangle A_1A_2A_3 - \sphericalangle A_1A_2C = 156^\circ - 2 \cdot 12^\circ = 132^\circ.$$

U jednakokračnom trokutu $A_0A_2A_4$ $(|A_0A_2| = |A_2A_4| = b)$ imamo

$$\sphericalangle A_0A_2A_4 = \sphericalangle A_1A_2A_3 - (\sphericalangle A_1A_2A_0 + \sphericalangle A_3A_2A_4) = 156^\circ - (12^\circ + 12^\circ) = 132^\circ.$$

Zaključujemo da su jednakokračni trokutovi A_2A_3C i $A_2A_4A_0$ slični, pa je

$$|A_2A_3| : |A_0A_2| = |CA_3| : |A_0A_4| \quad \text{ili} \quad a : b = (c - a) : d.$$

Odavde slijedi $ad = b\,(c - a)$, odnosno

$$ab + ad = bc. \tag{3}$$

Dijeljenjem jednakosti (2) i (3) dobivamo

$$\begin{aligned} \frac{dg - ad}{ab + ad} = \frac{cg}{bc} &\iff \frac{d\,(g - a)}{a\,(b + d)} = \frac{g}{b} \\ &\iff \frac{g - a}{ag} = \frac{b + d}{bd} \\ &\iff \frac{1}{a} = \frac{1}{b} + \frac{1}{d} + \frac{1}{g}. \end{aligned}$$

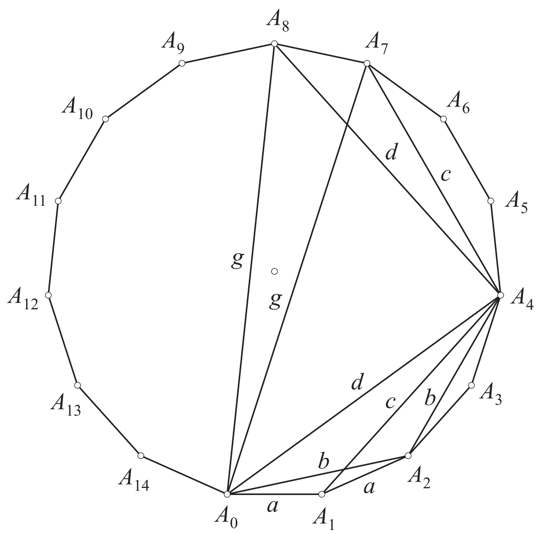
Drugi način. Primjenom Ptolemejevog poučka na tetivne četverokute $A_0A_4A_7A_8$ i $A_0A_1A_2A_4$ (slika 2), dobivamo $|A_0A_7| \cdot |A_4A_8| = |A_0A_8| \cdot |A_4A_7| + |A_0A_4| \cdot |A_7A_8|$ i

$|A_0A_2| \cdot |A_1A_4| = |A_0A_1| \cdot |A_2A_4| + |A_1A_2| \cdot |A_0A_4|$, odnosno

$$gd = gc + da \qquad \text{i} \qquad bc = ab + ad,$$

ili

$$dg - ad = cg \qquad \text{i} \qquad ab + ad = bc.$$



Slika 2.

Nakon dijeljenja posljednje dvije jednakosti imamo $\dfrac{d\,(g-a)}{a\,(b+d)} = \dfrac{g}{b}$, a odavde lako dobivamo željenu jednakost (1).

Treći način. Koristit ćemo Molweideove formule

$$\frac{a+b}{c} = \frac{\cos\dfrac{\alpha-\beta}{2}}{\sin\dfrac{\gamma}{2}} \qquad \text{i} \qquad \frac{a-b}{c} = \frac{\sin\dfrac{\alpha-\beta}{2}}{\cos\dfrac{\gamma}{2}},$$

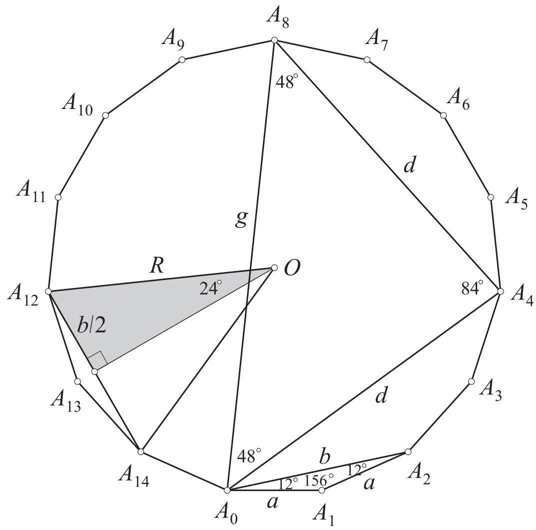
gdje su a, b, c duljine stranica i α, β, γ veličine unutarnjih kutova trokuta ABC.

Primjenom prve formule na $\Delta A_0A_4A_8$ (slika 3), imamo

$$\frac{d+g}{d} = \frac{\cos\dfrac{84^\circ - 48^\circ}{2}}{\sin\dfrac{48^\circ}{2}} = \frac{\cos 18^\circ}{\sin 24^0} = \frac{\sin 72^\circ}{\sin 24^\circ},$$

a iz druge na $\Delta A_0A_1A_2$:

$$\frac{b-a}{a} = \frac{\sin\dfrac{156^\circ - 12^\circ}{2}}{\cos\dfrac{12^\circ}{2}} = \frac{\sin 72^\circ}{\cos 6^\circ} = \frac{\sin 72^\circ}{\sin 84^\circ}.$$



Slika 3.

Iz teorema o sinusima dobivamo

$$\frac{b}{g} = \frac{2R\sin 24^\circ}{2R\sin 84^\circ} = \frac{\sin 24^\circ}{\sin 84^\circ},$$

gdje je R polumjer kružnice opisane oko pravilnog 15-erokuta (slika 3).

Sada je

$$\frac{b-a}{a} \cdot \frac{d}{d+g} \cdot \frac{g}{b} = \frac{\sin 72^\circ}{\sin 84^\circ} \cdot \frac{\sin 24^\circ}{\sin 72^\circ} \cdot \frac{\sin 84^\circ}{\sin 24^\circ} = 1,$$

a odavde je

$$\frac{b-a}{ab} = \frac{d+g}{dg} \quad \text{ili} \quad \frac{1}{a} = \frac{1}{b} + \frac{1}{d} + \frac{1}{g}, \quad \text{tj.} \quad (1).$$

Četvrti način. Kako je $a = 2R\sin 12^\circ$, $b = 2R\sin 24^\circ$, $d = 2R\sin 48^\circ$ i $g = 2R\sin 84^\circ$, jednakost (1) ekvivalentna je sa

$$\frac{1}{\sin 12^\circ} = \frac{1}{\sin 24^\circ} + \frac{1}{\sin 48^\circ} + \frac{1}{\sin 84^\circ},$$

tj. sa

$$\frac{\sin 84^\circ - \sin 12^\circ}{\sin 84^\circ \sin 12^\circ} = \frac{\sin 48^\circ + \sin 24^\circ}{\sin 48^\circ \sin 24^\circ}. \tag{4}$$

Korištenjem trigonometrijskih identiteta

$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$$

i

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}, \quad (x, y \in \mathbf{R}),$$

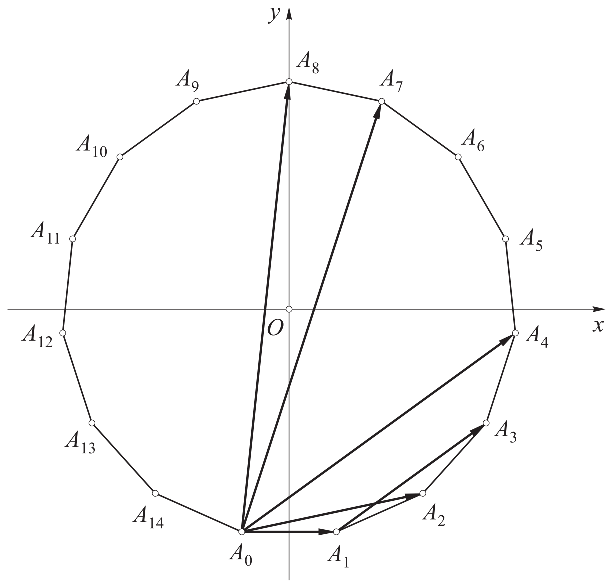
jednakost (4) prelazi u

$$\frac{\cos 48^\circ}{\sin 84^\circ \sin 12^\circ} = \frac{\cos 12^\circ}{\sin 48^\circ \sin 24^\circ},$$

odnosno u
$$\sin 24^\circ\left(\sin 48^\circ \cos 48^\circ\right)=\sin 48^\circ\left(\sin 12^\circ \cos 12^\circ\right). \tag{5}$$
Kako je
$$2\sin x\cos x=\sin 2x, \quad (x\in\mathbf{R})$$
jednakost (5) ekvivalentna je sa
$$\sin 24^\circ\cdot\sin 96^\circ=\sin 84^\circ\cdot\sin 24^\circ,$$
što vrijedi zbog $\sin 96^\circ=\sin\left(180^\circ-84^\circ\right)=\sin 84^\circ$.

Peti način. Korištenjem trigonometrijskog identiteta
$$\sin^2 x-\sin^2 y=\sin(x-y)\sin(x+y), \quad (x,y\in\mathbf{R})$$
dobivamo redom
$$\begin{aligned}
e^2-b^2&=(2R\sin 60^\circ)^2-(2R\sin 24^\circ)^2=2R\sin 36^\circ\cdot 2R\sin 84^\circ\\
&=cg, &(6)\\
g^2-e^2&=(2R\sin 84^\circ)^2-(2R\sin 60^\circ)^2=2R\sin 24^\circ\cdot 2R\sin 144^\circ\\
&=2R\sin 24^\circ\cdot 2R\sin 36^\circ=bc, &(7)\\
f^2-b^2&=(2R\sin 72^\circ)^2-(2R\sin 24^\circ)^2=2R\sin 48^\circ\cdot 2R\sin 96^\circ\\
&=2R\sin 48^\circ\cdot 2R\sin 84^\circ=dg, &(8)\\
f^2-e^2&=(2R\sin 72^\circ)^2-(2R\sin 60^\circ)^2=2R\sin 12^\circ\cdot 2R\sin 132^\circ\\
&=2R\sin 12^\circ\cdot 2R\sin 48^\circ=ad &(9)
\end{aligned}$$
i
$$\begin{aligned}
g^2-f^2&=(2R\sin 84^\circ)^2-(2R\sin 72^\circ)^2=2R\sin 12^\circ\cdot 2R\sin 156^\circ\\
&=2R\sin 12^\circ\cdot 2R\sin 24^\circ=ab, &(10)
\end{aligned}$$
gdje smo upotrijebili identitet
$$\sin(180^\circ-x)=\sin x, \quad x\in\mathbf{R}.$$
Sada imamo redom
$$\begin{aligned}
\frac{g}{b}&=\frac{cg}{bc}=\frac{e^2-b^2}{g^2-e^2} \quad \text{(zbog jednakosti (6) i (7))}\\
&=\frac{f^2-b^2-\left(f^2-e^2\right)}{g^2-f^2+\left(f^2-e^2\right)}\\
&=\frac{dg-ad}{ab+ad} \quad \text{(zbog jednakosti (8)–(10),}
\end{aligned}$$
tj.
$$\frac{1}{a}=\frac{1}{b}+\frac{1}{d}+\frac{1}{g}.$$

Šesti način. Neka je pravilan petnaestorokut $A_0A_1\ldots A_{14}$ u Gaussovoj ravnini upisan u jediničnu kružnicu sa središtem u koordinatnom ishodištu (slika 4). Tada su brojevi
$$\varepsilon_k=\cos\frac{2k\pi}{15}+i\sin\frac{2k\pi}{15}=\varepsilon_1^k \quad (k=0,1,2,\ldots,14)$$
redom koordinate točaka A_0, A_1,..., A_{14}.



Slika 4.

Kako je

$$\frac{1}{|A_0A_2|}+\frac{1}{|A_0A_4|}=\frac{1}{|A_1A_3|}+\frac{1}{|A_0A_4|}$$

i kako su vektori $\overrightarrow{A_1A_3}$ i $\overrightarrow{A_0A_4}$ kolinearni i istog smjera i vrijedi $|A_1A_3|=|\varepsilon_3-\varepsilon_1|$ i $|A_0A_4|=|\varepsilon_4-\varepsilon_0|$, to je

$$\begin{aligned}\frac{1}{|A_1A_3|}+\frac{1}{|A_0A_4|}&=\frac{|A_0A_4|+|A_1A_3|}{|A_1A_3|\cdot|A_0A_4|}=\frac{|(\varepsilon_4-\varepsilon_0)+(\varepsilon_3-\varepsilon_1)|}{|\varepsilon_3-\varepsilon_1|\,|\varepsilon_4-\varepsilon_0|}=\frac{\left|\varepsilon_1^4-1+\varepsilon_1^3-\varepsilon_1\right|}{\left|\varepsilon_1^3-\varepsilon_1\right|\left|\varepsilon_1^4-1\right|}\\&=\frac{\left|\left(\varepsilon_1^2-1\right)\left(\varepsilon_1^2+1\right)+\varepsilon_1\left(\varepsilon_1^2-1\right)\right|}{\left|\varepsilon_1\left(\varepsilon_1^2-1\right)\right|\left|\left(\varepsilon_1^2-1\right)\left(\varepsilon_1^2+1\right)\right|}=\frac{\left|\left(\varepsilon_1^2-1\right)\left(\varepsilon_1^2+1+\varepsilon_1\right)\right|}{|\varepsilon_1|\left|\varepsilon_1^2-1\right|\left|\varepsilon_1^2-1\right|\left|\varepsilon_1^2+1\right|}\\&=\frac{\left|\varepsilon_1^2-1\right|\left|\varepsilon_1^2+\varepsilon_1+1\right|}{\left|\varepsilon_1^2-1\right|^2\left|\varepsilon_1^2+1\right|}=\frac{\left|\varepsilon_1^2+\varepsilon_1+1\right|}{\left|\varepsilon_1^2-1\right|\left|\varepsilon_1^2+1\right|}.\end{aligned}$$

tj.

$$\frac{1}{|A_0A_2|}+\frac{1}{|A_0A_4|}=\frac{\left|\varepsilon_1^2+\varepsilon_1+1\right|}{\left|\varepsilon_1^2-1\right|\left|\varepsilon_1^2+1\right|} \tag{11}$$

Također

$$\frac{1}{|A_0A_1|}-\frac{1}{|A_0A_7|}=\frac{1}{|A_3A_4|}-\frac{1}{|A_0A_7|}.$$

Kako su vektori $\overrightarrow{A_3A_4}$ i $\overrightarrow{A_0A_7}$ kolinearni i istog smjera i vrijedi $|A_3A_4|=|\varepsilon_4-\varepsilon_3|$ i $|A_0A_7|=|\varepsilon_7-\varepsilon_0|$, to je

$$\begin{aligned}\frac{1}{|A_3A_4|}-\frac{1}{|A_0A_7|}&=\frac{|A_0A_7|-|A_3A_4|}{|A_3A_4|\cdot|A_0A_7|}=\frac{|(\varepsilon_7-\varepsilon_0)-(\varepsilon_4-\varepsilon_3)|}{|\varepsilon_4-\varepsilon_3|\,|\varepsilon_7-\varepsilon_0|}=\frac{\left|\varepsilon_1^7-1-\varepsilon_1^4+\varepsilon_1^3\right|}{\left|\varepsilon_1^4-\varepsilon_1^3\right|\left|\varepsilon_1^7-1\right|}\\
&=\frac{\left|\varepsilon_1^4\left(\varepsilon_1^3-1\right)+\varepsilon_1^3-1\right|}{\left|\varepsilon_1^3\left(\varepsilon_1-1\right)\right|\left|\varepsilon_1^7-1\right|}=\frac{\left|\left(\varepsilon_1^3-1\right)\left(\varepsilon_1^4+1\right)\right|}{\left|\varepsilon_1^3\right|\left|\varepsilon_1-1\right|\left|\varepsilon_1^7-1\right|}\\
&=\frac{\left|\varepsilon_1^3-1\right|\left|\varepsilon_1^4+1\right|}{\left|\varepsilon_1\right|^3\left|\varepsilon_1-1\right|\left|\varepsilon_1^7-1\right|}=\frac{\left|\left(\varepsilon_1-1\right)\left(\varepsilon_1^2+\varepsilon_1+1\right)\right|\left|\varepsilon_1^4+1\right|}{\left|\varepsilon_1-1\right|\left|\varepsilon_1^7-1\right|}\\
&=\frac{\left|\varepsilon_1-1\right|\left|\varepsilon_1^2+\varepsilon_1+1\right|\left|\varepsilon_1^4+1\right|}{\left|\varepsilon_1-1\right|\left|\varepsilon_1^7-1\right|}=\frac{\left|\varepsilon_1^2+\varepsilon_1+1\right|\left|\varepsilon_1^4+1\right|}{\left|\varepsilon_1^7-1\right|}. \qquad (12)\end{aligned}$$

Kako je $|A_0A_7|=|A_0A_8|$, imamo $|\varepsilon_7-\varepsilon_0|=|\varepsilon_8-\varepsilon_0|$, tj. $\left|\varepsilon_1^7-1\right|=\left|\varepsilon_1^8-1\right|$. Sada (12) postaje

$$\begin{aligned}\frac{1}{|A_0A_1|}-\frac{1}{|A_0A_7|}&=\frac{\left|\varepsilon_1^2+\varepsilon_1+1\right|\left|\varepsilon_1^4+1\right|}{\left|\varepsilon_1^8-1\right|}=\frac{\left|\varepsilon_1^2+\varepsilon_1+1\right|\left|\varepsilon_1^4+1\right|}{\left|\left(\varepsilon_1^4-1\right)\left(\varepsilon_1^4+1\right)\right|}\\
&=\frac{\left|\varepsilon_1^2+\varepsilon_1+1\right|\left|\varepsilon_1^4+1\right|}{\left|\varepsilon_1^4-1\right|\left|\varepsilon_1^4+1\right|}=\frac{\left|\varepsilon_1^2+\varepsilon_1+1\right|}{\left|\left(\varepsilon_1^2-1\right)\left(\varepsilon_1^2+1\right)\right|}\\
&=\frac{\left|\varepsilon_1^2+\varepsilon_1+1\right|}{\left|\varepsilon_1^2-1\right|\left|\varepsilon_1^2+1\right|} \qquad (13)\end{aligned}$$

Iz (13) i (11) dobivamo

$$\frac{1}{|A_0A_1|}=\frac{1}{|A_0A_2|}+\frac{1}{|A_0A_4|}+\frac{1}{|A_0A_7|}.$$

Literatura

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