

SOME WEIGHTED TRAPEZOIDAL TYPE INEQUALITIES VIA h -PREINVELOCITY

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ABSTRACT. In this paper, a new identity is given, some weighted trapezoidal type inequalities via h -preinvexity are established, and several known results are derived.

1. INTRODUCTION

Let f be a convex function on the finite interval $[a, b]$, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2},$$

The inequality (1.1) is known in the literature as Hermite-Hadamard inequality.

The above inequality has never ceased to intrigue researchers, several variants, extensions, generalizations and improvements have been established.

In [4], Dragomir and Agarwal established the following Hermite-Hadamard type inequalities

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|),$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{(p+1)^{\frac{1}{p}}}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

In [6], Kirmaci et al. gave the following result connected with Hermite-Hadamard type inequalities

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\frac{2^s s + 1}{2^s (s+1)(s+2)}\right)^{\frac{1}{q}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.$$

2020 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.

Key words and phrases. Hermite-Hadamard inequality, Hölder inequality, h -preinvex functions.

In [12], Pearce and Pečarić showed the following inequality

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

In [5] Hua et al. gave the following weighted trapezoidal inequalities for s -convex functions

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b f(x) w(x) dx \right| \\ & \leq \frac{(b-a)^2}{4} \|w\|_{\infty} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left(\frac{1}{2^s(s+1)} \right)^{\frac{1}{q}} \left\{ [(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}} \right. \\ & \quad \left. + [|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q]^{\frac{1}{q}} \right\}, \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b f(x) w(x) dx \right| \\ & \leq \frac{b-a}{2} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \int_0^1 \left[\int_{\varphi(t)}^{\Psi(t)} w(x) dx \right] dt. \end{aligned}$$

Motivated by the above and others existing in the literature, in this study we start by establishing a new equality as a partial result then we derive some new inequalities of weighted Hermite-Hadamard type for h -preinvex functions. Several known results are also derived.

2. PRELIMINARIES

In this section, we recall some definitions

DEFINITION 2.1 ([8]). *A set $I \subseteq \mathbb{R}^n$ is said to be convex if for any $x, y \in I$, and $\forall t \in [0, 1]$, we have*

$$tx + (1-t)y \in I.$$

DEFINITION 2.2 ([13]). *A function $f : I \rightarrow \mathbb{R}$ is said to be convex on I where I is an interval of \mathbb{R} , if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

DEFINITION 2.3 ([3]). *A nonnegative function $f : I \rightarrow \mathbb{R}$ is said to be P -convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

DEFINITION 2.4 ([2]). A nonnegative function $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

DEFINITION 2.5 ([16]). Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative function, where $(0, 1) \subseteq J$. A nonnegative function $f : I \rightarrow \mathbb{R}$ is said to be h -convex function on I , if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

DEFINITION 2.6 ([17]). A set $K \subset \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \rightarrow \mathbb{R}^n$, if

$$x + t\eta(y, x) \in K$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

DEFINITION 2.7 ([17]). A function $f : K \subset (0, +\infty) \rightarrow \mathbb{R}$ is said to be preinvex with respect to η , if

$$f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

DEFINITION 2.8 ([11]). A nonnegative function $f : K \rightarrow \mathbb{R}$ is said to be P -preinvex function with respect to η , if

$$f(x + t\eta(y, x)) \leq f(x) + f(y)$$

holds for all $x, y \in K$ and all $t \in [0, 1]$.

DEFINITION 2.9 ([7]). A nonnegative function $f : K \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be s -preinvex in the second sense with respect to η for some fixed $s \in (0, 1]$, if

$$f(x + t\eta(y, x)) \leq (1-t)^s f(x) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

DEFINITION 2.10 ([9]). Let $h : [0, 1] \rightarrow \mathbb{R}$ be a nonnegative function $h \neq 0$. A nonnegative function f on the invex set K is said to be h -preinvex function with respect to η , if

$$f(x + t\eta(y, x)) \leq h(1-t)f(x) + h(t)f(y)$$

holds for all $x, y \in K$ and $t \in (0, 1)$.

3. MAIN RESULTS

LEMMA 3.1. *Let $f : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be differentiable on $(a, a + \eta(b, a))$ with $\eta(b, a) > 0$, and let $w : [a, a + \eta(b, a)] \rightarrow [0, +\infty)$ be continuous function and symmetric to $\frac{2a + \eta(b, a)}{2}$. If $f' \in L([a, a + \eta(b, a)])$, then one has the following equality*

$$(3.1) \quad \begin{aligned} & \frac{f(a) + f(a + \eta(b, a))}{2} \int_a^{a + \eta(b, a)} w(x) dx - \int_a^{a + \eta(b, a)} w(x) f(x) dx \\ &= \frac{\eta(b, a)}{2} \int_0^1 \left(\int_{a + (1-t)\eta(b, a)}^{a + t\eta(b, a)} w(x) dx \right) f'(a + t\eta(b, a)) dt. \end{aligned}$$

PROOF. Integrating by parts the right side of (3.1), using a change of variable and the symmetry of w , we obtain

$$\begin{aligned} & \frac{\eta(b, a)}{2} \int_0^1 \left(\int_{a + (1-t)\eta(b, a)}^{a + t\eta(b, a)} w(x) dx \right) f'(a + t\eta(b, a)) dt \\ &= \frac{1}{2} \left(\int_{a + (1-t)\eta(b, a)}^{a + t\eta(b, a)} w(x) dx \right) f(a + t\eta(b, a)) \Big|_{t=0}^{t=1} \\ & \quad - \frac{\eta(b, a)}{2} \int_0^1 (w(a + t\eta(b, a)) + w(a + (1-t)\eta(b, a))) f(a + t\eta(b, a)) dt \\ &= \frac{1}{2} \left(\int_a^{a + \eta(b, a)} w(x) dx \right) f(a + \eta(b, a)) + \frac{1}{2} \left(\int_a^{a + \eta(b, a)} w(x) dx \right) f(a) \\ & \quad - \frac{\eta(b, a)}{2} \int_0^1 (w(a + t\eta(b, a)) + w(a + (1-t)\eta(b, a))) f(a + t\eta(b, a)) dt \\ &= \frac{1}{2} \left(\int_a^{a + \eta(b, a)} w(x) dx \right) (f(a + \eta(b, a)) + f(a)) \\ & \quad - \frac{1}{2} \int_a^{a + \eta(b, a)} (w(x) + w(2a + \eta(b, a) - x)) f(x) dx \end{aligned}$$

$$= \frac{f(a+\eta(b,a))+f(a)}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx.$$

The proof is completed. \square

In what follows we assume that $h : [0, 1] \rightarrow \mathbb{R}$ be a nonnegative function and $h \neq 0$, $\eta(b, a) > 0$, and $K = [a, a + \eta(b, a)] \subset [0, +\infty)$.

THEOREM 3.2. *Let $f : K \rightarrow \mathbb{R}$ be differentiable on K° with $f' \in L(K)$ where $a, b \in K^\circ$, and let $w : K \rightarrow [0, +\infty)$ be continuous and symmetric to $a + \frac{1}{2}\eta(b, a)$. If $|f'|$ is h -preinvex, then one has the following inequality*

$$\begin{aligned} & \left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ & \leq \frac{(\eta(b,a))^2}{2} \|w\|_\infty \left((|f'(a)| + |f'(a + \eta(b,a))|) \int_{\frac{1}{2}}^1 (2t-1) h(t) dt \right. \\ & \quad \left. + 2 \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| \int_0^{\frac{1}{2}} (1-2t) h(t) dt \right). \end{aligned}$$

PROOF. From Lemma 3.1, properties of modulus, and h -preinvexity of $|f'|$, we have

$$\begin{aligned} & \left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ & \leq \frac{\eta(b,a)}{2} \int_0^1 \left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right| |f'(a+t\eta(b,a))| dt \\ & = \frac{\eta(b,a)}{2} \left(\int_0^{\frac{1}{2}} \left(\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)} w(x) dx \right) |f'(a+t\eta(b,a))| dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right) |f'(a+t\eta(b,a))| dt \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\eta(b,a))^2}{2} \left(\int_0^{\frac{1}{2}} \|w(x)\|_{[a+t\eta(b,a), a+(1-t)\eta(b,a)], \infty} (1-2t) |f'(a+t\eta(b,a))| dt \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \|w(x)\|_{[a+(1-t)\eta(b,a), a+t\eta(b,a)], \infty} (2t-1) |f'(a+t\eta(b,a))| dt \right) \\
&\leq \frac{(\eta(b,a))^2}{2} \|w\|_{\infty} \left(\int_0^{\frac{1}{2}} (1-2t) |f'(a+t\eta(b,a))| dt \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 (2t-1) |f'(a+t\eta(b,a))| dt \right) \\
&\leq \frac{(\eta(b,a))^2}{2} \|w\|_{\infty} \left(\int_0^{\frac{1}{2}} (1-2t) \left(h(1-t) |f'(a)| + h(t) \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| \right) dt \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 (2t-1) \left(h(1-t) \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| + h(t) |f'(a+\eta(b,a))| \right) dt \right) \\
&= \frac{(\eta(b,a))^2}{2} \|w\|_{\infty} \left((|f'(a)| + |f'(a+\eta(b,a))|) \int_{\frac{1}{2}}^1 (2t-1) h(t) dt \right. \\
&\quad \left. + 2 \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| \int_0^{\frac{1}{2}} (1-2t) h(t) dt \right).
\end{aligned}$$

□

COROLLARY 3.3. *In Theorem 3.2, if we choose $w(x) = \frac{1}{\eta(b,a)}$, we obtain*

$$\begin{aligned}
&\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \\
&\leq \frac{\eta(b,a)}{2} \left((|f'(a)| + |f'(a+\eta(b,a))|) \int_{\frac{1}{2}}^1 (2t-1) h(t) dt \right.
\end{aligned}$$

$$+ 2 \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| \int_0^{\frac{1}{2}} (1-2t) h(t) dt \Bigg).$$

COROLLARY 3.4. In Theorem 3.2, taking $\eta(b, a) = b - a$, we get

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \leq \frac{(b-a)^2}{2} \|w\|_\infty \\ \times \left((|f'(a)| + |f'(b)|) \int_{\frac{1}{2}}^1 (2t-1) h(t) dt + 2 \left| f' \left(\frac{a+b}{2} \right) \right| \int_0^{\frac{1}{2}} (1-2t) h(t) dt \right).$$

Moreover, if we choose $w(x) = \frac{1}{b-a}$, we obtain

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \\ \times \left((|f'(a)| + |f'(b)|) \int_{\frac{1}{2}}^1 (2t-1) h(t) dt + 2 \left| f' \left(\frac{a+b}{2} \right) \right| \int_0^{\frac{1}{2}} (1-2t) h(t) dt \right).$$

COROLLARY 3.5. In Theorem 3.2, if we assume that $|f'|$ is P -preinvex function we obtain

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ \leq \frac{(\eta(b,a))^2}{8} \|w\|_\infty \left(|f'(a)| + 2 \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| + |f'(a+\eta(b,a))| \right).$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \\ \leq \frac{\eta(b,a)}{8} \left(|f'(a)| + 2 \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| + |f'(a+\eta(b,a))| \right).$$

COROLLARY 3.6. In Corollary 3.5, if we take $\eta(b, a) = b - a$, we get

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \\ \leq \frac{(b-a)^2}{8} \|w\|_\infty \left(|f'(a)| + 2 \left| f' \left(\frac{a+b}{2} \right) \right| + |f'(b)| \right).$$

Moreover, if we take $w(x) = \frac{1}{b-a}$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} (|f'(a)| + 2|f'(\frac{a+b}{2})| + |f'(b)|).$$

COROLLARY 3.7. In Theorem 3.2, if we assume that $|f'|$ is preinvex function

$$\begin{aligned} & \left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ & \leq \frac{(\eta(b,a))^2}{48} \|w\|_\infty \left(5|f'(a)| + 2 \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| + 5|f'(a+\eta(b,a))| \right). \end{aligned}$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$(3.2) \quad \left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \leq \frac{\eta(b,a)}{48} \left(5|f'(a)| + 2 \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| + 5|f'(a+\eta(b,a))| \right).$$

REMARK 3.8. In inequality (3.2), using the fact that $2 \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| \leq |f'(a)| + |f'(b)|$ and $|f'(a+\eta(b,a))| \leq |f'(b)|$, we obtain Theorem 2.1 from [1].

COROLLARY 3.9. In Corollary 3.7, if we take $\eta(b,a) = b-a$, we get

$$(3.3) \quad \left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \leq \frac{(b-a)^2}{48} \|w\|_\infty (5|f'(a)| + 2|f'(\frac{a+b}{2})| + 5|f'(b)|).$$

Moreover, if we take $w(x) = \frac{1}{b-a}$

$$(3.4) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{48} (5|f'(a)| + 2|f'(\frac{a+b}{2})| + 5|f'(b)|).$$

REMARK 3.10. In inequality (3.3), using the convexity of $|f'|$, we obtain inequality (2) of Corollary 3.1.1 from [5]. Also Corollary 8 from [15].

REMARK 3.11. In inequality (3.4), using the convexity of $|f'|$, we obtain Theorem 2.2 from [4].

COROLLARY 3.12. *In Theorem 3.2, if we assume that $|f'|$ is s -preinvex function*

$$\begin{aligned} & \left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ & \leq \frac{(\eta(b,a))^2}{2(s+1)(s+2)} \|w\|_\infty \left(\frac{s2^{1+s}+1}{2^{1+s}} |f'(a)| + \frac{s2^{1+s}+1}{2^{1+s}} |f'(a+\eta(b,a))| \right. \\ & \quad \left. + \frac{1}{2^s} \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| \right). \end{aligned}$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\begin{aligned} (3.5) \quad & \left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \\ & \leq \frac{\eta(b,a)}{2(s+1)(s+2)} \left(\frac{s2^{1+s}+1}{2^{1+s}} |f'(a)| + \frac{1}{2^s} \left| f' \left(\frac{2a+\eta(b,a)}{2} \right) \right| + \frac{s2^{1+s}+1}{2^{1+s}} |f'(a+\eta(b,a))| \right). \end{aligned}$$

REMARK 3.13. In inequality (3.5), using the preinvexity of $|f'|$, we obtain the correct result of Theorem 2 from [14].

COROLLARY 3.14. *In Corollary 3.12, if we take $\eta(b,a) = b - a$, we get*

$$\begin{aligned} (3.6) \quad & \left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2(s+1)(s+2)} \|w\|_\infty \left(\frac{s2^{1+s}+1}{2^{1+s}} |f'(a)| + \frac{1}{2^s} |f' \left(\frac{a+b}{2} \right)| + \frac{s2^{1+s}+1}{2^{1+s}} |f'(b)| \right). \end{aligned}$$

Moreover, if we take $w(x) = \frac{1}{b-a}$

$$\begin{aligned} (3.7) \quad & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2(s+1)(s+2)} \left(\frac{s2^{1+s}+1}{2^{1+s}} |f'(a)| + \frac{1}{2^s} |f' \left(\frac{a+b}{2} \right)| + \frac{s2^{1+s}+1}{2^{1+s}} |f'(b)| \right). \end{aligned}$$

REMARK 3.15. In inequality (3.6), using the convexity of $|f'|$, we obtain inequality (1) of Corollary 3.1.1 from [5].

REMARK 3.16. In inequality (3.7), using the convexity of $|f'|$, we obtain inequality (1) of Corollary 3.1.2 from [5], Also Theorem 2 from [10].

THEOREM 3.17. *Let $f : K \rightarrow \mathbb{R}$ be differentiable on K° with $f' \in L(K)$, and let $w : K \rightarrow [0, +\infty)$ be continuous and symmetric to $a + \frac{1}{2}\eta(b,a)$. If*

$|f'|^q$ is h -preinvex, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then one has the following inequality

$$\begin{aligned} & \left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ & \leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right)^p dt \right)^{\frac{1}{p}} \\ & \quad \times (|f'(a)|^q + |f'(a+\eta(b,a))|^q)^{\frac{1}{q}} \left(\int_0^1 h(t) dt \right)^{\frac{1}{q}}. \end{aligned}$$

PROOF. From Lemma 3.1, properties of modulus, Hölder inequality, and h -preinvexity of $|f'|^q$, we have

$$\begin{aligned} & \left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ & \leq \frac{\eta(b,a)}{2} \int_0^1 \left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right| |f'(a+t\eta(b,a))| dt \\ & \leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right)^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 (h(1-t)|f'(a)|^q + h(t)|f'(a+\eta(b,a))|^q) dt \right)^{\frac{1}{q}} \\ & = \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right)^p dt \right)^{\frac{1}{p}} \\ & \quad \times (|f'(a)|^q + |f'(a+\eta(b,a))|^q)^{\frac{1}{q}} \left(\int_0^1 h(t) dt \right)^{\frac{1}{q}}. \end{aligned}$$

The proof is achieved. \square

COROLLARY 3.18. In Theorem 3.17, if we choose $w(x) = \frac{1}{\eta(b,a)}$, we obtain

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \\ \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}} (|f'(a)|^q + |f'(a+\eta(b,a))|^q)^{\frac{1}{q}} \left(\int_0^1 h(t) dt \right)^{\frac{1}{q}}.$$

COROLLARY 3.19. In Theorem 3.17, taking $\eta(b,a) = b-a$, we get

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \\ \leq \frac{b-a}{2} \left(\int_0^1 \left(\int_{at+(1-t)b}^{a(1-t)+tb} w(x) dx \right)^p dt \right)^{\frac{1}{p}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\int_0^1 h(t) dt \right)^{\frac{1}{q}}.$$

Moreover, if we choose $w(x) = \frac{1}{b-a}$, we obtain

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\int_0^1 h(t) dt \right)^{\frac{1}{q}}.$$

COROLLARY 3.20. In Theorem 3.17, if we assume that $|f'|^q$ is P -preinvex function

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ \leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right)^p dt \right)^{\frac{1}{p}} (|f'(a)|^q + |f'(a+\eta(b,a))|^q)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \\ \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}} (|f'(a)|^q + |f'(a+\eta(b,a))|^q)^{\frac{1}{q}}.$$

COROLLARY 3.21. In Corollary 3.20, if we take $\eta(b, a) = b - a$, we get

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \\ & \leq \frac{b-a}{2} \left(\int_0^1 \left(\int_{at+(1-t)b}^{a(1-t)+tb} w(x) dx \right)^p dt \right)^{\frac{1}{p}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}. \end{aligned}$$

Moreover, if we take $w(x) = \frac{1}{b-a}$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.$$

COROLLARY 3.22. In Theorem 3.17, if we assume that $|f'|^q$ is preinvex function

$$\begin{aligned} & \left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ & \leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right)^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(a+\eta(b,a))|^q}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$, we get Theorem 2.2 from [1].

COROLLARY 3.23. In Corollary 3.22, if we take $\eta(b, a) = b - a$, we get

$$(3.8) \quad \begin{aligned} & \left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \\ & \leq \frac{b-a}{2} \left(\int_0^1 \left(\int_{at+(1-t)b}^{a(1-t)+tb} w(x) dx \right)^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

Moreover, if we take $w(x) = \frac{1}{b-a}$, we obtain Theorem 2.3 from [4].

REMARK 3.24. In inequality (3.9), using the fact that $w(x) \leq \|w\|_\infty$, we obtain Corollary 13 from [15].

COROLLARY 3.25. In Theorem 3.17, if we assume that $|f'|^q$ is s -preinvex function

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ \leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right)^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(a+\eta(b,a))|^q}{s+1} \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$(3.9) \quad \left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^q + |f'(a+\eta(b,a))|^q}{s+1} \right)^{\frac{1}{q}}.$$

REMARK 3.26. In inequality (3.9), using the fact that $|f'(a+\eta(b,a))| \leq |f'(b)|$, we obtain Theorem 4 from [14].

COROLLARY 3.27. In Corollary 3.25, if we take $\eta(b,a) = b-a$, we get

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \\ \leq \frac{b-a}{2} \left(\int_0^1 \left(\int_{at+(1-t)b}^{a(1-t)+tb} w(x) dx \right)^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{b-a}$, we obtain Theorem 4 from [10].

THEOREM 3.28. Under the assumptions of Theorem 3.17, one has

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \\ \leq \frac{(\eta(b,a))^2}{2^{2-\frac{1}{q}}} \|w\|_{\infty} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\int_0^1 |2t-1| h(t) dt \right)^{\frac{1}{q}}.$$

PROOF. Using Lemma 3.1, properties of modulus, Power mean inequality, and h -preinvexity of $|f'|^q$, we have

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right|$$

$$\begin{aligned}
&\leq \frac{\eta(b,a)}{2} \int_0^1 \left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right| |f'(a+t\eta(b,a))| dt \\
&\leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right| dt \right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x) dx \right| |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{(\eta(b,a))^2}{2} \|w\|_\infty \left(\int_0^1 |2t-1| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |2t-1| |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{(\eta(b,a))^2}{2} \|w\|_\infty \left(\int_0^1 |2t-1| dt \right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 |2t-1| [h(1-t)|f'(a)|^q + h(t)|f'(b)|^q] dt \right)^{\frac{1}{q}} \\
&\leq \frac{(\eta(b,a))^2}{2^{2-\frac{1}{q}}} \|w\|_\infty (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\int_0^1 |2t-1| h(t) dt \right)^{\frac{1}{q}}.
\end{aligned}$$

□

COROLLARY 3.29. In Theorem 3.28, if we choose $w(x) = \frac{1}{\eta(b,a)}$, we obtain

$$\begin{aligned}
&\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \\
&\leq \frac{\eta(b,a)}{2^{2-\frac{1}{q}}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\int_0^1 |2t-1| h(t) dt \right)^{\frac{1}{q}}.
\end{aligned}$$

COROLLARY 3.30. In Theorem 3.28, taking $\eta(b,a) = b-a$, we get

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \\
&\leq \frac{(b-a)^2}{2^{2-\frac{1}{q}}} \|w\|_\infty (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\int_0^1 |2t-1| h(t) dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Moreover, if we choose $w(x) = \frac{1}{b-a}$, we obtain

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{\frac{1}{q}-1}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\int_0^1 |2t-1| h(t) dt \right)^{\frac{1}{q}}.$$

COROLLARY 3.31. In Theorem 3.28, if we assume that $|f'|^q$ is P -preinvex function

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \leq \frac{(\eta(b,a))^2}{4} \|w\|_{\infty} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \leq \frac{\eta(b,a)}{4} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.$$

COROLLARY 3.32. In Corollary 3.31, if we take $\eta(b,a) = b-a$, we get

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \leq \frac{(b-a)^2}{4} \|w\|_{\infty} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{b-a}$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.$$

COROLLARY 3.33. In Theorem 3.28, if we assume that $|f'|^q$ is preinvex function

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \leq \frac{(\eta(b,a))^2}{4} \|w\|_{\infty} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right| \leq \frac{\eta(b,a)}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

COROLLARY 3.34. In Corollary 3.33, if we take $\eta(b, a) = b - a$, we get

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \leq \frac{(b-a)^2}{4} \|w\|_\infty \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{b-a}$, we obtain Theorem 1 from [12].

COROLLARY 3.35. In Theorem 3.28, if we assume that $|f'|^q$ is s -preinvex function

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x) dx - \int_a^{a+\eta(b,a)} w(x) f(x) dx \right| \leq \frac{(\eta(b,a))^2}{2^{2-\frac{1}{q}}} \|w\|_\infty (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\frac{1+s2^s}{(1+s)(2+s)2^s} \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$, we obtain the correct result of Theorem 7 from [14].

COROLLARY 3.36. In Corollary 3.35, if we take $\eta(b, a) = b - a$, we get

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x) dx - \int_a^b w(x) f(x) dx \right| \leq \frac{(b-a)^2}{2^{2-\frac{1}{q}}} \|w\|_\infty (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \left(\frac{1+s2^s}{(1+s)(2+s)2^s} \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{b-a}$, we obtain Theorem 1 from [6].

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Neke nejednakosti težinskog trapezoidnog tipa preko h -preinveksnosti

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SAŽETAK. U ovom članku, dan je jedan novi identitet, dobivene su neke nejednakosti težinskog trapezoidnog tipa te su izvedeni neki poznati rezultati.

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Received: 9.5.2020.

Accepted: 9.6.2020.