Formule rekurzije za alternativne konačne sume potencija prirodnih brojeva

Petar Svirčević[1]

Većini srednjoškolaca je poznato, kako je C. F. Gauss (1777. – 1855.), jedan od najvećih matematičara svih vremena, u osnovnoj školi brzo zbrojio prirodne brojeve od 1 do 100. Naime, njegov rezon je bio

$$\begin{aligned}1+2+3+\ldots+100 &= \underbrace{(1+100)+(2+99)+(3+98)+\ldots+(50+51)}_{50}\\ &= 50\cdot 101 = 5050.\end{aligned}$$

No, opća formula za zbroj prvih n prirodnih brojeva, koja se može izvesti na više načina, dana je u obliku $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$. Jednostavno se izvodi i formula rekurzije pomoću koje se nalaze sume: $1^2+2^2+3^2+\ldots+n^2$, $1^3+2^3+3^3+\ldots+n^3$, … Svakako, da pomoću ovih formula možemo naći i ove složenije sume: $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)$, $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\ldots+n(n+1)(n+2)$, …

U ovom članku izvodimo formule rekurzije pomoću kojih možemo naći ove alternativne sume: $1-2+3-\ldots+(-1)^{n+1}n$, $1^2-2^2+3^2-\ldots+(-1)^{n+1}n^2$, …; i složenije oblike: $1\cdot 2-2\cdot 3+3\cdot 4-\ldots+(-1)^{n+1}n(n+1)$, $1\cdot 2\cdot 3-2\cdot 3\cdot 4+3\cdot 4\cdot 5-\ldots+(-1)^{n+1}n(n+1)(n+2)$, …

Prije nego prijeđemo na izvođenje rekurzija uvedimo mnemotehničke oznake:

$$S_m(n)=1^m+2^m+3^m+4^m+\ldots+n^m \tag{1}$$

$$\overline{S}_m(n)=1^m-2^m+3^m-4^m+\ldots+(-1)^{n+1}n^m, \tag{2}$$

za $n=1,2,3,\ldots$ i $m=0,1,2,3,\ldots$, gdje je

$$S_0(n)=n, \tag{3}$$

$$\overline{S}_0(n)=\frac{1-(-1)^n}{2}. \tag{4}$$

Naime, vidljivo je da gornji potez na lijevoj strani (2) asocira na alternativnu sumu.

Na osnovu definiranih suma (1) i (2) za $k=1,2,3,\ldots$ izvodimo alternirajuće formule suma potencija istih eksponenata uzastopnih prirodnih brojeva: (eksponenti su neparni)

$$\begin{aligned}\overline{S_{2k-1}}(n)=\frac{1}{2k}\Bigg\{\left[(n+1)^{2k}-n^{2k}\right]\frac{1-(-1)^n}{2}-\Bigg[\binom{2k}{0}S_0(n)+\binom{2k}{1}\overline{S_1}(n)\\ +\binom{2k}{2}S_2(n)+\ldots+\binom{2k}{2k-3}\overline{S}_{2k-3}(n)+\binom{2k}{2k-2}S_{2k-2}(n)\Bigg]\Bigg\}\end{aligned} \tag{5}$$

i (eksponenti su parni)

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[1] Autor je profesor u mirovini na Željezničkoj tehničkoj školi u Zagrebu,
e-pošta: `petar.svircevic@zg.t-com.hr`

$$\overline{S_{2k}}(n)=\frac{1}{2k+1}\left\{\left[(n+1)^{2k+1}-n^{2k+1}\right]\frac{1-(-1)^n}{2}-\left[\binom{2k+1}{0}\overline{S}_0(n)+\binom{2k+1}{1}S_1(n)\right.\right.$$
$$\left.\left.+\binom{2k+1}{2}\overline{S}_2(n)+\ldots+\binom{2k+1}{2k-2}\overline{S}_{2k-2}(n)+\binom{2k+1}{2k-1}S_{2k-1}(n)\right]\right\}. \tag{6}$$

U navedenim formulama koristimo dobro poznatu rekurziju

$$S_k(n)=\frac{1}{k+1}\left\{(n+1)^{k+1}-1-\left[\binom{k+1}{0}S_0(n)+\binom{k+1}{1}S_1(n)+\ldots+\binom{k+1}{k-1}S_{k-1}(n)\right]\right\}, \tag{7}$$

koju dobijemo ako u razvijenu binomnu formulu $(1+a)^{k+1}$ supstituiramo vrijednosti po redu za $a=1,2,3,\ldots,n$, te onda te razvoje zbrojimo.

Da dokažemo formulu (5) polazimo od razvoja

$$[1-(-1)^a a]^b=\binom{b}{0}-\binom{b}{1}(-1)^a a^1+\binom{b}{2}(-1)^{2a}a^2-\binom{b}{3}(-1)^{3a}a^3$$
$$+\ldots+(-1)^b\binom{b}{b}(-1)^{ba}a^b, \tag{8}$$

gdje je b paran broj. Sada u (8) izvršimo po redu uvrštavanje za $a=1,2,3,\ldots,n$, i te jednakosti zbrojimo. Dobivamo zbroj lijevih strana

$$S_b(n)\frac{1+(-1)^n}{2}+\left[S_b(n)-n^b+(n+1)^b\right]\frac{1-(-1)^n}{2}=S_b(n)+\left[(n+1)^b-n^b\right]\frac{1-(-1)^n}{2}, \tag{9}$$

a zbroj desnih strana

$$\binom{b}{0}S_0(n)+\binom{b}{1}\overline{S}_1(n)+\binom{b}{2}S_2(n)+\ldots+\binom{b}{b-2}S_{b-2}(n)+\binom{b}{b-1}\overline{S}_{b-1}(n)+\binom{b}{b}S_b(n). \tag{10}$$

Nakon izjednačavanja i sređivanja (9) i (10), uz uvažavanje (3) i (4), dobivamo

$$\overline{S_{b-1}}(n)=\frac{1}{b}\left\{\left[(n+1)^b-n^b\right]\frac{1-(-1)^n}{2}-\left[\binom{b}{0}S_0(n)+\binom{b}{1}\overline{S_1}(n)+\binom{b}{2}S_2(n)\right.\right.$$
$$\left.\left.+\ldots+\binom{b}{b-3}\overline{S}_{b-3}(n)+\binom{b}{b-2}S_{b-2}(n)\right]\right\}. \tag{11}$$

I napokon, ako u (11) supstituiramo $b=2k$ dobivamo (5), što je i trebalo pokazati.

Za dokaz rekurzije (6) polazimo od razvoja

$$[a-(-1)^a]^b=\binom{b}{0}a^b-\binom{b}{1}(-1)^{1a}a^{b-1}+\binom{b}{2}(-1)^{2a}a^{b-2}-\ldots+(-1)^b\binom{b}{b}(-1)^{ba}a^0. \tag{12}$$

Analognom procedurom kao i u prethodnom slučaju, uz uvjet da je b neparan broj, dolazimo do tražene formule.

Sada idemo na primjenu rekurzija (5) i (6) određivati neke sume.

Primjer 1.
$$\overline{S_1}(n)=\frac{1+(-1)^{n+1}(2n+1)}{4}. \tag{13}$$

Rješenje. Ako je $k=1$, iz (5) slijedi

$$\overline{S_1}(n)=\frac{1}{2}\left\{\left[(n+1)^2-n^2\right]\frac{1-(-1)^n}{2}-\binom{2}{0}S_0(n)\right\}=\frac{1}{2}\left\{[2n+1]\frac{1-(-1)^n}{2}-n\right\}$$

ili, nakon sređivanja, dobivamo (13).

Za formule: $\overline{S_2}(n)$, $\overline{S_3}(n)$, $\overline{S_4}(n)$, ... moramo znati formule: $S_1(n)$, $S_2(n)$, $S_3(n)$, ... Prema tome navedimo samo neke formule, koje možemo dobiti iz (7):

$$S_1(n) = \frac{1}{2}n(n+1), \tag{14}$$

$$S_2(n) = \frac{1}{6}n(n+1)(2n+1), \tag{15}$$

$$S_3(n) = \frac{1}{4}n^2(n+1)^2 = S_1^2(n), \tag{16}$$

$$S_4(n) = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1), \tag{17}$$

$$S_5(n) = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1), \tag{18}$$

$$S_6(n) = \frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1), \tag{19}$$

$$S_7(n) = \frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2), \tag{20}$$

$$S_8(n) = \frac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3), \tag{21}$$

$$S_9(n) = \frac{1}{20}n^2(n+1)^2(2n^6+6n^5+n^4-8n^3+n^2+6n-3). \tag{22}$$

Primjer 2.
$$\overline{S_2}(n) = \frac{1}{2}(-1)^{n+1}n(n+1). \tag{23}$$

Rješenje. Ako u (6) stavimo $k=1$ dobivamo

$$\begin{aligned}
\overline{S_2}(n) &= \frac{1}{3}\left\{\left[(n+1)^3-n^3\right]\frac{1-(-1)^n}{2} - \left[\binom{3}{0}\overline{S}_0(n) + \binom{3}{1}S_1(n)\right]\right\}\\
&= \frac{1}{3}\left\{\left[(n+1)^3-n^3\right]\frac{1-(-1)^n}{2} - \left[\frac{1-(-1)^n}{2} + \frac{3}{2}n(n+1)\right]\right\}\\
&= \ldots = -\frac{(-1)^n n(n+1)}{2}.
\end{aligned}$$

Primjer 3.
$$\overline{S_3}(n) = \frac{1}{8}\left[-1+(-1)^{n+1}(4n^3+6n^2-1)\right]. \tag{24}$$

Rješenje. Za $k=2$ iz (5) dobivamo

$$\overline{S_3}(n) = \frac{1}{4}\left\{\left[(n+1)^4-n^4\right]\frac{1-(-1)^n}{2} - \left[\binom{4}{0}S_0(n) + \binom{4}{1}\overline{S_1}(n) + \binom{4}{2}S_2(n)\right]\right\}.$$

Ako sada uvrstimo: (3), (13) i (15), tada nakon sređivanja dobivamo (24).

Primjer 4.
$$\overline{S_4}(n) = \frac{1}{2}(-1)^{n+1}n(n+1)(n^2+n-1). \tag{25}$$

Uputa. Za $k=2$ iz (6) dobivamo

$$\begin{aligned}
\overline{S_4}(n) = \frac{1}{5}\Bigg\{&\left[(n+1)^5-n^5-1\right]\frac{1-(-1)^n}{2}\\
&-\left[\binom{5}{0}\overline{S}_0(n) + \binom{5}{1}S_1(n) + \binom{5}{2}\overline{S}_2(n) + \binom{5}{3}S_3(n)\right]\Bigg\} = \ldots
\end{aligned}$$

Primjer 5.

$$\overline{S_5}(n) = \frac{1}{4}\left[1 + (-1)^{n+1}(2n+1)(n^4 + 2n^3 - n^2 - 2n + 1)\right]. \tag{26}$$

Uputa. Za $k = 3$ iz (5) dobivamo

$$\begin{aligned}\overline{S_5}(n) = \frac{1}{6}\Bigg\{&\left[(n+1)^6 - n^6\right]\frac{1-(-1)^n}{2}\\ &- \left[\binom{6}{0}S_0(n) + \binom{6}{1}\overline{S}_1(n) + \binom{6}{2}S_2(n) + \binom{6}{3}\overline{S}_3(n) + \binom{6}{4}S_4(n)\right]\Bigg\} = \ldots\\ = \frac{1}{4}&\left[1 + (-1)^{n+1}(2n^5 + 5n^4 - 5n^2 + 1)\right]\\ = \ldots = &\frac{1}{4}\left[1 + (-1)^{n+1}(2n+1)(n^4 + 2n^3 - n^2 - 2n + 1)\right].\end{aligned}$$

Napomenimo, da jednadžba $n^4 + 2n^3 - n^2 - 2n + 1 = 0$ ima dva realna rješenja, koja nisu racionalna, te dva konjugirano kompleksna. Prema tome nema smisla polinom u (26) dalje rastavljati u faktore.

Primjer 6.

$$S(n) = 1 - 3 + 5 - \ldots (-1)^{n+1}(2n-1) = (-1)^{n+1}n. \tag{27}$$

Uputa. $S(n) = 2\overline{S}_1(n) - \overline{S}_0(n) = \ldots$

Primjer 7.

$$S(n) = 1^2 - 3^2 + 5^2 - \ldots (-1)^{n+1}(2n-1)^2 = \frac{1}{2}\left[(-1)^{n+1}\left(4n^2 - 1\right) - 1\right]. \tag{28}$$

Uputa. $S(n) = 4\overline{S}_2(n) - 4\overline{S}_1(n) + \overline{S}_0(n) = \ldots$

Primjer 8.

$$S(n) = 1\cdot 2 - 2\cdot 3 + \ldots + (-1)^{n+1}n(n+1) = \frac{1-(-1)^n(2n^2+4n+1)}{4}. \tag{29}$$

Uputa. $S(n) = \overline{S}_2(n) + \overline{S}_1(n) = \ldots$

Primjer 9.

$$S(n) = 1\cdot 3 - 3\cdot 5 + \ldots + (-1)^{n+1}(2n-1)(2n+1) = \frac{(-1)^{n+1}(4n^2+4n-1)-1}{2}. \tag{30}$$

Uputa. $S(n) = 4\overline{S}_2(n) - \overline{S}_0(n) = \ldots$

Primjer 10.

$$S(n) = 1{\cdot}2{\cdot}3 - 2{\cdot}3{\cdot}4 + \ldots + (-1)^{n+1}n(n+1)(n+2) = \frac{3 + (-1)^{n+1}(2n+3)(2n^2+6n+1)}{8}. \tag{31}$$

Uputa. Iz $n(n+1)(n+2) = n^3 + 3n^2 + 2n$ slijedi

$$\begin{aligned} S(n) &= \overline{S}_3(n) + 3\overline{S}_2(n) + 2\overline{S}_1(n) = \ldots = \frac{3 - (-1)^n(4n^3 + 18n^2 + 20n + 3)}{8} \\ &= \frac{3 - (-1)^n(2n+3)(2n^2+6n+1)}{8}. \end{aligned}$$

Primjer 11.

$$S(n) = \sum_{i=1}^{n} (-1)^{i+1}(2i-1)(2i+1)(2i+3) = -3 + (-1)^{n+1}(n+1)(4n^2+8n-3). \tag{32}$$

Uputa. Iz $(2n-1)(2n+1)(2n+3) = 8n^3 + 12n^2 - 2n - 3$ slijedi

$$S(n) = 8\overline{S}_3(n) + 12\overline{S}_2(n) - 2\overline{S}_1(n) - 3\overline{S}_0(n) = \ldots$$

Primjer 12.

$$\sum_{k=1}^{n} (-1)^{k+1}(ak+b)(ck+d) = \frac{ad+4bd+bc-(-1)^n\left[2acn^2+2(ac+ad+bc)n+ad+bc\right]}{4}. \tag{33}$$

Uputa. $\ldots = \sum_{k=1}^{n} (-1)^{k+1}\left[ack^2 + (ad+bc)k + bd\right] = ac\overline{S}_2(n) + (ad+bc)\overline{S}_1(n) + bd\overline{S}_0(n) = \ldots$

Napomenimo, ako uzmemo $a = c = 1$ i $b = d = 0$, tada (33) prelazi u $\overline{S}_2(n)$.

Primjer 13.

$$\sum_{i=1}^{n} (-1)^k(ai+b)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a^{k-j} b^j \overline{S}_{k-j}(n). \tag{34}$$

Primjer 14.

a)

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} (-1)^{i+j} ij = \frac{1}{2}\left[\overline{S}_1^2 - S_2(n)\right], \tag{35}$$

b)

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{n} (-1)^{i+j} ij = \overline{S}_1^2 - S_2(n). \tag{36}$$

Primjer 15. Ako je $k = 1, 2, 3, \ldots$ tada je:

a)

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} (-1)^{i+j} (ij)^k = \frac{1}{2}\left[\overline{S}_k^2 - S_{2k}(n)\right], \tag{37}$$

b)

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{n} (-1)^{i+j} (ij)^k = \overline{S}_k^2 - S_{2k}(n). \tag{38}$$

Sada ćemo naše sume generalizirati za konačne sume, odnosno alternativne sume, potencija istih eksponenata uzastopnih članova općeg aritmetičkog reda. Dakle trebamo promotriti sume

$$S_{k,a,d}(n)=\sum_{i=1}^{n}(a+(i-1)d)^k, \tag{39}$$

$$\overline{S}_{k,a,d}(n)=\sum_{i=1}^{n}(-1)^{i-1}(a+(i-1)d)^k, \tag{40}$$

gdje je

$$S_{k,1,1}(n)=S_k(n), \tag{41}$$

$$\overline{S}_{k,1,1}(n)=\overline{S}_k(n). \tag{42}$$

Primjer 16. Ako je $k=2$, iz (40) imamo:

$$\overline{S}_{2,a,d}(n)=\sum_{i=1}^{n}(-1)^{i+1}(a+(i-1)d)^2=\frac{a^2-d}{2}-\frac{(-1)^n}{2}\left[n^2-(2d^2-2d-1)n+a^2-d\right]. \tag{43}$$

Uputa. Za $k=2$ iz (40) dobivamo

$$\begin{aligned}\overline{S}_{2,a,d}(n)&=\sum_{i=1}^{n}(-1)^{i+1}\left[a^2+2a(i-1)d+(i-1)^2d^2\right]=\ldots\\&=(a^2-2d+d^2)\overline{S}_0(n)+2(d-d^2)\overline{S}_1(n)+d^2\overline{S}_2(n)=\text{(uvrstimo (4), (13), (23))}\\&=(a^2-2d+d^2)\frac{1-(-1)^n}{2}+2(d-d^2)\frac{1-(-1)^n(2n-1)}{4}-d^2\frac{(-1)^n n(n+1)}{2}=\ldots\end{aligned}$$

Primjer 17. $S_{k,a,d}(n)=\sum_{i=1}^{n}(a+(i-1)d)^k=\sum_{j=0}^{k}\binom{k}{j}a^{k-j}S_j(n-1)$.

Uputa. $\ldots=\sum_{i=1}^{n}\sum_{j=0}^{k}\binom{k}{j}a^{k-j}(i-1)^j=\sum_{j=0}^{k}\binom{k}{j}a^{k-j}S_j(n-1)=\ldots$

Primjer 18. $\overline{S}_{k,a,d}(n)=\sum_{i=1}^{n}(-1)^{i-1}(a+(i-1)d)^k=\sum_{j=0}^{k}\binom{k}{j}a^{k-j}\overline{S}_j(n-1)$.

Uputa. $\ldots=\sum_{i=1}^{n}\sum_{j=0}^{k}(-1)^{i-1}\binom{k}{j}a^{k-j}(i-1)^j=\sum_{j=0}^{k}\binom{k}{j}a^{k-j}\overline{S}_j(n-1)=\ldots$

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