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COMBINATORIAL EXTENSIONS OF POPOVICIU'S INEQUALITY VIA ABEL-GONTSCHAROFF POLYNOMIAL WITH APPLICATIONS IN INFORMATION THEORY

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Abstract. We establish new refinements and improvements of Popoviciu's inequality for n-convex functions using Abel-Gontscharoff interpolating polynomial along with the aid of new Green functions. We construct new inequalities for n-convex functions and compute new upper bounds for Ostrowski and Grüss type inequalities. As an application of our work in information theory, we give new estimations for Shannon, Relative and Zipf-Mandelbrot entropies using generalized Popoviciu's inequality.

1. Introduction and preliminary results

Popoviciu result has received a great deal of attention and many improvements and extensions have been obtained. Two easy extensions of Popoviciu's inequality that escaped unnoticed refer to the case of convex functions with values in a Banach lattice and that of semiconvex functions (i.e., of the functions that become convex after the addition of a suitable smooth function). Popoviciu's inequality has widely studied and many refinements and extensions have been obtained.

In 1965, T. Popoviciu [27] gives the following characterization of convex function:

Theorem 1.1. *Let $n \geq 3$ and k is a positive integers where $2 \leq k \leq n-1$. Suppose λ is continuous on I, then λ is convex iff*

(1.1)
$$\sum_{1\leq i_1<\ldots<i_k\leq n} \lambda\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \leq \frac{1}{k}\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n}\lambda(x_i) + n\lambda\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big)\right)$$

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holds for all $x_1, x_2, \cdots x_n \in I$.

THEOREM 1.2 ([25]). *Let* $\lambda : [\beta_1, \beta_2] \to \mathbb{R}$ *be convex. Then for each* $x, y, z \in [\beta_1, \beta_2]$ *and all* $p, q, r > 0$*, it holds*

$$(1.2)\quad (p+q+r)\lambda\left(\frac{px+qy+rz}{p+q+r}\right) - (q+r)\lambda\left(\frac{qy+rz}{q+r}\right) - (r+p)\lambda\left(\frac{rz+px}{r+p}\right) - (p+q)\lambda\left(\frac{px+qy}{p+q}\right) + p\lambda(x) + q\lambda(y) + r\lambda(z) \geq 0.$$

An axiom of convex function which was proved by T. Popoviciu in [27] is widely studied these days (see [25] and references with in). In 2016, M. V. Mihai introduced new extensions of Popoviciu's inequality (see [19]). In 2010, M. Bencze et al. in [2] gave Popoviciu's inequality for functions of several variables. C. P. Niculescu in 2009 gave the integral version of Popoviciu's inequality (see [21]). In 2006, C. P. Niculescu also gave refinement of Popoviciu's inequality in [22].

This form of Popoviciu's inequality was given by Vasić and Stanković (see [25, page 173]):

THEOREM 1.3. *Let* $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\beta_1, \beta_2] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_m) \in [\beta_1, \beta_2]^m$, $\mathbf{q} = (q_1, ..., q_m)$ *be positive* m*-tuple in such a way that* $\sum_{i=1}^{m} q_i = 1$. *Also let* $\lambda : [\beta_1, \beta_2] \to \mathbb{R}$. *Then*

$$(1.3)\quad \frac{1}{C_{k-1}^{m-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \left(\sum_{j=1}^{k} q_{i_j}\right) \lambda \left(\frac{\sum_{j=1}^{k} q_{i_j} x_{i_j}}{\sum_{j=1}^{k} q_{i_j}} \right) \leq \frac{m-k}{m-1} \sum_{i=1}^{m} q_i \lambda(x_i) + \frac{k-1}{m-1} \lambda\Big(\sum_{i=1}^{m} q_i x_i\Big)$$

or

$$(1.4)\qquad P_k^m(\mathbf{q}; \lambda) \leq \frac{m-k}{m-1} P_1^m(\mathbf{q}; \lambda) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; \lambda),$$

where

$$P_k^m(\mathbf{q}; \lambda) = P_k^m(\mathbf{q}; \lambda(x)) := \frac{1}{C_{k-1}^{m-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \left(\sum_{j=1}^{k} q_{i_j}\right) \lambda \left(\frac{\sum_{j=1}^{k} q_{i_j} x_{i_j}}{\sum_{j=1}^{k} q_{i_j}} \right)$$

is linear w.r.t. λ.

For two point right focal problem, the Abel-Gontscharoff theorem (see [1]) is given as

THEOREM 1.4. *Let* $n, s \in \mathbb{N}$, $n \geq 2$, $0 \leq s \leq n-2$ *and* $f \in C^n([\beta_1, \beta_2])$. *Then*

$$f(x) = \sum_{u=0}^{s} \frac{(x-\beta_1)^u}{u!} f^{(u)}(\beta_1) + \sum_{v=0}^{n-s-2} \left[\sum_{u=0}^{v} \frac{(x-\beta_1)^{s+1+u}(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!} \right] f^{(s+1+v)}(\beta_2) + \int_{\beta_1}^{\beta_2} AG_{(n)}(x,w) f^{(n)}(w)dw, \tag{1.5}$$

where $AG_{(n)}(x,w)$ *is defined by*

$$AG_{(n)}(x,w) = \frac{1}{(n-1)!} \begin{cases} \sum_{u=0}^{s} \binom{n-1}{u} (x-\beta_1)^u (\beta_1-w)^{n-u-1}, & \beta_1 \leq w \leq x, \\ -\sum_{u=s+1}^{n-1} \binom{n-1}{u} (x-\beta_1)^u (\beta_1-w)^{n-u-1}, & x \leq w \leq \beta_2. \end{cases} \tag{1.6}$$

Further, for $\beta_1 \leq w, \ \ x \leq \beta_2$ the following inequalities hold

$$(-1)^{n-s-1} \frac{\partial^u AG_{(n)}(x,w)}{\partial x^u} \geq 0, \qquad 0 \leq u \leq s, \tag{1.7}$$

$$(-1)^{n-u} \frac{\partial^u AG_{(n)}(x,w)}{\partial x^u} \geq 0, \qquad s+1 \leq u \leq n-1. \tag{1.8}$$

As a special choice for "two-point right focal" the Abel-Gontscharoff polynomial is given as:

$$\lambda(x) = \lambda(\beta_1) + (x-\beta_1)\lambda^{'}(\beta_2) + \int_{\beta_1}^{\beta_2} AG_{(2)}(x,w)\lambda^{''}(w)dw, \tag{1.9}$$

where

$$G_1(x,w) = AG_{(2)}(x,w) = \begin{cases} (\beta_1 - w), & \beta_1 \leq w \leq x, \\ (\beta_1 - x), & x \leq w \leq \beta_2. \end{cases} \tag{1.10}$$

In [18], authors gave the following new types of Green functions $G_d : [\beta_1, \beta_2] \times [\beta_1, \beta_2] \to \mathbb{R}$, $(d = 2, 3, 4,)$ considering Abel-Gontscharoff Green function for ' two-point right focal problem':

$$G_2(x,w) = \begin{cases} (x - \beta_2), & \beta_1 \leq w \leq x, \\ (w - \beta_2), & x \leq w \leq \beta_2. \end{cases} \tag{1.11}$$

$$G_3(x,w)=\begin{cases}(x-\beta_1), & \beta_1\le w\le x,\\ (w-\beta_1), & x\le w\le \beta_2.\end{cases}\tag{1.12}$$

$$G_4(x,w)=\begin{cases}(\beta_2-w), & \beta_1\le w\le x,\\ (\beta_2-x), & x\le w\le \beta_2.\end{cases}\tag{1.13}$$

The Green functions G_d, $(d=1,2,3,4)$ are symmetric and continuous. Moreover, with respect to both the variables x and w, all the functions are convex. In the following lemma, they introduced new identities by using new Green functions:

LEMMA 1.5 ([18]). *Let $\lambda:[\beta_1,\beta_2]\to\mathbb{R}$ be twice differentiable, G_d $(d=1,2,3,4)$ represents the new Green functions defined above. Then along with* (1.9)*, the following identities hold:*

$$\lambda(x)=\lambda(\beta_2)+(\beta_2-x)\lambda^{'}(\beta_1)+\int\limits_{\beta_1}^{\beta_2}G_2(x,w)\lambda^{''}(w)dw,\tag{1.14}$$

$$\lambda(x)=\lambda(\beta_2)-(\beta_2-\beta_1)\lambda^{'}(\beta_2)+(x-\beta_1)\lambda^{'}(\beta_1)+\int\limits_{\beta_1}^{\beta_2}G_3(x,w)\lambda^{''}(w)dw,\tag{1.15}$$

$$\lambda(x)=\lambda(\beta_1)+(\beta_2-\beta_1)\lambda^{'}(\beta_1)-(\beta_2-x)\lambda^{'}(\beta_2)+\int\limits_{\beta_1}^{\beta_2}G_4(x,w)\lambda^{''}(w)dw.\tag{1.16}$$

The known Čebyšev functional given for $\mathbb{F}_1,\mathbb{F}_2:[\beta_1,\beta_2]\to\mathbb{R}$ as

$$\mathbb{C}(\mathbb{F}_1,\mathbb{F}_2)=\frac{1}{\beta_2-\beta_1}\int\limits_{\beta_1}^{\beta_2}\mathbb{F}_1(\xi)\mathbb{F}_2(\xi)d\xi-\frac{1}{\beta_2-\beta_1}\int\limits_{\beta_1}^{\beta_2}\mathbb{F}_1(\xi)d\xi.\frac{1}{\beta_2-\beta_1}\int\limits_{\beta_1}^{\beta_2}\mathbb{F}_2(\xi)d\xi,$$

is extremely helpful to construct some new upper bounds.

Cerone and Dragomir in [6] utilized Čebyšev functional to established the following inequalities of Grüss and Ostrowski type:

THEOREM 1.6. *Let $\mathbb{F}_1\in L[\beta_1,\beta_2]$ and $\mathbb{F}_2:[\beta_1,\beta_2]\to\mathbb{R}$ be absolutely continuous with $(.-\beta_1)(\beta_2-.)[\mathbb{F}_2']^2\in L[\beta_1,\beta_2]$. Then, inequality*

$$|\mathbb{C}(\mathbb{F}_1,\mathbb{F}_2)|\le\frac{1}{\sqrt{2}}\left[\frac{\mathbb{C}(\mathbb{F}_1,\mathbb{F}_1)}{(\beta_2-\beta_1)}\right]^{\frac{1}{2}}\left(\int\limits_{\beta_1}^{\beta_2}(x-\beta_1)(\beta_2-x)[\mathbb{F}_2'(x)]^2dx\right)^{\frac{1}{2}},\tag{1.17}$$

holds with $\frac{1}{\sqrt{2}}$ be most appropriate constant.

THEOREM 1.7. *Let* $\mathbb{F}_1 : [\beta_1, \beta_2] \to \mathbb{R}$ *be absolutely continuous function for* $\mathbb{F}_1' \in L_\infty[\beta_1, \beta_2]$, $\mathbb{F}_2 : [\beta_1, \beta_2] \to \mathbb{R}$ *is nondecreasing and monotonic. Then*

$$|\mathbb{C}(\mathbb{F}_1, \mathbb{F}_2)| \leq \frac{||\mathbb{F}_1'||_\infty}{2(\beta_2 - \beta_1)} \int_{\beta_1}^{\beta_2} (x - \beta_1)(\beta_2 - x) d\mathbb{F}_2(x), \tag{1.18}$$

is valid, where $\frac{1}{2}$ *is the best possible constant.*

In this paper we will formulate new refinements and generalizations of Popoviciu's inequality for n-convex functions using Abel-Gontscharoff interpolating polynomial and compute new upper bounds for Ostrowski and Grüss type inequalities. We will also give new upper bounds for Shannon, relative and Zipf-Mandelbrot entropies.

2. New generalizations of Popoviciu's inequality

Before giving our main results, we consider the following assumptions:

A_1 $\lambda : [\beta_1, \beta_2] \to \mathbb{R}$ such that $\lambda \in C^n([\beta_1, \beta_2])$
A_2 $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $\mathbf{x} \in [\beta_1, \beta_2]^m$
A_3 For any $1 \leq i_1 < ... < i_k \leq m$, $\frac{\sum_{j=1}^{k} q_{i_j} x_{i_j}}{\sum_{j=1}^{k} q_{i_j}} \in [\beta_1, \beta_2]$.
A_4 For any $f \in AC([\beta_1, \beta_2]) \implies f : [\beta_1, \beta_2] \to \mathbb{R}$ that is absolutely continuous.

THEOREM 2.1. *Consider the assumptions* A_1, A_2, A_3 *and* $\sum_{j=1}^{k} q_{i_j} \neq 0$ *for any* $1 \leq i_1 < ... < i_k \leq m$ *and* $\sum_{i=1}^{m} q_i = 1$. *Also for* $n \geq 4$, *let* $AG_{(n)}(\cdot, w)$ *and* $G_d(\cdot, w)$ $(d = 1, 2, 3, 4)$ *be defined in* (1.6) *and* (1.10)–(1.13). *If* λ *is* n*-convex function and*

$$\left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) \geq 0 \tag{2.1}$$

holds, provided that $(n = even,\ s = odd)$ *or* $(n = odd,\ s = even)$, *then*

$$\frac{m-k}{m-1} P_1^m(\mathbf{q}; \lambda) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; \lambda) - P_k^m(\mathbf{q}; \lambda) \geq \sum_{u=0}^{s} \frac{\lambda^{(u+2)}(\beta_1)}{u!} \times \tag{2.2}$$

$$\int_{\beta_1}^{\beta_2} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) (w - \beta_1)^u dw$$

$$+\sum_{v=0}^{n-s-2}\left[\sum_{u=0}^{v}\frac{\lambda^{(s+1+u)}(\beta_2)(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!}\right]\times$$

$$\int_{\beta_1}^{\beta_2}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_d)+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_d)-P_k^m(\mathbf{q};G_d)\right)(w-\beta_1)^{s+1+u}dw,$$

where

$$P_k^m(\mathbf{q};G_d)=P_k^m(\mathbf{q};G_d(x,w)):= \tag{2.3}$$

$$\frac{1}{C_{k-1}^{m-1}}\sum_{1\le i_1<\ldots<i_k\le m}\left(\sum_{j=1}^{k}q_{i_j}\right)G_d\left(\frac{\sum_{j=1}^{k}q_{i_j}x_{i_j}}{\sum_{j=1}^{k}q_{i_j}},w\right),$$

for the function $G_d:[\beta_1,\beta_2]\times[\beta_1,\beta_2]\to\mathbb{R}$ *and* $2\le k\le m$.

Proof. For fixed $d=4$, using (1.4) in (1.16) , we get

$$\begin{aligned}&\frac{m-k}{m-1}P_1^m(\mathbf{q};\lambda)+\frac{k-1}{m-1}P_m^m(\mathbf{q};\lambda)-P_k^m(\mathbf{q};\lambda)\\&=\int_{\beta_1}^{\beta_2}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_4)+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_4)-P_k^m(\mathbf{q};G_4)\right)\lambda''(w)dw.\end{aligned} \tag{2.4}$$

Applying "two-point right focal" Abel-Gontscharoff polynomial for λ'', we get

$$\begin{aligned}\lambda''(x)&=\sum_{u=0}^{s}\frac{(x-\beta_1)^u}{u!}\lambda^{(u+2)}(\beta_1)\\&+\sum_{v=0}^{n-s-4}\left[\sum_{u=0}^{v}\frac{(x-\beta_1)^{s+1+u}(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!}\right]\lambda^{(s+3+v)}(\beta_2)\\&+\int_{\beta_1}^{\beta_2}AG_{(n-2)}(x,w)\lambda^{(n)}(w)dw.\end{aligned} \tag{2.5}$$

Now, using (2.5) in (2.4), we get the generalized Popoviciu identity involving "two-point right focal" Abel-Gontscharoff polynomial

$$\begin{aligned}&\frac{m-k}{m-1}P_1^m(\mathbf{q};\lambda)+\frac{k-1}{m-1}P_m^m(\mathbf{q};\lambda)-P_k^m(\mathbf{q};\lambda)=\sum_{u=0}^{s}\frac{\lambda^{(u+2)}(\beta_1)}{u!}\\&\times\int_{\beta_1}^{\beta_2}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_4)+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_4)-P_k^m(\mathbf{q};G_4)\right)(w-\beta_1)^u dw\end{aligned} \tag{2.6}$$

$$+\sum_{v=0}^{n-s-2}\left[\sum_{u=0}^{v}\frac{\lambda^{(s+1+u)}(\beta_2)(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!}\right]$$

$$\times\int_{\beta_1}^{\beta_2}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_4)+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_4)-P_k^m(\mathbf{q};G_4)\right)(w-\beta_1)^{s+1+u}dw$$

$$+\int_{\beta_1}^{\beta_2}\int_{\beta_1}^{\beta_2}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_4)+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_4)-P_k^m(\mathbf{q};G_4)\right)$$

$$\times AG_{(n-2)}(w,t)\lambda^n(t)dwdt.$$

Now from (1.7), we get $(-1)^{n-s-3}AG_{(n-2)}(w,t)\geq 0$. Hence utilizing our assumptions ($n=even$, $s=odd$) or ($n=odd$, $s=even$), we get $AG_{(n-2)}(w,t)\geq 0$. Now applying n-covexity of the function λ and using (2.1), we get (2.2). We can treat $d=1,2,3$ analogously. □

Now we relax the conditions on the weights to be positive and give generalization of Popoviciu's inequality for n-convex functions.

Corollary 2.2. *If the conditions of Theorem 2.1 are satisfied with additional conditions that $q_1,\dots,q_m$ is a nonnegative tuple such that $\sum_{i=1}^{m}q_i=1$, then for $\lambda:[\beta_1,\beta_2]\to\mathbb{R}$ being n-convex, we obtain the following results:*

(a) For ($n=even,\ s=odd$) or ($n=odd,\ s=even$) (2.2) holds.

(b) For

$$\sum_{u=0}^{s}\frac{(w-\beta_1)^u}{u!}\lambda^{(u+2)}(\beta_1)+\sum_{v=0}^{n-s-4}\left[\sum_{u=0}^{v}\frac{(w-\beta_1)^{s+1+u}(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!}\right]\lambda^{(s+3+v)}(\beta_2)\geq 0, \tag{2.7}$$

the right side of (2.2) is non negative, particularly

$$\frac{m-k}{m-1}P_1^m(\mathbf{q};\lambda)+\frac{k-1}{m-1}P_m^m(\mathbf{q};\lambda)-P_k^m(\mathbf{q};\lambda)\geq 0. \tag{2.8}$$

Proof.

(a) We have assumed positive weights and $G_d(\cdot,w)$, ($d=1,2,3,4$) are convex. Thus by applying Popoviciu's inequality for convex function $G_d(\cdot,w)$, ($d=1,2,3,4$), (2.1) is established. Since λ is n-convex, so by using Theorem 2.1, we get (2.2).

(b) Now taking into account the positivity of (2.7) and Popoviciu's inequality for convex function $G_d(\cdot,w)$, ($d=1,2,3,4$) in (2.2), we get (2.8).

□

In the next results, we use above theorem in order to form some novel estimates for Grüss and Ostrowski type inequalities using generalized identity (2.6). In what follows we let $t \in [\beta_1, \beta_2]$, for $(d = 1, 2, 3, 4)$,

$$\mathfrak{A}_d(t) = \int_{\beta_1}^{\beta_2} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) AG_{(n-2)}(w, t) dw. \tag{2.9}$$

THEOREM 2.3. *With the hypothesis of Theorem 2.1, suppose* $|\lambda^{(n)}|^r : [\beta_1, \beta_2] \to \mathbb{R}$ *is R-integrable, where* $n \geq 4$ *while* $r, r' \in [1, \infty]$ *and* $\frac{1}{r} + \frac{1}{r'} = 1$. *Then we have*

$$\begin{aligned} &\Bigg| \frac{m-k}{m-1} P_1^m(\mathbf{q}; \lambda) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; \lambda) - P_k^m(\mathbf{q}; \lambda) - \sum_{u=0}^{s} \frac{\lambda^{(u+2)}(\beta_1)}{u!} \\ &\times \int_{\beta_1}^{\beta_2} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) (w-\beta_1)^u dw \\ &- \sum_{w=0}^{n-s-2} \left[\sum_{u=0}^{v} \frac{\lambda^{(s+1+u)}(\beta_2)(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!} \right] \\ &\times \int_{\beta_1}^{\beta_2} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) (w-\beta_1)^{s+1+u} dw \Bigg| \\ &\leq ||\lambda^{(n)}||_r \left(\int_{\beta_1}^{\beta_2} \left| \mathfrak{A}_d(t) \right|^{r'} dt \right)^{1/r'} \end{aligned} \tag{2.10}$$

where $\mathfrak{A}_d(t)$ $(d = 1, 2, 3, 4)$ *is defined in* (2.9). *The constants on the right hand side of* (2.10) *are good when* $1 < r \leq \infty$ *while most appropriate choice is* $r = 1$.

PROOF. We rearrange identity (2.6) in such a way that

$$\begin{aligned} &\Bigg| \frac{m-k}{m-1} P_1^m(\mathbf{q}; \lambda) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; \lambda) - P_k^m(\mathbf{q}; \lambda) - \sum_{u=0}^{s} \frac{\lambda^{(u+2)}(\beta_1)}{u!} \\ &\times \int_{\beta_1}^{\beta_2} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) (w-\beta_1)^u dw \end{aligned} \tag{2.11}$$

$$- \sum_{w=0}^{n-s-2} \left[\sum_{u=0}^{v} \frac{\lambda^{(s+1+u)}(\beta_2)(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!} \right]$$

$$\times \int_{\beta_1}^{\beta_2} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) (w-\beta_1)^{s+1+u} dw \Bigg|$$

$$= \left| \int_{\beta_1}^{\beta_2} \mathfrak{A}_d(t) \lambda^{(n)}(t) dt \right|.$$

Using Holder's inequality on right hand side of (2.11) we obtain (2.10). For sharpness, the proof is same as that of Theorem 3.5 in [4] (see also [3]).

□

We now give some upper bounds of the Grüss type inequality.

Theorem 2.4. *With the assumptions of Theorem 2.1 and absolute continuity of* $\lambda^{(n)}$ *while* $(. - \beta_1)(\beta_2 - .)[\lambda^{(n+1)}]^2 \in L[\beta_1, \beta_2]$ *such that* $\mathfrak{A}_d$, $(d = 1, 2, 3, 4)$ *are defined in* (2.9)*, the remainders* $Rem(\beta_1, \beta_2, \mathfrak{A}_d, \lambda^{(n)})$*, given in the following identity*

$$\frac{m-k}{m-1} P_1^m(\mathbf{q}; \lambda) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; \lambda) - P_k^m(\mathbf{q}; \lambda) - \sum_{u=0}^{s} \frac{\lambda^{(u+2)}(\beta_1)}{u!} \tag{2.12}$$

$$\times \int_{\beta_1}^{\beta_2} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) (w-\beta_1)^{u} dw$$

$$- \sum_{w=0}^{n-s-2} \left[\sum_{u=0}^{v} \frac{\lambda^{(s+1+u)}(\beta_2)(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!} \right]$$

$$\times \int_{\beta_1}^{\beta_2} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d) - P_k^m(\mathbf{q}; G_d) \right) (w-\beta_1)^{s+1+u} dw$$

$$= \frac{\lambda^{(n-1)}(\beta_2) - \lambda^{(n-1)}(\beta_1)}{(\beta_2 - \beta_1)} \int_{\beta_1}^{\beta_2} \mathfrak{A}_d(t) dt + Rem(\beta_1, \beta_2, \mathfrak{A}_d, \lambda^{(n)}),$$

satisfy the bound

$$|Rem(\beta_1, \beta_2, \mathfrak{A}_d, \lambda^{(n)})| \leq$$

$$\left[\mathbb{C}(\mathfrak{A}_d, \mathfrak{A}_d)\right]^{\frac{1}{2}} \sqrt{\frac{(\beta_2-\beta_1)}{2}} \left| \int_{\beta_1}^{\beta_2} (t-\beta_1)(\beta_2-t)[\lambda^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}.$$

PROOF. Using Čebyšev functional for $\mathbb{F}_1 = \mathfrak{A}_d$, $(d = 1, 2, 3, 4)$, $\mathbb{F}_2 = \lambda^{(n)}$ and by comparing (2.12) with (2.6), we have

$$Rem(\beta_1, \beta_2, \mathfrak{A}_d, \lambda^{(n)}) = (\beta_2 - \beta_1)\mathbb{C}(\mathfrak{A}_d, \lambda^{(n)}).$$

Using Theorem 1.6 the desired bound can be obtained. □

THEOREM 2.5. *Consider assumptions of Theorem 2.1, while* $\lambda^{(n+1)} \geq 0$ *on* $[\beta_1, \beta_2]$ *with* $\mathfrak{A}_d$ $(d = 1, 2, 3, 4)$ *given by* (2.9). *Then in* (2.12) $Rem(\beta_1, \beta_2, \mathfrak{A}_d, \lambda^{(n)})$ *fulfills estimation*

$$(2.13) \quad |Rem(\beta_1, \beta_2, \mathfrak{A}_d, \lambda^{(n)})| \leq (\beta_2 - \beta_1)||\mathfrak{A}'_d||_\infty \times \left[\frac{\lambda^{(n-1)}(\beta_2) + \lambda^{(n-1)}(\beta_1)}{2} - \frac{\lambda^{(n-2)}(\beta_2) - \lambda^{(n-2)}(\beta_1)}{\beta_2 - \beta_1}\right].$$

PROOF. We have established

$$Rem(\beta_1, \beta_2, \mathfrak{A}_d, \lambda^{(n)}) = (\beta_2 - \beta_1)\mathbb{C}(\mathfrak{A}_d, \lambda^{(n)}).$$

Now using Theorem 1.7, for $\mathbb{F}_1 \to \mathfrak{A}_d$, $\mathbb{F}_2 \to \lambda^{(n)}$, gives

$$|Rem(\beta_1, \beta_2, \mathfrak{A}_d, \lambda^{(n)})| = (\beta_2 - \beta_1)|\mathbb{C}(\mathfrak{A}_d, \lambda^{(n)})|$$

$$(2.14) \quad \leq \frac{||\mathfrak{A}'_d||_\infty}{2} \int_{\beta_1}^{\beta_2} (\xi - \beta_1)(\beta_2 - \xi)\lambda^{(n+1)}(\xi).$$

Using the right hand side of (2.14), (2.13) is obtained. □

3. NEW ENTROPIC BOUNDS IN INFORMATION THEORY

As Jensen's inequality plays a key role in information theory to construct bounds for some notable inequalities, here we will use Popoviciu's inequality to make connections between inequalities in information theory.

Let $\lambda : (0, \infty) \to (0, \infty)$ be convex, let $p := (p_1, ..., p_m)$ and $q := (q_1, ..., q_m)$ represent positive probability distributions. Then λ-divergence functional is defined as follows

$$I_\lambda(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} q_i \lambda\left(\frac{p_i}{q_i}\right).$$

L. Horváth et al. in [10] defined a new Csiszár divergence functional:

DEFINITION 3.1. *Let* $\lambda : I \to \mathbb{R}$ *be a function with* I *an interval in* $\mathbb{R}$. *Let* $\mathbf{p} := (p_1, \ldots, p_m), \mathbf{q} := (q_1, \ldots, q_m) \in \mathbb{R}^m$*, such that*

$$\frac{p_i}{q_i} \in I, \; q_i \neq 0 \quad i = 1, \ldots, m.$$

Then we define

$$\widetilde{I}_{\lambda}(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{m} q_i \lambda\left(\frac{p_i}{q_i}\right). \tag{3.1}$$

We now give the first application of Theorem 2.1:

THEOREM 3.1. *Under the assumptions of Theorem 2.1, let* $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $\mathbf{p} := (p_1, \ldots, p_m), \mathbf{q} := (q_1, \ldots, q_m) \in \mathbb{R}^m$, *such that*

$$\frac{p_i}{q_i} \in [\beta_1, \beta_2], \; q_i \neq 0 \quad i = 1, \ldots, m.$$

If $\lambda : [\beta_1, \beta_2] \to \mathbb{R}$ *is an* n*-convex function, then we obtain the following bound for our new Csiszár divergence functional:*

$$\begin{aligned}\widetilde{I}_{\lambda}(\mathbf{p},\mathbf{q}) \geq{}& \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q},\mathbf{p};\lambda) - \frac{k-1}{m-k}\lambda\left(P_m\right) + \sum_{u=0}^{s} \frac{\lambda^{(u+2)}(\beta_1)}{u!} \times \\ & \int_{\beta_1}^{\beta_2} \left(\sum_{i=1}^{m} q_i G_d\left(\frac{p_i}{q_i}, w\right) + \frac{k-1}{m-k} G_d\left(P_m, w\right) - \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q},\mathbf{p};G_d)\right)(w-\beta_1)^u dw \\ & + \sum_{v=0}^{n-s-2}\left[\sum_{u=0}^{v} \frac{\lambda^{(s+1+u)}(\beta_2)(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!}\right] \times \\ & \int_{\beta_1}^{\beta_2} \left(\sum_{i=1}^{m} q_i G_d\left(\frac{p_i}{q_i}, w\right) + \frac{k-1}{m-k} G_d\left(P_m, w\right) - \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q},\mathbf{p};G_d)\right) \\ & \times (w-\beta_1)^{s+1+u} dw,\end{aligned} \tag{3.2}$$

where $\sum\limits_{i=1}^{m} p_i = P_m$ *and*

$$\mathbb{C}_k^m(\mathbf{q},\mathbf{p};\lambda) := \frac{1}{C_{k-1}^{m-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \left(\sum_{j=1}^{k} q_{i_j}\right) \lambda\left(\frac{\sum_{j=1}^{k} p_{i_j}}{\sum_{j=1}^{k} q_{i_j}}\right). \tag{3.3}$$

PROOF. By taking into account the assumptions of Theorem 2.1, write (2.2) in explicit form as:

(3.4)

$$\frac{m-k}{m-1}\sum_{i=1}^{m} q_i \lambda(x_i) + \frac{k-1}{m-1}\lambda\Big(\sum_{i=1}^{m} q_i x_i\Big) - \frac{1}{C_{k-1}^{m-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \left(\sum_{j=1}^{k} q_{i_j}\right) \lambda\left(\frac{\sum_{j=1}^{k} q_{i_j} x_{i_j}}{\sum_{j=1}^{k} q_{i_j}}\right) \geq$$

$$\sum_{u=0}^{s}\frac{\lambda^{(u+2)}(\beta_1)}{u!}\int_{\beta_1}^{\beta_2}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_d)+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_d)-P_k^m(\mathbf{q};G_d)\right)(w-\beta_1)^u dw$$

$$+\sum_{v=0}^{n-s-2}\left[\sum_{u=0}^{v}\frac{\lambda^{(s+1+u)}(\beta_2)(\beta_1-\beta_2)^{v-u}}{(s+1+u)!(v-u)!}\right]\times$$

$$\int_{\beta_1}^{\beta_2}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_d)+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_d)-P_k^m(\mathbf{q};G_d)\right)(w-\beta_1)^{s+1+u}dw,$$

where

$$P_k^m(\mathbf{q};G_d)=P_k^m(\mathbf{q};G_d(x,w)):=$$

$$\frac{1}{C_{k-1}^{m-1}}\sum_{1\le i_1<\ldots<i_k\le m}\left(\sum_{j=1}^{k}q_{i_j}\right)G_d\left(\frac{\sum_{j=1}^{k}q_{i_j}x_{i_j}}{\sum_{j=1}^{k}q_{i_j}},w\right).$$

Now replacing $x_i\to\frac{p_i}{q_i}$ in (3.4), after some calculations we get required result (3.2). □

The next result is the application of the Corollary 2.2 for positive probability distributions.

COROLLARY 3.2. *Under the assumptions of Corollary 2.2 for* (n = *even*, s = *odd*) *assume that* (2.7) *holds. If* $\lambda:[\beta_1,\beta_2]\to\mathbb{R}$ *is an* n*-convex function, then the above bound* (3.2) *takes the shape*

$$\frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q},\mathbf{p};\lambda)-\frac{k-1}{m-k}\lambda(1)\le\widetilde{I}_\lambda(\mathbf{p},\mathbf{q}). \tag{3.5}$$

PROOF. It is the direct consequence of Corollary 2.2 by substituting $x_i=\frac{p_i}{q_i}$ in (2.8). □

Shannon entropy and the measures related to it are frequently applied in fields like population genetics and molecular ecology, information theory, dynamical systems and statistical physics (see [7, 16]). For positive n-tuple $\mathbf{q}=(q_1,\ldots,q_m)$ such that $\sum_{i=1}^{m}q_i=1$, the *Shannon entropy* is defined by

$$S(\mathbf{q})=-\sum_{i=1}^{m}q_i\ln q_i. \tag{3.6}$$

Some recent bounds for Shannon entropy can be seen in [10, 13]. We propose the following results:

COROLLARY 3.3. *Let* $m,k\in\mathbb{N}$*,* $m\ge 3$*,* $2\le k\le m-1$*.*

(a) If $\mathbf{q} := (q_1, \ldots, q_m) \in (0, \infty)^m$ *and* n *is even, then for* $d = 1, 2, 3, 4$

(3.7)

$$\sum_{i=1}^{m} q_i \ln(q_i) \geq \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q}, \mathbf{1}; -\ln(\cdot)) + \frac{k-1}{m-k}\ln(m) + \sum_{u=0}^{s} \frac{(-1)^{u+2}(u+1)}{(\beta_1)^{u+2}} \times$$
$$\int_{\beta_1}^{\beta_2} \left(\sum_{i=1}^{m} q_i G_d\left(\frac{1}{q_i}, w\right) + \frac{k-1}{m-k} G_d(m, w) - \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q}, \mathbf{1}; G_d)\right)(w-\beta_1)^u dw$$
$$+ \sum_{v=0}^{n-s-2} \left[\sum_{u=0}^{v} \frac{(-1)^{s+1+v}(\beta_1 - \beta_2)^{v-u}}{(\beta_2)^{s+1+v}(s+1+u)(v-u)!} \right] \times$$
$$\int_{\beta_1}^{\beta_2} \left(\sum_{i=1}^{m} q_i G_d\left(\frac{1}{q_i}, w\right) + \frac{k-1}{m-k} G_d(m, w) - \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q}, \mathbf{1}; G_d)\right)$$
$$\times (w-\beta_1)^{s+1+u} dw.$$

(b) If $\mathbf{q} := (q_1, \ldots, q_m)$ *is a positive probability distribution and* n *is even, then we get the following bounds for Shannon entropy of* $\mathbf{q}$*:*

(3.8)

$$S(\mathbf{q}) \leq -\left(\frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q}, \mathbf{1}; -\ln(\cdot)) + \frac{k-1}{m-k}\ln(m)\right) - \sum_{u=0}^{s} \frac{(-1)^{u+2}(u+1)}{(\beta_1)^{u+2}} \times$$
$$\int_{\beta_1}^{\beta_2} \left(\sum_{i=1}^{m} q_i G_d\left(\frac{1}{q_i}, w\right) + \frac{k-1}{m-k} G_d(m, w) - \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q}, \mathbf{1}; G_d)\right)(w-\beta_1)^u dw$$
$$- \sum_{v=0}^{n-s-2} \left[\sum_{u=0}^{v} \frac{(-1)^{s+1+v}(\beta_1 - \beta_2)^{v-u}}{(\beta_2)^{s+1+v}(s+1+u)(v-u)!} \right] \times$$
$$\int_{\beta_1}^{\beta_2} \left(\sum_{i=1}^{m} q_i G_d\left(\frac{1}{q_i}, w\right) + \frac{k-1}{m-k} G_d(m, w) - \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q}, \mathbf{1}; G_d)\right)$$
$$\times (w-\beta_1)^{s+1+u} dw.$$

If n *is odd, then* (3.7) *and* (3.8) *hold in reverse directions.*

PROOF.

(a) Using $\lambda(x) := -\ln x$, and $\mathbf{p} := (1, 1, \ldots, 1)$ in Theorem 3.1, we obtain the desired results.
(b) It is a special case of (a).

∎

REMARK 3.4. Using positive probability distributions along with the function $\lambda(x) := -\ln x$ in (3.5), we get the bound

$$S(\mathbf{q}) \leq -\left(\frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q},\mathbf{p};-\ln(\cdot)) + \sum_{i=1}^{m} q_i \ln(p_i)\right). \tag{3.9}$$

The second case is corresponding to the relative entropy also known as Kullback-Leibler divergence between the two probability distributions. One of the most famous distance functions used in information theory, mathematical statistics and signal processing is Kullback-Leibler distance. The *Kullback-Leibler* distance [15] between the positive probability distributions $\mathbf{q} = (q_1, ..., q_m)$ and $\mathbf{p} = (p_1, ..., p_m)$ is defined by

$$D(\mathbf{q} \parallel \mathbf{p}) = \sum_{i=1}^{m} q_i \ln\left(\frac{q_i}{p_i}\right). \tag{3.10}$$

Some recent bounds for relative entropy can be seen in [10, 13]. We propose the following results:

COROLLARY 3.5. *Let $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$.*

(a) If $\mathbf{q} := (q_1, \ldots, q_m), \mathbf{p} := (p_1, \ldots, p_m) \in (0,\infty)^m$ and n is even, then for $d = 1, 2, 3, 4$

$$\begin{aligned}
&\sum_{i=1}^{m} q_i \ln\left(\frac{q_i}{p_i}\right) \geq \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q},\mathbf{p};-\ln(\cdot)) + \frac{k-1}{m-k}\ln(P_m) + \sum_{u=0}^{s}\frac{(-1)^{u+2}(u+1)}{(\beta_1)^{u+2}} \times \\
&\int_{\beta_1}^{\beta_2}\left(\sum_{i=1}^{m} q_i G_d\left(\frac{p_i}{q_i}, w\right) + \frac{k-1}{m-k}G_d(P_m, w) - \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q},\mathbf{p};G_d)\right)(w-\beta_1)^u dw \\
&\qquad + \sum_{v=0}^{n-s-2}\left[\sum_{u=0}^{v}\frac{(-1)^{s+1+v}(\beta_1-\beta_2)^{v-u}}{(\beta_2)^{s+1+v}(s+1+u)(v-u)!}\right] \times \\
&\int_{\beta_1}^{\beta_2}\left(\sum_{i=1}^{m} q_i G_d\left(\frac{p_i}{q_i}, w\right) + \frac{k-1}{m-k}G_d(P_m, w) - \frac{m-1}{m-k}\mathbb{C}_k^m(\mathbf{q},\mathbf{p};G_d)\right)(w-\beta_1)^{s+1+u} dw
\end{aligned} \tag{3.11}$$

where $\sum_{i=1}^{m} p_i = P_m$.

(b) If $\mathbf{q} := (q_1, \ldots, q_m), \mathbf{p} := (p_1, \ldots, p_m)$ are positive probability distributions and n is even, then we have the following bound for Kullback-Leibler distance

$$D(\mathbf{q} \parallel \mathbf{p}) \geq \frac{m-1}{m-k}D_k^m(\mathbf{q} \parallel \mathbf{p}) + \sum_{u=0}^{s}\frac{(-1)^{u+2}(u+1)}{(\beta_1)^{u+2}} \times \tag{3.12}$$

$$\int_{\beta_1}^{\beta_2} \left(\sum_{i=1}^{m} q_i G_d \left(\frac{p_i}{q_i}, w \right) + \frac{k-1}{m-k} G_d(1,w) - \frac{m-1}{m-k} \mathbb{C}_k^m(\mathbf{q},\mathbf{p};G_d) \right) (w-\beta_1)^u dw$$

$$+ \sum_{v=0}^{n-s-2} \left[\sum_{u=0}^{v} \frac{(-1)^{s+1+v}(\beta_1-\beta_2)^{v-u}}{(\beta_2)^{s+1+v}(s+1+u)(v-u)!} \right] \times$$

$$\int_{\beta_1}^{\beta_2} \left(\sum_{i=1}^{m} q_i G_d \left(\frac{p_i}{q_i}, w \right) + \frac{k-1}{m-k} G_d(1,w) - \frac{m-1}{m-k} \mathbb{C}_k^m(\mathbf{q},\mathbf{p};G_d) \right) (w-\beta_1)^{s+1+u} dw,$$

where

$$D_k^m(\mathbf{q} \parallel \mathbf{p}) = \frac{1}{C_{k-1}^{m-1}} \sum_{1 \le i_1 < \ldots < i_k \le m} \left(\sum_{j=1}^{k} q_{i_j} \right) \ln \left(\frac{\sum_{j=1}^{k} q_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \right).$$

If n is odd, then (3.11) *and* (3.12) *hold in reverse directions.*

PROOF.

(a) Using $\lambda(x) := -\ln x$, in Theorem 3.1, we obtain the desired results.
(b) It is a special case of (a).

□

REMARK 3.6. Using positive probability distributions along with the function $\lambda(x) := -\ln x$ in (3.5), we get the bound

$$\frac{m-1}{m-k} D_k^m(\mathbf{q} \parallel \mathbf{p}) \le D(\mathbf{q} \parallel \mathbf{p}). \tag{3.13}$$

One of the basic laws in information sciences, which is excessively applied in linguistics is Zipf's law [26] named by George Zipf (1932), who discovered the counting problem of each word appearing in the text. Besides the application of this law in linguistics and information science, Zipf's law has a mythical impact in economics, where its distribution is called Pareto's law, which analyze the distribution of the wealthiest members in the community ([8], p. 125). Although in mathematical sense these two laws are same, but they are utilized in a different way ([9, p. 294]).

For $m \in \{1, 2, \ldots\}$, $t \ge 0$ and $w > 0$ the *Zipf-Mandelbrot* law (probability mass function) is stated as

$$\psi(i; m, t, w) = \frac{1}{((i+t)^s H_{m,t,w})}, \quad i = (1, 2, \ldots, m) \tag{3.14}$$

where

$$H_{m,t,w} = \sum_{j=1}^{m} \frac{1}{(j+t)^w}.$$

The probability mass function can be given as in (3.14) and $H_{m,t,w}$ which can also be taken as a generalization of a harmonic number. In the formula, i represents rank of the data, t and w are parameters of the distribution. In the limit as m approaches infinity, this becomes the Hurwitz zeta function $\zeta(w,t)$. For finite m and $t=0$ the Zipf-Mandelbrot law becomes Zipf's law. For infinite m and $t=0$ it becomes a Zeta distribution.

Let $m \in \{1,2,\ldots\}$, $t \geq 0$, $w > 0$, then *Zipf-Mandelbrot entropy* can be given as

$$Z(H,t,w) = \frac{w}{H_{m,t,w}} \sum_{i=1}^{m} \frac{\ln(i+t)}{(i+t)^s} + \ln(H_{m,t,w}). \tag{3.15}$$

Consider

$$q_i = \psi(i;m,t,w) = \frac{1}{((i+t)^w H_{m,t,w})}. \tag{3.16}$$

Application of Zipf-Mandelbrot law can be found in linguistics [17, 20, 26], information sciences and also is often applicable in ecological field studies [17]. Some of the recent study regarding Zipf-Mandelbrot law can be seen in the listed references (see [10, 11, 13, 14]). Now we state our results involving entropy introduced by Mandelbrot Law for positive probability distributions:

THEOREM 3.7. *Let $m,k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$ and $\mathbf{q}$ be as defined in* (3.16) *by Zipf-Mandelbrot law with parameters $t \geq 0$, $w > 0$. For n even, the following holds*

$$\begin{aligned} S(\mathbf{q}) = Z(H,t,w) \leq -\sum_{i=1}^{m} \frac{1}{((i+t)^w H_{m,t,w})} \ln(p_i) \\ -\frac{m-1}{m-k}\left(\frac{1}{C_{k-1}^{m-1}} \sum_{1\leq i_1<\ldots<i_k\leq m}\left(\sum_{j=1}^{k}\frac{1}{((i_j+t)^w H_{m,t,w})}\right)\ln\left(\frac{\sum_{j=1}^{k}\frac{1}{((i_j+t)^w H_{m,t,w})}}{\sum_{j=1}^{k} p_{i_j}}\right)\right) \end{aligned} \tag{3.17}$$

PROOF. Substituting $q_i = \frac{1}{((i+t)^w H_{m,t,w})}$ in Remark 3.4, we get the desired result. It is interesting to see that $\sum_{i=1}^{m} q_i = 1$. Moreover using above q_i in Shannon entropy (3.6), we get Mandelbrot entropy $Z(H,t,w)$ (3.15). ∎

The next result establishes the relationship of Mandelbrot entropy (3.15) with Kullback-Leibler distance (3.10).

REMARK 3.8. Let $m,k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $t_1,t_2 \in [0,\infty)$, $w_1,w_2 > 0$, $H_{m,t_1,w_1} = \frac{1}{(i+t_1)^{w_1}}$ and $H_{m,t_2,w_2} = \frac{1}{(i+t_2)^{w_2}}$. Then using $q_i = \dfrac{1}{(i+t_1)^{w_1} H_{m,t_1,w_1}}$ and $p_i = \dfrac{1}{(i+t_2)^{w_2} H_{m,t_2,w_2}}$ in Remark 3.6, we get

$$D(\mathbf{q}\parallel\mathbf{p}) = Z(H,t_1,w_1) \leq \frac{w_2}{H_{m,t_1,w_1}}\sum_{i=1}^{m}\frac{\ln((i+t_2))}{(i+t_1)^{w_1}} + \ln(H_{m,t_2,w_2}) - \frac{m-1}{m-k}D_k^m(\mathbf{q}\parallel\mathbf{p}) = \tag{3.18}$$

$$-\frac{m-1}{m-k}\left(\frac{1}{C_{k-1}^{m-1}}\sum_{1\leq i_1<\ldots<i_k\leq m}\left(\sum_{j=1}^{k}\frac{1}{(i_j+t_1)^{w_1}H_{m,t_1,w_1}}\right)\ln\left(\frac{\sum_{j=1}^{k}\frac{1}{(i_j+t_1)^{w_1}H_{m,t_1,w_1}}}{\sum_{j=1}^{k}\frac{1}{(i_j+t_2)^{w_2}H_{m,t_2,w_2}}}\right)\right)$$

4. Monotonicity of Jensen Type Linear Functionals

Now we present related results for the class of n-convex functions at a point introduced in [24] which is more general class of n-convex functions.

Definition 4.1. *Let $I \subseteq R$, $c \in I^o$ and $n \in \mathbb{N}$. A function $\lambda : I \to \mathbb{R}$ is called $(n+1)$-convex at point c if there exists a constant X_c so that the function*

$$\Upsilon(x) = \lambda(x) - \frac{X_c}{n!}x^n \tag{4.1}$$

is n-concave on $I\cap(-\infty, c]$ and n-convex on $I\cap[c,\infty)$. A function λ is called $(n+1)$-concave at point c if the function $-\lambda$ is $(n+1)$-convex at point c.

A function is $(n+1)$-convex on an interval if and only if it is $(n+1)$-convex at every point of the interval (see [24]). Pečarić, Praljak and Witkowski in [24] studied the conditions which are necessary and sufficient on two linear functionals $\Omega_d : C([\beta_1, c]) \to \mathbb{R}$ and $\Lambda_d : C([c, \beta_2]) \to \mathbb{R}$, for $d = 1, 2, 3, 4$, so that the inequality $\Omega_d(\lambda) \leq \Lambda_d(\lambda)$ is valid for every function λ which is $(n+1)$-convex at point c. For the particular linear functionals obtained from the inequalities in the previous section, we shall introduce inequalities of such type in this section section . Suppose σ_i represents the monomials $\sigma_i(x) = x^i$, $i \in \mathbb{N}_0$. For the remaining part of the present section, $\Omega_d(\lambda)$ and $\Lambda_d(\lambda)$ will represent the linear functionals which we get by taking the difference of the left hand size and right hand side of the inequality (2.2), applied to the intervals $[\beta_1, c]$ and $[c, \beta_2]$ respectively, i.e., for $\mathbf{x} \in [\beta_1, c]^m$, $\mathbf{q} \in \mathbb{R}^m$, $\mathbf{y} \in [c, \beta_2]^{\bar{m}}$ and $\bar{\mathbf{q}} \in \mathbb{R}^{\bar{m}}$, also for $d = 1, 2, 3, 4$ let

$$\Omega_d(\lambda) := \frac{m-k}{m-1}P_1^m(\mathbf{q};\lambda) + \frac{k-1}{m-1}P_m^m(\mathbf{q};\lambda) - P_k^m(\mathbf{q};\lambda) - \sum_{u=0}^{s}\frac{\lambda^{(u+2)}(\beta_1)}{u!}\times \tag{4.2}$$

$$\int_{\beta_1}^{c}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_d(x,w)) + \frac{k-1}{m-1}P_m^m(\mathbf{q};G_d(x,w)) - P_k^m(\mathbf{q};G_d(x,w))\right)(w-\beta_1)^u dw$$

$$-\sum_{v=0}^{n-s-2}\left[\sum_{u=0}^{v}\frac{\lambda^{(s+1+u)}(c)(\beta_1-c)^{v-u}}{(s+1+u)!(v-u)!}\right]\times$$

$$\int_{\beta_1}^{c}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_d(x,w))+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_d(x,w))-P_k^m(\mathbf{q};G_d(x,w))\right)$$
$$\times(w-\beta_1)^{s+1+u}dw,$$

$$(4.3)\quad \Lambda_d(\lambda):=\frac{\bar{m}-\bar{k}}{\bar{m}-1}P_1^{\bar{m}}(\bar{\mathbf{q}};\lambda)+\frac{\bar{k}-1}{\bar{m}-1}P_{\bar{m}}^{\bar{m}}(\bar{\mathbf{q}};\lambda)-P_{\bar{k}}^{\bar{m}}(\bar{\mathbf{q}};\lambda)-\sum_{u=0}^{s}\frac{\lambda^{(u+2)}(c)}{u!}\times$$

$$\int_{c}^{\beta_2}\left(\frac{\bar{m}-\bar{k}}{\bar{m}-1}P_1^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))+\frac{\bar{k}-1}{\bar{m}-1}P_{\bar{m}}^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))-P_{\bar{k}}^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))\right)(w-c)^u dw$$

$$-\sum_{v=0}^{n-s-2}\left[\sum_{u=0}^{v}\frac{\lambda^{(s+1+u)}(\beta_2)(c-\beta_2)^{v-u}}{(s+1+u)!(v-u)!}\right]\times$$

$$\int_{c}^{\beta_2}\left(\frac{\bar{m}-\bar{k}}{\bar{m}-1}P_1^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))+\frac{\bar{k}-1}{\bar{m}-1}P_{\bar{m}}^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))-P_{\bar{k}}^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))\right)$$
$$\times(w-c)^{s+1+u}dw.$$

It is significant to observe that by giving the new linear functionals $\Omega_d(\lambda)$ and $\Lambda_d(\lambda)$, for $(d=1,2,3,4)$ identity (2.6) applied to the respective intervals $[\beta_1,c]$ and $[c,\beta_2]$ takes the shape:

$$(4.4)\quad \Omega_d(\lambda)=\int_{\beta_1}^{c}\int_{\beta_1}^{c}\left(\frac{m-k}{m-1}P_1^m(\mathbf{q};G_d(x,w))+\frac{k-1}{m-1}P_m^m(\mathbf{q};G_d(x,w))\right.$$
$$\left.-P_k^m(\mathbf{q};G_d(x,w))\right)AG_{(n-2)}(w,t)\lambda^n(t)dwdt,$$

$$(4.5)\quad \Lambda_d(\lambda)=\int_{c}^{\beta_2}\int_{c}^{\beta_2}\left(\frac{\bar{m}-\bar{k}}{\bar{m}-1}P_1^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))+\frac{\bar{k}-1}{\bar{m}-1}P_{\bar{m}}^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))\right.$$
$$\left.-P_{\bar{k}}^{\bar{m}}(\bar{\mathbf{q}};G_d(y,w))\right)AG_{(n-2)}(w,t)\lambda^n(t)dwdt.$$

For the inequalities involving $(n+1)$-convex function at a point, we now state the following theorem:

THEOREM 4.1. *Suppose* $\mathbf{x} \in [\beta_1, c]^m$, $\mathbf{q} \in R^m$, $\mathbf{y} \in [c, \beta_2]^{\bar{m}}$ *and* $\bar{\mathbf{q}} \in R^{\bar{m}}$ *so that*

$$\left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d(x,w)) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d(x,w)) - P_k^m(\mathbf{q}; G_d(x,w))\right) \geq 0, \tag{4.6}$$

$$\left(\frac{\bar{m}-\bar{k}}{\bar{m}-1} P_1^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w)) + \frac{\bar{k}-1}{\bar{m}-1} P_{\bar{m}}^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w)) - P_{\bar{k}}^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w))\right) \geq 0 \tag{4.7}$$

provided that $(n = even, s = odd)$ *or* $(n = odd, s = even)$, *and*

$$\begin{aligned} &\int_{\beta_1}^{c}\int_{\beta_1}^{c} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d(x,w)) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d(x,w)) - P_k^m(\mathbf{q}; G_d(x,w))\right) \\ &\qquad\qquad \times AG_{(n-2)}(w,t)\,dw\,dt \\ &= \int_{c}^{\beta_2}\int_{c}^{\beta_2} \left(\frac{\bar{m}-\bar{k}}{\bar{m}-1} P_1^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w)) + \frac{\bar{k}-1}{\bar{m}-1} P_{\bar{m}}^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w)) - P_{\bar{k}}^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w))\right) \\ &\qquad\qquad \times AG_{(n-2)}(w,t)\,dw\,dt, \end{aligned} \tag{4.8}$$

where $\Omega_d(\lambda)$, $\Lambda_d(\lambda)$, *for* $d = 1,2,3,4$, *be the linear functionals given by* (4.2) *and* (4.3). *If* $\lambda_d : [\beta_1, \beta_2] \to \mathbb{R}$ *is* $(n+1)$*-convex at point* c, *then the following monotonicity is obtained*

$$\Omega_d(\lambda) \leq \Lambda_d(\lambda). \tag{4.9}$$

By reversing the inequalities in (4.6) *and* (4.7), (4.9) *is established with the reversed sign of inequality.*

PROOF. With the help of Definition 4.1, we construct function $\Upsilon(x) = \lambda(x) - \frac{X_c}{n!}\sigma_n$ so that the function Υ is n-concave on $[\beta_1, c]$ and n-convex on $[c, \beta_2]$. Applying Theorem 2.1 to Υ on the interval $[\beta_1, c]$, we get

$$0 \geq \Omega_d(\Upsilon) = \Omega_d(\lambda) - \frac{X_c}{n!}\Omega_d(\sigma_n). \tag{4.10}$$

Similarly, by applying Theorem 2.1 to Υ on the interval $[c, \beta_2]$, we obtain

$$0 \leq \Lambda_d(\Upsilon) = \Lambda_d(\lambda) - \frac{X_c}{n!}\Lambda_d(\sigma^n). \tag{4.11}$$

Also, by applying the identities (4.4) and (4.5) to the function σ^n, for $d = 1,2,3,4$, we get

$$\Omega_d(\sigma^n) = n! \int_{\beta_1}^{c}\int_{\beta_1}^{c} \left(\frac{m-k}{m-1} P_1^m(\mathbf{q}; G_d(x,w)) + \frac{k-1}{m-1} P_m^m(\mathbf{q}; G_d(x,w))\right. \tag{4.12}$$

$$- P_k^m(\mathbf{q}; G_d(x,w))\Big)AG_{(n-2)}(w,t)dwdt,$$

$$(4.13)\quad \Lambda_d(\sigma^n) = n!\int_c^{\beta_2}\int_c^{\beta_2}\Big(\frac{\bar{m}-\bar{k}}{\bar{m}-1}P_1^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w)) + \frac{\bar{k}-1}{\bar{m}-1}P_{\bar{m}}^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w)) - P_{\bar{k}}^{\bar{m}}(\bar{\mathbf{q}}; G_d(y,w))\Big)AG_{(n-2)}(w,t)dwdt.$$

Hence the assumption (4.8) is equivalent to

$$\Omega_d(\sigma^n) = \Lambda_d(\sigma^n).$$

Therefore from (4.10) and (4.11), we get the required result. □

REMARK 4.2. In the proof of Theorem 4.1, for $(d = 1,2,3,4)$ we have shown that

$$\Omega_d(\lambda) \le \frac{X_c}{n!}\Omega_d(\sigma^n) = \frac{X_c}{n!}\Lambda_d(\sigma^n) \le \Lambda_d(\lambda).$$

It is also significant to observe that the inequality (4.9) remains valid on replacing assumption (4.8) with a weaker assumption that is $X_c\big(\Lambda_d(\sigma^n) - \Omega_d(\sigma^n)\big) \ge 0$.

We give the following remark to conclude our paper.

REMARK 4.3. We may form non-trivial examples for exponentially convex functions and n-exponentially for positive linear functional for n-convex function coming form the difference of the left hand side and right hand side of (2.2), with the help of n-exponentially techniques given by Pečarić et al. in [12] and [23] (see also [5], [4] and [3]). Most importantly, it is known that Jensen inequality has an elegant connection with its applications in information theory. But we are also able to find applications of our generalized Popoviciu's inequality in information theory as we define new divergence functional and can employ it to give new combinatorial bounds for different entropies, specially the famous Shannon, Kullback and Mandelbrot etc.

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