RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 24 = 542 (2020): 39-48
DOI: https://doi.org/10.21857/90836c7nxy

SQUARE-FULL PRIMITIVE ROOTS IN SHORT INTERVALS

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Abstract. Using the character sum method of Shapiro and the work of Liu based on the exponent pair technique, an asymptotic formula for the number of square-full primitive roots modulo a prime in short intervals is obtained.

1. Introduction

Throughout, let p be an odd prime, let ε denote a fixed sufficiently small positive constant, let $\phi(n)$ be the Euler's totient function, let $\mu(n)$ be the Möbius function, and let $\omega(n)$ denote the number of distinct prime divisors of $n \in \mathbb{N}$.

An integer $n > 1$ is called square-full, if in its prime factorization each prime appears with exponent ≥ 2; the integer 1 is square-full by convention. Let $Q_2(x)$ denote the number of square-full integers $n \leq x$. The investigation of the distribution of square-full integers was originated by Erdös and Szekeres [6] who proved that

$$Q_2(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3}). \tag{1.1}$$

Bateman and Grosswald [1] in 1958 improved upon (1.1) by showing that

$$Q_2(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O\Big(x^{1/6} \exp(-C(\log x)^{3/5}(\log\log x)^{-1/5})\Big),$$

for some absolute constant $C > 0$. Any improvement on the exponent $1/6$ would imply that $\zeta(s) \neq 0$ for $\Re(s) > \sigma\,(1/2 \leq \sigma < 1)$. There are many other works on the improvement of the error terms under the Riemann Hypothesis, see e.g. [3], [4], [5], [9], [13], [16] and [17].

Concerning the distribution of square-full integers which are primitive roots, Shapiro [12] proved that the number of square-full integers which are

2020 *Mathematics Subject Classification.* 11L70, 11N69.
Key words and phrases. Primitive roots, square-full integers, short intervals.

primitive roots modulo an odd prime p, and not exceeding x is equal to

$$\frac{\phi(p-1)}{p-1}\left(cx^{1/2}+O(x^{1/3}p^{1/6}(\log p)^{1/3}2^{\omega(p-1)})\right), \tag{1.2}$$

where the constant $c = 2(1-1/p)\sum_{(q|p)=-1}\mu^2(q)/q^{3/2}$ with $(q|p)$ being the Legendre's symbol. In [10], Liu and Zhang improved upon (1.2) with the error term $O(x^{1/4+\varepsilon}p^{9/44+\varepsilon})$ by using Perron's formula. In 2018, Munsch and Trudgian [11] further refined the result of Liu and Zhang by showing that (1.2) can be replaced by

(1.3)
$$\frac{\phi(p-1)}{p-1}\left(\left(1+\frac{1}{p}+\frac{1}{p^2}\right)^{-1}\frac{C_p x^{1/2}}{\zeta(3)}+O(x^{1/3}(\log x)p^{1/9}(\log p)^{1/6}2^{\omega(p-1)})\right),$$

where $C_p \gg p^{-1/8\sqrt{e}}$. Recently, the second author [15] used the concept of exponent pair (in the problem of exponential sum estimates) and the lemmas used in the proof of Theorem 2.1 in [14], to improve the estimate (1.3) with the following result: for a given odd prime $p \le x^{1/5}$, the number of square-full integers which are primitive roots mod p and $\le x$ is equal to

(1.4)
$$\frac{\phi(p-1)}{p}\left\{\left(\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0)}\right)x^{1/2}+\left(\frac{L(2/3,\chi_0)-L(2/3,\chi_2^2)}{L(2,\chi_0)}\right)x^{1/3}\right\}$$
$$+O\left(x^{1/6}\phi(p-1)3^{\omega_{1,3}(p-1)}p^{1/2+\varepsilon}\right).$$

Here, χ_0, $\chi_1 \ne \chi_0, \chi_2 \ne \chi_0$ denote, respectively, the principal, quadratic, cubic characters mod p, $L(s,\chi)$ their corresponding Dirichlet L-functions, and $\omega_{1,3}(n)$ denotes the number of distinct primes $q \equiv 1 \pmod 3$ which are divisors of n.

Regarding the distribution of primitive roots in an interval, Burgess [2] proved that in an interval $[N, N+H]$ with $H > p^{1/4+\varepsilon}$, the number of primitive roots modulo p is

$$\frac{\phi(p-1)}{p-1}H\Big(1+O(p^{-\delta})\Big),$$

where $\delta > 0$ is a constant depending only on ε. In 2006, Zhai and Liu [18] studied square-free primitive roots in an interval and proved the existence of small square-free primitive roots.

It thus seems natural to search for some estimate on the number of square-full integers which are primitive roots mod p in short intervals. We derive here such an asymptotic estimate. Our main result reads:

Theorem 1.1. *Let $T_2(n)$ be the characteristic function of the square-full integers which are primitive roots modulo an odd prime p. For $\varepsilon > 0$ and θ*

in the range $\frac{14}{107} + 2\varepsilon \leq \theta < \frac{1}{6}$, *we have*

$$\sum_{x<n\leq x+x^{1/2+\theta}} T_2(n) = \frac{\phi(p-1)}{2p}\Big(\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0)}\Big)x^\theta(1+O(2^{\omega(p-1)}px^{-\varepsilon/4})), \tag{1.5}$$

where χ_0 *and* χ_1 *denote the principal, respectively, quadratic characters mod* p *with* $L(s,\chi)$ *being their corresponding Dirichlet L-functions.*

Our approach combines two methods; one is due to Liu [8] based on the exponent pair technique and the other is the formula for the characteristic function of primitive roots mod p due to Shapiro [12]. Let us first recall the notion exponent pair taken from [7, Chapter 2].

Definition 1.2. *Let* $A \geq 1, B \geq 1$, *and suppose that, for all* C *in* $[B, 2B]$,

$$\sum_{B\leq n\leq C\leq 2B} e^{2\pi i f(n)} = O(A^\kappa B^\lambda)$$

for some pair (κ,λ) *of real numbers satisfying* $0 \leq \kappa \leq 1/2 \leq \lambda \leq 1$, *and for any real function*

$$f \in C^\infty[B, 2B]$$

satisfying, for all $r \geq 1$ *and for* $x \in [B, 2B]$

$$AB^{1-r} \ll |f^{(r)}(x)| \ll AB^{1-r},$$

where the constants implied by $\ll$ *depend only on* r. *Then we call* (κ,λ) *an exponent pair.*

Lemma 1.3 ([8, Proposition 2]). *For* $x \in \mathbb{R}$, *let*

$$\psi(x) = x - \lfloor x \rfloor - \frac{1}{2},$$

where $\lfloor x \rfloor$ *is the integer part, and for* $\beta \in \mathbb{R}$, $\beta > 0$, *let*

$$R(x,\beta) = \sum_{n\leq x^\alpha} \psi\left(\frac{x}{n^\beta}\right), \quad \alpha = \frac{1}{\beta+1}. \tag{1.6}$$

We have

$$R(x,\beta) \ll x^{\tau(\beta)+\varepsilon}. \tag{1.7}$$

Here

$$\tau(\beta) = \begin{cases} \frac{7}{11(\beta+1)} & \text{if } 0 < \beta \leq 1, \\ \max(\tau_1(\beta), \tau_2(\beta)) & \text{if } \beta > 1, \end{cases}$$

with

$$\tau_1(\beta) = \inf_{(\kappa,\lambda)\in E} \Big(\frac{7(\lambda-\kappa)}{22\lambda - (15\beta+7)\kappa + 7(\beta-1)} \Big),$$
$$\tau_2(\beta) = \inf_{(\kappa,\lambda)\in E} \Big(\frac{3\lambda+\kappa}{4\lambda + (1-\beta)\kappa + 3\beta + 1} \Big),$$

where

$$E := E(\beta) = \{(\kappa,\lambda) | (\kappa,\lambda) \text{ is an exponent pair such that } \lambda \geq \beta\kappa\},$$

and the infima are taken over all exponent pairs belonging to E.

Lemma 1.4 ([12, Lemma 8.5.1]). *For a given odd prime* p*, the characteristic function of the primitive roots mod* p *is*

$$\frac{\phi(p-1)}{p-1} \sum_{d|p-1} \frac{\mu(d)}{\phi(d)} \sum_{\chi\in\Gamma_d} \chi(n) = \begin{cases} 1 & \text{if } n \text{ is a primitive root mod } p \\ 0 & \text{otherwise,} \end{cases}$$

where Γ_d *denotes the set of characters of the character group mod* p *that are of order* d.

2. Main results

Let our main character sum to be analyzed be

$$Q(x,\chi) = \sum_{\substack{n\leq x \\ n \text{ square-full}}} \chi(n).$$

Our first auxiliary result, whose proof proceeds along the line similar to [8, Theorem 1], is:

Lemma 2.1. *Let*

- χ *be a Dirichlet character modulo an odd prime* p *with* χ_0 *and* χ_1 *being the principal and quadratic characters, respectively;*
- $L(s,\chi)$ *be the associated Dirichlet* L*-function;*
- $R(\cdot,\cdot)$ *be as defined in* (1.6).

If $\sigma \in \mathbb{R}$ *is such that for any* $\varepsilon > 0$ *and any* $y > 1$*, the following estimates hold*

$$R(y^{1/2}, 3/2) \ll y^{\sigma+\varepsilon}, \quad R(y^{1/3}, 2/3) \ll y^{\sigma+\varepsilon}, \tag{2.1}$$

then, for any number θ *with* $\sigma + 2\varepsilon < \theta < \frac{1}{6}$*, we have*

$$Q(x + x^{1/2+\theta}, \chi_0) - Q(x,\chi_0) = \frac{p-1}{2p} \cdot \frac{L(3/2,\chi_0)}{L(3,\chi_0)} x^\theta (1 + O(x^{-\varepsilon/2})), \tag{2.2}$$

$$Q(x + x^{1/2+\theta}, \chi_1) - Q(x,\chi_1) = \frac{p-1}{2p} \cdot \frac{L(3/2,\chi_1)}{L(3,\chi_0)} x^\theta (1 + O(x^{-\varepsilon/2})), \tag{2.3}$$

and for $\chi \neq \chi_0, \chi_1$,

$$Q(x + x^{1/2+\theta}, \chi) - Q(x, \chi) = O(p\, x^{\theta-\varepsilon}). \tag{2.4}$$

PROOF. For brevity, let $B = x^{\theta-\varepsilon}$ and $h = x^{1/2+\theta}$. Since a square-full integer has a unique representation in the form $n = a^2 b^3$, where b is square-free, we have

$$\begin{aligned} &Q(x+h, \chi) - Q(x, \chi) = \\ &\quad \sum_{\substack{x < a^2 b^3 \le x+h \\ b \le B}} |\mu(b)| \chi^2(a) \chi^3(b) + \sum_{\substack{x < a^2 b^3 \le x+h \\ b > B}} |\mu(b)| \chi^2(a) \chi^3(b). \end{aligned} \tag{2.5}$$

First we bound the second sum in (2.5). We have

$$\left| \sum_{\substack{x < a^2 b^3 \le x+h \\ b > B}} |\mu(b)| \chi^2(a) \chi^3(b) \right| \le \sum_{\substack{x < a^2 b^3 \le x+h \\ b > B}} 1 = \Sigma_1 + \Sigma_2 \;;$$

we split the sum into two subsums Σ_1 and Σ_2 corresponding to $b \le (x+h)^{1/5}$ and $b > (x+h)^{1/5}$; in Σ_2 we have $x + h \ge a^2 b^3 > a^2 (x+h)^{3/5}$ yielding $a < (x+h)^{1/5}$. Thus

$$\begin{aligned} \Sigma_1 &= \sum_{B < b \le (x+h)^{1/5}} \; \sum_{(x/b^3)^{1/2} < a \le ((x+h)/b^3)^{1/2}} 1, \\ \Sigma_2 &= \sum_{a < (x+h)^{1/5}} \; \sum_{(x/a^2)^{1/3} < b \le ((x+h)/a^2)^{1/3}} 1. \end{aligned}$$

As

$$\sum_{\alpha < n \le \beta} 1 = \beta - \alpha + \psi(\alpha) - \psi(\beta),$$

$$(x+h)^{1/2} - x^{1/2} = \frac{1}{2} x^{\theta} (1 + O(x^{\theta - 1/2})) \tag{2.6}$$

and

$$(x+h)^{1/3} - x^{1/3} = \frac{1}{3} x^{\theta - 1/6} (1 + O(x^{\theta - 1/2})),$$

we have

$$\begin{aligned} \Sigma_1 &= \sum_{B < b \le (x+h)^{1/5}} \Big(\frac{(x+h)^{1/2} - x^{1/2}}{b^{3/2}} + \psi\Big(\frac{x^{1/2}}{b^{3/2}}\Big) - \psi\Big(\frac{(x+h)^{1/2}}{b^{3/2}}\Big)\Big) \\ &= R(x^{1/2}, 3/2) - R((x+h)^{1/2}, 3/2) + O(x^{\theta-\varepsilon}), \end{aligned}$$

and

$$\Sigma_2 = \sum_{a<(x+h)^{1/5}} \Big(\frac{(x+h)^{1/3}-x^{1/3}}{a^{2/3}} + \psi\Big(\frac{x^{1/3}}{a^{2/3}}\Big) - \psi\Big(\frac{(x+h)^{1/3}}{a^{2/3}}\Big)\Big)$$
$$= R(x^{1/3}, 2/3) - R((x+h)^{1/3}, 2/3) + O(x^{\theta-\varepsilon}).$$

From the assumption (2.1), we see that

$$\Sigma_1 = O(x^{\theta-\varepsilon}), \quad \Sigma_2 = O(x^{\theta-\varepsilon}). \tag{2.7}$$

Returning to the first term of (2.5), we write it as

$$\sum_{\substack{x<a^2b^3\le x+x^{1/2+\theta}\\ b\le B}} |\mu(b)|\chi^2(a)\chi^3(b) = \tag{2.8}$$
$$\sum_{b\le B} |\mu(b)|\chi^3(b) \sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} \chi^2(a).$$

For the case of principal character χ_0, the right hand side becomes

$$\sum_{b\le B} |\mu(b)|\chi_0^3(b) \sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} \chi_0^2(a)$$
$$= \sum_{b\le B} |\mu(b)|\chi_0(b) \sum_{\substack{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}\\ \gcd(a,p)=1}} 1$$
$$= \sum_{b\le B} |\mu(b)|\chi_0(b) \left(\sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} 1 - \sum_{(x/b^3)^{1/2}<ap\le((x+h)/b^3)^{1/2}} 1\right)$$
$$= \sum_{b\le B} |\mu(b)|\chi_0(b) \left(\frac{(x+h)^{1/2}-x^{1/2}}{b^{3/2}} - \frac{(x+h)^{1/2}-x^{1/2}}{pb^{3/2}} + O(1)\right)$$
$$= \frac{p-1}{p}((x+h)^{1/2}-x^{1/2}) \sum_{b\le B} \frac{|\mu(b)|\chi_0(b)}{b^{3/2}} + O(B).$$

Using (2.6), and

$$\sum_{b\le B} \frac{|\mu(b)|\chi_0(b)}{b^{3/2}} = \sum_{b=1}^{\infty} \frac{|\mu(b)|\chi_0(b)}{b^{3/2}} + O(B^{-1/2}), \quad \sum_{b=1}^{\infty} \frac{|\mu(b)|\chi_0(b)}{b^{3/2}} = \frac{L(3/2,\chi_0)}{L(3,\chi_0)},$$

we get

$$\sum_{b\le B} \chi_0^3(b) \sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} \chi_0^2(a) = \frac{p-1}{2p}\, x^{\theta}\, \frac{L(3/2,\chi_0)}{L(3,\chi_0)} \left(1 + O(x^{-\varepsilon/2})\right). \tag{2.9}$$

The assertion (2.2) follows from (2.5), (2.7) and (2.9).

The estimate (2.3) is proved in a similar manner.

Lastly, consider the case where $\chi \notin \{\chi_0, \chi_1\}$. From the relation (2.8), we have

$$\begin{aligned}
&\sum_{b\le B} |\mu(b)|\chi^3(b) \sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} \chi^2(a) \\
&= \sum_{b\le B} |\mu(b)|\chi^3(b) \left(\sum_{a\le((x+h)/b^3)^{1/2}} \chi^2(a) - \sum_{a\le(x/b^3)^{1/2}} \chi^2(a) \right) \\
&= \sum_{b\le B} |\mu(b)|\chi^3(b) \left(\sum_{j\le p} \sum_{\substack{a\le((x+h)/b^3)^{1/2} \\ a\equiv j \bmod p}} \chi^2(a) - \sum_{j\le p} \sum_{\substack{a\le(x/b^3)^{1/2} \\ a\equiv j \bmod p}} \chi^2(a) \right) \\
&= \sum_{b\le B} |\mu(b)|\chi^3(b) \left(\sum_{j\le p} \sum_{\substack{a\le((x+h)/b^3)^{1/2} \\ a\equiv j \bmod p}} \chi^2(j) - \sum_{j\le p} \sum_{\substack{a\le(x/b^3)^{1/2} \\ a\equiv j \bmod p}} \chi^2(j) \right) \\
&= \sum_{j\le p} \chi^2(j) \sum_{b\le B} |\mu(b)|\chi^3(b) \left(\sum_{\substack{a\le((x+h)/b^3)^{1/2} \\ a\equiv j \bmod p}} 1 - \sum_{\substack{a\le(x/b^3)^{1/2} \\ a\equiv j \bmod p}} 1 \right) \\
&= \sum_{j\le p} \chi^2(j) \sum_{b\le B} |\mu(b)|\chi^3(b) \left(\left\lfloor \frac{(x+h)^{1/2}}{pb^{3/2}} - \frac{j}{p} + 1 \right\rfloor - \left\lfloor \frac{x^{1/2}}{pb^{3/2}} - \frac{j}{p} + 1 \right\rfloor \right) \\
&= \sum_{j\le p} \chi^2(j) \sum_{b\le B} |\mu(b)|\chi^3(b) \left(\psi\left(\frac{x^{1/2}}{pb^{3/2}} - \frac{j}{p} \right) - \psi\left(\frac{(x+h)^{1/2}}{pb^{3/2}} - \frac{j}{p} \right) \right. \\
&\quad \left. + \frac{(x+h)^{1/2} - x^{1/2}}{pb^{3/2}} \right) \\
&= \sum_{j\le p} \chi^2(j) \sum_{b\le B} |\mu(b)|\chi^3(b) \left(\psi\left(\frac{x^{1/2}}{pb^{3/2}} - \frac{j}{p} \right) - \psi\left(\frac{(x+h)^{1/2}}{pb^{3/2}} - \frac{j}{p} \right) \right) \\
&= O(px^{\theta-\epsilon}),
\end{aligned}$$

where the second last equality follows from the identity $\sum_{j\le p} \chi^2(j) = 0$, which holds when $\chi^2 \ne \chi_0$. From this bound, (2.5) and (2.7), the assertion (2.4) follows. ∎

Our second main auxiliary result is:

Lemma 2.2. *If σ is a number such that for $\varepsilon > 0$,*

$$R(y^{1/2}, 3/2) \ll y^{\sigma+\varepsilon}, \quad R(y^{1/3}, 2/3) \ll y^{\sigma+\varepsilon} \quad \textit{for all } y > 1,$$

then, for any number θ with $\sigma+2\varepsilon<\theta<1/6$, we have

$$\sum_{x<n\le x+x^{1/2+\theta}} T_2(n) = \frac{\phi(p-1)}{2p}\Big(\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0)}\Big)x^{\theta} + O(2^{\omega(p-1)}p\,x^{\theta-\varepsilon/2}). \tag{2.10}$$

Proof. Since (Lemma 1.4) the characteristic function of the primitive roots mod p is $\frac{\phi(p-1)}{p-1}\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d}\chi(n)$, for $t>0$, we see that

$$\sum_{n\le t} T_2(n) = \frac{\phi(p-1)}{p-1}\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d} Q(t,\chi).$$

Separating out the first two values 1 and 2 of d, which correspond to the characters χ_0 and χ_1, respectively, we get

$$\begin{aligned}\sum_{x<n\le x+x^{1/2+\theta}} T_2(n) = {} & \frac{\phi(p-1)}{p-1}\Big\{Q(x+x^{1/2+\theta},\chi_0)-Q(x,\chi_0)\\ & -Q(x+x^{1/2+\theta},\chi_1)+Q(x,\chi_1)\Big\}\\ & +\frac{\phi(p-1)}{p-1}\sum_{\substack{d|p-1\\ d>2}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d}\Big(Q(x+x^{1/2+\theta},\chi)-Q(x,\chi)\Big).\end{aligned}$$

Using the estimates (2.2) and (2.3) in Lemma 2.1, the first portion on the right hand side is equal to

$$\frac{\phi(p-1)}{p-1}\,\frac{p-1}{2p}\,x^{\theta}\,\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0)}\Big(1+O(x^{-\varepsilon/2})\Big),$$

and using (2.4) in Lemma 2.1, the second portion is bounded by

$$\left|\sum_{\substack{d|p-1\\ d>2}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d}\Big(Q(x+x^{1/2+\theta},\chi)-Q(x,\chi)\Big)\right| \ll 2^{\omega(p-1)}px^{\theta-\varepsilon}.$$

The assertion now follows after simple simplifications. □

Proof of Theorem 1.1. We follow closely the arguments used in the proof of [8, Theorem 2]. By Lemma 1.3, we have

$$R(y^{1/3},2/3)\ll y^{7/55+\varepsilon}.$$

Choosing the pair $(2/7,4/7)\in E(3/2)$, which, by [7, p. 77], is an exponent pair, we get $\tau_1(3/2)\le 28/107$ and $\tau_2(3/2)\le 28/107$ yielding

$$R(y^{1/2},3/2)\ll y^{14/107+\varepsilon}.$$

Invoking upon Lemma 2.2 with $\sigma=14/107$, Theorem 1.1 follows. □

Acknowledgements.

This work is supported by the Departments of Mathematics, Faculty of Science, Kasetsart University, and the Department of Mathematics, Faculty of Science and Technology, Phranakhon Rajabhat University.

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Kvadratno puni primitivni korijeni u kratkim intervalima

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Sažetak. Korištenjem Shapirove metode sume karaktera te rad Liua zasnovan na tehnici parova eksponenata, dobivena je asimptotska formula za broj kvadratno punih primitivnih korijena po prostom modulu u kratkim intervalima.

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Received: 10.4.2020.
Revised: 11.6.2020.
Accepted: 15.9.2020.