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ESTIMATES OF THE LOGARITHMIC DERIVATIVE NEAR A SINGULAR POINT AND APPLICATIONS

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Abstract. In this paper, we will give estimates near $z = 0$ for the logarithmic derivative $\left|\frac{f^{(k)}(z)}{f(z)}\right|$ where f is a meromorphic function in a region of the form $D(0, R) = \{z \in \mathbb{C} : 0 < |z| < R\}$. Some applications on the growth of solutions of linear differential equations near a singular point are given.

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [11,17,21]). The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna Theory to annuli have been made by [2, 13, 15, 16, 18]. Recently in [5, 10], Hamouda and Fettouch investigated the growth of solutions of a class of linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0 \tag{1.1}$$

near a singular point where the coefficients $A_j(z)$ $(j = 0, 1, ..., k-1)$ are meromorphic or analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ and for that they gave estimates of the logarithmic derivative $\left|\frac{f^{(k)}(z)}{f(z)}\right|$ for a meromorphic function f in $\overline{\mathbb{C}} \setminus \{z_0\}$, $\left(\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\right)$. A question was asked in [5] as the following: can we get similar estimates near z_0 of $\left|\frac{f^{(k)}(z)}{f(z)}\right|$ where f is a meromorphic function in a region of the form $D_{z_0}(0, R) = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$? Naturally, this allows us to study the solutions of (1.1) with meromorphic coefficients in $D_{z_0}(0, R)$. The same question was asked in [10] for another problem concerning the case when the coefficients of (1.1) are analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, the

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solutions may be non analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$. In this paper, we will answer this question and give some applications. Without loss of generality, we will study the case $z_0 = 0$ and for $z_0 \neq 0$ we may use the change of variable $w = z - z_0$.

Throughout this paper, we will use the following notation:

$$D\left(R\right) = \left\{z \in \mathbb{C} : |z| < R\right\},$$

$$D\left(R_1, R_2\right) = \left\{z \in \mathbb{C} : R_1 < |z| < R_2\right\},$$

$$D\left[R_1, R_2\right] = \left\{z \in \mathbb{C} : R_1 \leq |z| \leq R_2\right\}.$$

Analogously, we can define $D\left[R_1, R_2\right)$, $D\left(R_1, R_2\right]$. We recall the appropriate definitions [5,16,18]. Suppose that $f\left(z\right)$ is meromorphic in $D\left(0, +\infty\right]$. Define the counting function near 0 by

$$N_0\left(r, f\right) = -\int_{\infty}^{r} \frac{n\left(t, f\right) - n\left(\infty, f\right)}{t} dt - n\left(\infty, f\right) \log r, \tag{1.2}$$

where $n\left(t, f\right)$ counts the number of poles of $f\left(z\right)$ in the region

$$\left\{z \in \mathbb{C} : t \leq |z|\right\} \cup \left\{\infty\right\}$$

each pole according to its multiplicity; and the proximity function by

$$m_0\left(r, f\right) = \frac{1}{2\pi} \int_{0}^{2\pi} \ln^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi. \tag{1.3}$$

The characteristic function of f is defined by

$$T_0\left(r, f\right) = m_0\left(r, f\right) + N_0\left(r, f\right). \tag{1.4}$$

For a meromorphic function $f\left(z\right)$ in $D\left(0, R\right)$, we define the counting function near 0 by

$$N_0\left(r, R', f\right) = \int_{r}^{R'} \frac{n\left(t, f\right)}{t} dt, \tag{1.5}$$

where $n\left(t, f\right)$ counts the number of poles of $f\left(z\right)$ in the region

$$\left\{z \in \mathbb{C} : t \leq |z| \leq R'\right\} \quad \left(0 < R' < R\right),$$

each pole according to its multiplicity; and the proximity function near the singular point 0 by

$$m_0\left(r, f\right) = \frac{1}{2\pi} \int_{0}^{2\pi} \ln^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi. \tag{1.6}$$

The characteristic function of f is defined in the usual manner by

$$T_0\left(r, R', f\right) = m_0\left(r, f\right) + N_0\left(r, R', f\right). \tag{1.7}$$

In addition, the order of growth of a meromorphic function $f(z)$ near 0 is defined by

$$\sigma_T(f,0)=\limsup_{r\to 0}\frac{\log^+ T_0(r,R',f)}{-\log r}. \tag{1.8}$$

For an analytic function $f(z)$ in $D(0,R)$, we have also the definition

$$\sigma_M(f,0)=\limsup_{r\to 0}\frac{\log^+\log^+ M_0(r,f)}{-\log r}, \tag{1.9}$$

where $M_0(r,f)=\max\{|f(z)|:|z|=r\}$.

If $f(z)$ is meromorphic in $D(0,R)$ of finite order $0<\sigma_T(f,0)=\sigma<\infty$, then we can define the type of f as the following:

$$\tau_T(f,0)=\limsup_{r\to 0}\, r^{\sigma}T_0(r,R',f).$$

If $f(z)$ is analytic in $D(0,R)$ of finite order $0<\sigma_M(f,0)=\sigma<\infty$, we have also another definition of the type of f as the following:

$$\tau_M(f,0)=\limsup_{r\to 0}\, r^{\sigma}\log^+ M_0(r,f). \tag{1.10}$$

Remark 1.1. The choice of R' in (1.2) does not have any influence in the values $\sigma_T(f,0)$ and $\tau_T(f,0)$. In fact, if we take two values of R', namely $0<R_1'<R_2'<R$, then we have

$$\int_{R_1'}^{R_2'}\frac{n(t,f)}{t}dt=n\log\frac{R_2'}{R_1'},$$

where n designates the number of poles of $f(z)$ in the region

$$\{z\in\mathbb{C}:R_1'\le|z|\le R_2'\}$$

which is bounded. Thus, $T_0(r,R_1',f)=T_0(r,R_2',f)+C$ where C is a real constant. So, we can write briefly $T_0(r,f)$ instead of $T_0(r,R',f)$.

Example 1.2. Consider the function $f(z)=\exp\left\{z^2+\frac{1}{z^2}\right\}$. We have

$$T_0(r,f)=m_0(r,f)=\frac{1}{\pi}\left(r^2+\frac{1}{r^2}\right),$$

then $\sigma_T(f,0)=2,\ \tau_T(f,0)=\frac{1}{\pi}$. Also we have

$$M_0(r,f)=\exp\left\{r^2+\frac{1}{r^2}\right\},$$

then $\sigma_M(f,0)=2,\ \tau_M(f,0)=1$.

In the usual manner of the complex plane case [1, 14], we define the iterated order near 0 as follows:

$$\sigma_{n,T}(f,0)=\limsup_{r\to 0}\frac{\log_n^+ T_0(r,f)}{-\log r}, \tag{1.11}$$

$$\sigma_{n,M}(f,0)=\limsup_{r\to 0}\frac{\log_{n+1}^+ M_0(r,f)}{-\log r}, \tag{1.12}$$

where $\log_1^+ x=\log^+ x=\max\{\log x,0\}$ and $\log_{n+1}^+ x=\log^+\log_n^+ x$ for $n\geq 1$.

Remark 1.3. It is shown in [5] that $\sigma_M(f,0)=\sigma_T(f,0)$; and then for any integer $n\geq 1$ we have $\sigma_{n,T}(f,0)=\sigma_{n,M}(f,0)$. So, we can use the notation $\sigma_n(f,0)$ in the two cases. For $n=2$, $\sigma_2(f,0)$ is called hyper-order.

We recall the following definitions. The linear measure of a set $E\subset(0,\infty)$ is defined as $\int_0^\infty \chi_E(t)\,dt$ and the logarithmic measure of E is defined by $\int_0^\infty \frac{\chi_E(t)}{t}dt$ where $\chi_E(t)$ is the characteristic function of the set E.

The main tool we use throughout this paper is the decomposition lemma of G. Valiron.

Lemma 1.4 ([18, 20] (Valiron's decomposition lemma)). *Let f be meromorphic function in $D(R_1,R_2)$, and set $R_1<R'<R_2$. Then f may be represented as*

$$f(z)=z^m\phi(z)\mu(z)$$

where

a) The poles and zeros of f in $D(R_1,R')$ are precisely the poles and zeros of $\phi(z)$. The poles and zeros of f in $D(R',R_2)$ are precisely the poles and zeros of $\mu(z)$.
b) $\phi(z)$ is meromorphic in $D(R_1,\infty)$ and analytic and nonzero in $D[R',\infty]$.
c) $\phi(z)$ satisfies

$$\left|\frac{\phi'(\xi e^{i\theta})}{\phi(\xi e^{i\theta})}\right|=O\left(\frac{1}{\xi^2}\right),\ \xi\to\infty.$$

d) $\mu(z)$ is meromorphic in $D(R)$ and analytic and nonzero in $D(R')$.

e) $m\in\mathbb{Z}$.

Remark 1.5. Let f be a non-constant meromorphic function in $D(0,R)$ and $f(z)=z^m\phi(z)\mu(z)$ is a Valiron's decomposition. Set $\tilde{\phi}(z)=z^m\phi(z)$. It is easy to see that

$$T_0(r,f)=T_0\left(r,\tilde{\phi}\right)+O(1). \tag{1.13}$$

If f be a non-constant analytic function in $D(0,R)$, then $\tilde{\phi}(z)$ is analytic in $D(0,\infty]$ and by [5] and (1.13), we obtain that $\sigma_{n,T}(f,0)=\sigma_{n,M}(f,0)$ for $n\geq 1$.

Now, we provide estimates near 0 of the logarithmic derivative for a meromorphic function in $D(0,R)$.

THEOREM 1.6. *Let f be meromorphic function in $D(0,R)$ with a singular point at the origin. Let k be a positive integer and $\alpha>1$ be given real constant; then*

(i) there exists a set $E_1^*\subset(0,R')$ $(0<R'<R)$ that has finite logarithmic measure and a constant $C>0$ such that for all $r=|z|$ satisfying $r\in(0,R')\setminus E_1^*$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq C\left[\frac{1}{r}T_0\left(\frac{r}{\alpha},f\right)\log^{\alpha}\left(\frac{1}{r}\right)\log T_0\left(\frac{r}{\alpha},f\right)\right]^k; \tag{1.14}$$

(ii) there exists a set $E_2^*\subset[0,2\pi)$ that has a linear measure zero such that for all $\theta\in[0,2\pi)\setminus E_2^*$ there exists a constant $r_0=r_0(\theta)>0$ such that(1.14) holds for all z satisfying $\arg z\in[0,2\pi)\setminus E_2^*$ *and $r=|z|<r_0$.*

The following two corollaries are consequences of Theorem 1.6 and have independent interest.

COROLLARY 1.7. *Let f be a non-constant meromorphic function in $D(0,R)$ with a singular point at the origin of finite order $\sigma(f,0)=\sigma<\infty$; let $\varepsilon>0$ be a given constant and k be a positive integer. Then the following two statements hold.*

i) There exists a set $E_1^*\subset(0,R')$ that has finite logarithmic measure such that for all $r=|z|$ satisfying $r\in(0,R')\setminus E_1^*$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq\frac{1}{r^{k(\sigma+1+\varepsilon)}}. \tag{1.15}$$

ii) There exists a set $E_2^*\subset[0,2\pi)$ that has a linear measure zero such that for all $\theta\in[0,2\pi)\setminus E_2^*$ there exists a constant $r_0=r_0(\theta)>0$ such that for all z satisfying $\arg(z)\in[0,2\pi)\setminus E_2^*$ *and $r=|z|<r_0$ the inequality (1.15) holds.*

COROLLARY 1.8. *Let f be a non-constant meromorphic function in $D(0,R)$ with a singular point at the origin of finite iterated order $\sigma_n(f,0)=\sigma<\infty$ $(n\geq 2)$; let $\varepsilon>0$ be a given constant and k be a positive integer. Then the following two statements hold.*

i) There exists a set $E_1^*\subset(0,R')$ that has finite logarithmic measure such that for all $r=|z|$ satisfying $r\in(0,R')\setminus E_1^*$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq\exp_{n-1}\left\{\frac{1}{r^{\sigma+\varepsilon}}\right\}.$$

ii) There exists a set $E_2^* \subset [0, 2\pi)$ *that has a linear measure zero such that for all* $\theta \in [0, 2\pi) \backslash E_2^*$ *there exists a constant* $r_0 = r_0(\theta) > 0$ *such that for all* z *satisfying* $\arg(z) \in [0, 2\pi) \backslash E_2^*$ *and* $r = |z| < r_0$ *the inequality (1.15) holds.*

As applications of Theorem 1.6, we have the following results.

THEOREM 1.9. *Let* $A_0(z) \not\equiv 0, A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* $D(0, R)$. *All solutions* f *of*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0 \tag{1.16}$$

satisfy $\sigma_{n+1}(f, 0) \le \alpha$ *if and only if* $\sigma_n(A_j, 0) \le \alpha$ *for all* $(j = 0, 1, ..., k-1)$, *where* n *is a positive integer. Moreover, if* $q \in \{0, 1, ..., k-1\}$ *is the largest index for which* $\sigma_n(A_q, 0) = \max\limits_{0 \le j \le k-1} \{\sigma_n(A_j, 0)\}$ *then there are at least* $k - q$ *linearly independent solutions* f *of (1.16) such that* $\sigma_{n+1}(f, 0) = \sigma_n(A_q, 0)$.

Similar result to Theorem 1.9 in the unit disc has been proved in [12, Theorem 1.1].

COROLLARY 1.10. *Let* $A_0(z) \not\equiv 0, A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* $D(0, R)$ *satisfying* $\sigma_n(A_j, 0) < \sigma_n(A_0, 0) < \infty$ $(j = 1, ..., k-1)$. *Then, every solution* $f(z) \not\equiv 0$ *of (1.16) satisfies* $\sigma_{n+1}(f, 0) = \sigma_n(A_0, 0)$.

COROLLARY 1.11. *Let* $b \neq 0$ *be complex constants and* n *be a positive integer. Let* $A(z), B(z) \not\equiv 0$ *be analytic functions in* $D(0, R)$ *with* $\max\{\sigma(A, 0), \sigma(B, 0)\} < n$. *Then, every solution* $f(z) \not\equiv 0$ *of the differential equation*

$$f'' + A(z) f' + B(z) \exp\left\{\frac{b}{z^n}\right\} f = 0, \tag{1.17}$$

satisfies $\sigma_2(f, 0) = n$.

EXAMPLE 1.12. Every solution $f(z) \not\equiv 0$ of the differential equation

$$f'' + \exp\left\{\frac{1}{(1-z)^m}\right\} f' + \exp\left\{\frac{1}{z^n}\right\} f = 0, \tag{1.18}$$

satisfies $\sigma_2(f, 0) = n$, where m and n are positive integers.

Similar equations to (1.17) and (1.18) with analytic coefficients in the unit disc are investigated in [8].

Now, we will study the case when $\sigma(A_j, 0) = \sigma(A_0, 0)$ for some $j \neq 0$.

THEOREM 1.13. *Let* $A_0(z) \not\equiv 0, A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* $D(0, R)$ *satisfying* $0 < \sigma(A_j, 0) \le \sigma(A_0, 0) < \infty$ *and*

$$\max\{\tau_M(A_j, 0) : \sigma(A_j, 0) = \sigma(A_0, 0)\} < \tau_M(A_0, 0) \ \ (j = 1, ..., k-1).$$

Then, every solution $f(z) \not\equiv 0$ *of (1.16) satisfies* $\sigma_2(f, 0) = \sigma(A_0, 0)$.

The analogs of this result in the complex plane and in the unit disc are investigated in [9, 19].

THEOREM 1.14. *Let* $a, b \neq 0$ *be complex constants such that* $\arg a \neq \arg b$ *or* $a = cb$ $(0 < c < 1)$ *and* n *be a positive integer. Let* $A(z), B(z) \not\equiv 0$ *be analytic functions in* $D(0, R)$ *with* $\max\{\sigma(A, 0), \sigma(B, 0)\} < n$. *Then, every solution* $f(z) \not\equiv 0$ *of the differential equation*

$$f'' + A(z) \exp\left\{\frac{a}{z^n}\right\} f' + B(z) \exp\left\{\frac{b}{z^n}\right\} f = 0, \tag{1.19}$$

satisfies $\sigma_2(f, 0) = n$.

Similar results to Theorem 1.14 are established in different situations in [3, 5, 8].

EXAMPLE 1.15. By Theorem 1.14, every solution $f(z) \not\equiv 0$ of the differential equation

$$f'' + \exp\left\{\frac{i}{z(z+1)}\right\} f' + \exp\left\{\frac{1}{z(z-1)^2}\right\} f = 0,$$

satisfies $\sigma_2(f, 0) = 1$ and $\sigma_2(f, 1) = 2$.

2. Preliminary lemmas

To prove these results we need the following lemmas.

LEMMA 2.1 ([6]). *Let* g *be a transcendental meromorphic function in* $\mathbb{C}$ *and* k *be a positive integer. Let* $\alpha > 1$ *and* $\varepsilon > 0$ *be given real constants; then*

i) there exists a set $E_1 \subset (1, \infty)$ *that has a finite logarithmic measure and a constant* $c > 0$ *that depends only on* k *and* α *such that for all* $R = |w|$ *satisfying* $R \notin [0, 1) \cup E_1$, *we have*

$$\left|\frac{g^{(k)}(w)}{g(w)}\right| \leq c\left[T(\alpha R, g)\frac{\log^{\alpha}(R)}{R}\log T(\alpha R, g)\right]^k; \tag{2.1}$$

ii) there exists a set $E_2 \subset [0, 2\pi)$ *that has a linear measure zero such that for all* $\theta \in [0, 2\pi) \backslash E_2$ *there exists a constant* $R_0 = R_0(\theta) > 0$ *such that (2.1) holds for all* z *satisfying* $\arg z \in [0, 2\pi) \backslash E_2$ *and* $r = |z| > R_0$.

Lemma 2.1 is valid also for rational meromorphic functions but as explained in [6, page 1]: for rational functions one can get better results than those of transcendental meromorphic functions case.

LEMMA 2.2. *[5] Let* ϕ *be a non-constant meromorphic function in* $D(0, \infty]$ *and set* $g(w) = \phi\left(\frac{1}{w}\right)$. *Then,* $g(w)$ *is meromorphic in* $\mathbb{C}$ *and we have*

$$T\left(\frac{1}{r}, g\right) = T_0(r, \phi),$$

and so $\sigma(f, 0) = \sigma(g)$.

LEMMA 2.3. *Let f be a non-constant analytic function in $D(0,R)$ of finite order $\sigma(f,0)=\sigma>0$ and a finite type $\tau(f,0)=\tau>0$. Then, for any given $0<\beta<\tau$ there exists a set $F\subset(0,1)$ of infinite logarithmic measure such that for all $r\in F$ we have*

$$\log M_0(r,f)>\frac{\beta}{r^{\sigma}},$$

where $M_0(r,f)=\max\{|f(z)|:|z|=r\}$.

PROOF. By the definition of $\tau(f,0)$, there exists a decreasing sequence $\{r_m\}\to 0$ satisfying $\frac{m}{m+1}r_m>r_{m+1}$ and

$$\lim_{m\to\infty} r_m^{\sigma}\log M_0(r_m,f)=\tau.$$

Then, there exists m_0 such that for all $m>m_0$ and for a given $\varepsilon>0$, we have

$$\log M_0(r_m,f)>\frac{\tau-\varepsilon}{r_m^{\sigma}}. \tag{2.2}$$

There exists m_1 such that for all $m>m_1$ and for a given $0<\varepsilon<\tau-\beta$, we have

$$\left(\frac{m}{m+1}\right)^{\sigma}>\frac{\beta}{\tau-\varepsilon}. \tag{2.3}$$

By (2.2) and (2.3), for all $m>m_2=\max\{m_0,m_1\}$ and for any $r\in\left[\frac{m}{m+1}r_m,r_m\right]$, we have

$$\log M_0(r,f)>\log M_0(r_m,f)>\frac{\tau-\varepsilon}{r_m^{\sigma}}>\frac{\tau-\varepsilon}{r^{\sigma}}\left(\frac{m}{m+1}\right)^{\sigma}>\frac{\beta}{r^{\sigma}}.$$

Set $F=\bigcup\limits_{m=m_2}^{\infty}\left[\frac{m}{m+1}r_m,r_m\right]$; then we have

$$\sum_{m=m_2}^{\infty}\int_{\frac{m}{m+1}r_m}^{r_m}\frac{dt}{t}=\sum_{m>m_2}\log\frac{m+1}{m}=\infty.$$

□

By the same method of the proof of Lemma 2.3, we can prove the following lemma.

LEMMA 2.4. *Let f be a non-constant analytic function in $D(0,R)$ of order $\sigma(f,0)>\alpha>0$. Then there exists a set $F\subset(0,1)$ of infinite logarithmic measure such that for all $r\in F$ we have*

$$\log M_0(r,f)>\frac{1}{r^{\alpha}}.$$

LEMMA 2.5 ([10, Theorem 8]). *Let f be non-constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Then, there exists a set $E \subset (0,1)$ that has finite logarithmic measure such that for all $j = 0, 1, ..., k$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V_{z_0}(r)}{z_r - z_0} \right)^j,$$

as $r \to 0$, $r \notin E$, where $V_{z_0}(r)$ is the central index of f and z_r is a point in the circle $|z_0 - z| = r$ that satisfies $|f(z_r)| = \max_{|z_0 - z| = r} |f(z)|$.

LEMMA 2.6. *Let f be a non-constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of iterated order $\sigma_n(f, z_0) = \sigma$ $(n \geq 2)$, and let $V_{z_0}(r)$ be the central index of f. Then*

$$\limsup_{r \to 0} \frac{\log_n^+ V_{z_0}(r)}{-\log r} = \sigma. \tag{2.4}$$

PROOF. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. Then $g(w)$ is entire function of iterated order $\sigma_n(g) = \sigma_n(f, z_0) = \sigma$, and if $V(R)$ denotes the central index of g, then $V_{z_0}(r) = V(R)$ with $R = \frac{1}{r}$. From [4, Lemma 2], we have

$$\limsup_{R \to +\infty} \frac{\log_n^+ V(R)}{\log R} = \sigma. \tag{2.5}$$

Substituting R by $\frac{1}{r}$ in (2.5), we get (2.4). □

LEMMA 2.7. *Let $A_j(z)$ $(j = 0, ..., k-1)$ be analytic functions in $D(0, R)$ such that 0 is a singular point for at least one of the coefficients $A_j(z)$ and $\sigma_n(A_j, 0) \leq \alpha < \infty$. If f is a solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \tag{2.6}$$

then $\sigma_{n+1}(f, 0) \leq \alpha$.

PROOF. Let $f \not\equiv 0$ be a solution of (2.6). It is clear that f is analytic in $D(0, R)$. Let $f(z) = z^m \phi(z) \mu(z)$ be a Valiron's decomposition and set $\tilde{\phi}(z) = z^m \phi(z)$. By Valiron's decomposition lemma and since $f(z)$ is analytic function in $D(0, R)$, $\tilde{\phi}(z)$ is analytic in $D(0, \infty]$. By Lemma 2.5, there exists a set $E \subset (0, 1)$ that has finite logarithmic measure, such that for all $j = 0, 1, ..., k$, we have

$$\frac{\tilde{\phi}^{(j)}(z_r)}{\tilde{\phi}(z_r)} = (1 + o(1)) \left(\frac{V_0(r)}{z_r} \right)^j, \tag{2.7}$$

as $r \to 0$, $r \notin E$, where $V_0(r)$ is the central index of f near the singular point 0, z_r is a point in the circle $|z| = r$ that satisfies $|f(z_r)| = \max_{|z| = r} |f(z)|$. Since $\mu(z)$ is analytic and non zero in $D(R')$, we have

$$\left| \frac{\mu^{(j)}(z)}{\mu(z)} \right| \leq M, \ (j \in \mathbb{N}). \tag{2.8}$$

Set $M_0(r) = \max_{|z|=r} \{|A_j(z)| : j = 0, 1, ..., k-1\}$. From (2.6), we can write

$$\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z) = 0. \tag{2.9}$$

We have $f(z) = \tilde{\phi}(z)\mu(z)$, and then

$$\frac{f^{(j)}(z)}{f(z)} = \sum_{i=0}^{i=j} \binom{j}{i} \frac{\tilde{\phi}^{(j-i)}(z)}{\tilde{\phi}(z)} \frac{\mu^{(i)}(z)}{\mu(z)}, \ j = 1, ..., k, \tag{2.10}$$

where $\binom{j}{i} = \frac{j!}{i!(j-i)!}$ is the binomial coefficient. By combining (2.7), (2.8) and (2.10) in (2.9), we get

$$(V_0(r))^k \leq Cr^k (V_0(r))^{k-1} M_0(r);$$

where r near enough to 0 and $C > 0$, and then

$$V_0(r) \leq Cr^k M_0(r). \tag{2.11}$$

By (2.11), we obtain $\sigma_2(f, 0) \leq \alpha$. □

LEMMA 2.8. *Let $A(z)$ be a non-constant analytic function in $D(0, R)$ with $\sigma(A, 0) < n$. Set $g(z) = A(z)\exp\left\{\frac{a}{z^n}\right\}$, ($n \geq 1$ is an integer) ,$a = \alpha + i\beta \neq 0$, $z = re^{i\varphi}$, $\delta_a(\varphi) = \alpha\cos(n\varphi) + \beta\sin(n\varphi)$, and $H = \{\varphi \in [0, 2\pi) : \delta_a(\varphi) = 0\}$, (obviously, H is of linear measure zero). Then for any given $\varepsilon > 0$ and for any $\varphi \in [0, 2\pi) \backslash H$, there exists $r_0 > 0$ such that for $0 < r < r_0$, the two following statements hold.*

(i) If $\delta_a(\varphi) > 0$, then

$$\exp\left\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\right\} \leq |g(z)| \leq \exp\left\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\right\}. \tag{2.12}$$

(ii) If $\delta_a(\varphi) < 0$, then

$$\exp\left\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\right\} \leq |g(z)| \leq \exp\left\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\right\}. \tag{2.13}$$

PROOF. Let $A(z) = z^m\phi(z)\mu(z)$ be a Valiron's decomposition and set $\tilde{\phi}(z) = z^m\phi(z)$. By Valiron's decomposition lemma and since $A(z)$ is analytic function in $D(0, R)$, $\tilde{\phi}(z)$ is analytic in $D(0, \infty]$. By Remark 1.5, $\sigma(\tilde{\phi}, 0) = \sigma(A, 0) < n$. Since $\mu(z)$ is analytic and nonzero in $D(R')$, we have

$$0 < c_1 \leq |\mu(z)| \leq c_2 \text{ as } r \text{ is close enough to } 0. \tag{2.14}$$

By applying [5, Lemma 2.9] for $\tilde{\phi}(z)$, and (2.14), we get (2.12) and (2.13). □

Now, we give the standard order reduction procedure of linear differential equations which is an adaptation of [7, Lemma 6.4].

LEMMA 2.9. *Let* $f_{0,1}, f_{0,2}, ..., f_{0,m}$ *be* m $(m \geq 2)$ *linearly independent meromorphic (in* $D(0,R)$*) solutions of an equation of the form*

$$y^{(k)} + A_{0,k-1}(z)\, y^{(k-1)} + ... + A_{0,0}(z)\, y = 0, \;\; k \geq m, \tag{2.15}$$

where $A_{0,0}(z), ..., A_{0,k-1}(z)$ *are meromorphic functions in* $D(0,R)$. *For* $1 \leq q \leq m-1$, *set*

$$f_{q,j} = \left(\frac{f_{q-1,j+1}}{f_{q-1,1}}\right)', \;\; j = 1, 2, ..., m-q. \tag{2.16}$$

Then, $f_{q,1}, f_{q,2}, ..., f_{q,m-q}$ *are* $m-q$ *linearly independent meromorphic (in* $D(0,R)$*) solutions of the equation*

$$y^{(k-q)} + A_{q,k-q-1}(z)\, y^{(k-q-1)} + ... + A_{q,0}(z)\, y = 0, \tag{2.17}$$

where

$$A_{q,j}(z) = \sum_{i=j+1}^{k-q-1} \binom{i}{j+1} A_{q-1,j}(z) \frac{f_{q-1,1}^{(i-j-1)}(z)}{f_{q-1,1}(z)} \tag{2.18}$$

for $j = 0, 1, ..., k-q-1$. *Here we set* $A_{i,k-i}(z) \equiv 1$ *for all* $i = 0, 1, ..., q$. *Moreover, let* $\varepsilon > 0$ *and suppose for each* $j \in \{0, 1, ..., k-1\}$, *there exists a real number* α_j *such that*

$$|A_{0,j}(z)| \leq \exp\left\{\frac{1}{r^{\alpha_j+\varepsilon}}\right\}, \;\; r = |z| \notin E. \tag{2.19}$$

Suppose further that each $f_{0,j}$ *is of finite hyper-order* $\sigma_2(f_{0,j}, 0)$. *Set* $\beta = \max\limits_{1\leq j\leq m}\{\sigma_2(f_{0,j}, 0)\}$ *and* $\tau_p = \max\limits_{p\leq j\leq k-1}\{\alpha_j\}$. *Then for any given* $\varepsilon > 0$, *we have*

$$|A_{q,j}(z)| \leq \exp\left\{\frac{1}{r^{\max\{\tau_{q+j},\beta\}+\varepsilon}}\right\}, \;\; r = |z| \notin E, \tag{2.20}$$

for $j = 0, 1, ..., k-q-1$.

PROOF. By [7, Lemma 6.2 and Lemma 6.3], we obtain (2.17) and (2.18). Therefore, we need only to prove (2.20). For this proof, we use mathematical induction over q. First suppose that $q = 1$. Then, from (2.18) we get

$$A_{1,j}(z) = \sum_{i=j+1}^{k} \binom{i}{j+1} A_{0,i}(z) \frac{f_{0,1}^{(i-j-1)}(z)}{f_{0,1}(z)}, \;\; j = 0, 1, ..., k-2. \tag{2.21}$$

Since $\sigma_2(f_{0,j}, 0) \leq \beta$, by Theorem 1.6, we have

$$\left|\frac{f_{0,1}^{(i-j-1)}(z)}{f_{0,1}(z)}\right| \leq \exp\left\{\frac{1}{r^{\beta+\varepsilon}}\right\}, \;\; r = |z| \notin E. \tag{2.22}$$

It follows from (2.19) and (2.21) that (2.20) holds for $q=1$. For the induction step, we make the assumption that (2.20) holds for $q-1$; i.e.

$$|A_{q-1,j}(z)| \leq \exp\left\{\frac{1}{r^{\max\{\tau_{q-1+j},\beta\}+\varepsilon}}\right\},\ r\notin E, \tag{2.23}$$

for $j=1,2,...,k-q-1$; and we show that (2.20) holds for q. From (2.18) we get

$$A_{q,j}(z) = \sum_{i=j+1}^{k-q-1}\binom{i}{j+1} A_{q-1,j}(z)\frac{f_{q-1,1}^{(i-j-1)}(z)}{f_{q-1,1}(z)}. \tag{2.24}$$

Since $\sigma_2(f_{0,j},0)$ and by elementary order considerations we get $\sigma_2(f_{q-1,1},0)\leq\beta$, and by Theorem 1.6, we obtain

$$\left|\frac{f_{q-1,1}^{(i-j-1)}(z)}{f_{q-1,1}(z)}\right| \leq \exp\left\{\frac{1}{r^{\beta+\varepsilon}}\right\},\ r=|z|\notin E. \tag{2.25}$$

From (2.23)-(2.25), we get

$$|A_{q,j}(z)| \leq \exp\left\{\frac{1}{r^{\max\{\tau_{q+j},\beta\}+\varepsilon}}\right\},\ r\notin E. \tag{2.26}$$

This proves the induction step, and therefore completes the proof of Lemma 2.9. □

LEMMA 2.10. *Under the assumptions of Lemma 2.9, we have*

$$A_{q,0} = A_{0,q} + G_q(z), \tag{2.27}$$

where $G_q(z)=\sum_{j=2}^{q+1} H_j$ *with*

$$H_j = \sum_{i=j}^{k-q+j-1}\binom{i}{j-1} A_{q-j+1,i}(z)\frac{f_{q-j+1,1}^{(i-j+1)}(z)}{f_{q-j+1,1}(z)}. \tag{2.28}$$

Moreover, $G_q(z)$ *satisfies*

$$|G_q(z)| \leq \exp\left\{\frac{1}{r^{\max\{\tau_{q+1},\beta\}+\varepsilon}}\right\},\ r=|z|\notin E. \tag{2.29}$$

PROOF. (2.27) and (2.28) are the same in [7, Lemma 6.5]. So, we need only to prove (2.29). We have

$$|G_q(z)| \leq \sum_{j=2}^{q+1}\sum_{i=j}^{k-q+j-1}\binom{i}{j-1}|A_{q-j+1,i}(z)|\left|\frac{f_{q-j+1,1}^{(i-j+1)}(z)}{f_{q-j+1,1}(z)}\right|.$$

By applying (2.20) for the coefficients $|A_{q-j+1,i}(z)|$ and Theorem 1.6 for the logarithmic derivatives $\left|\frac{f_{q-j+1,1}^{(i-j+1)}(z)}{f_{q-j+1,1}(z)}\right|$ by taking into account that $\sigma_2(f_{q-j+1,1},0)\leq\beta$, we obtain (2.29). □

3. Proof of theorems

Proof of Theorem 1.6. Suppose that f is meromorphic function in $D(0,R)$ with a singular point at the origin. By Valiron's decomposition lemma we have

$$f(z) = z^m \phi(z)\, \mu(z) \tag{3.1}$$

where

a) The poles and zeros of f in $D(0,R')$ are precisely the poles and zeros of $\phi(z)$. The poles and zeros of f in $D(R',R)$ are precisely the poles and zeros of $\mu(z)$.
b) $\phi(z)$ is meromorphic in $D(0,\infty]$ and analytic and nonzero in $D[R',\infty]$.
c) $\mu(z)$ is meromorphic in $D(R)$ and analytic and nonzero in $D(R')$.

Set $\tilde{\phi}(z) = z^m \phi(z)$. We have

$$\frac{f'(z)}{f(z)} = \frac{\tilde{\phi}'(z)}{\tilde{\phi}(z)} + \frac{\mu'(z)}{\mu(z)};$$

and thus

$$\left|\frac{f'(z)}{f(z)}\right| \leq \left|\frac{\tilde{\phi}'(z)}{\tilde{\phi}(z)}\right| + \left|\frac{\mu'(z)}{\mu(z)}\right|. \tag{3.2}$$

Since $\mu(z)$ is analytic and non zero in $D(R')$, we have

$$\left|\frac{\mu^{(j)}(z)}{\mu(z)}\right| \leq M,\ (j \in \mathbb{N}). \tag{3.3}$$

Set $g(w) = \tilde{\phi}\left(\frac{1}{w}\right)$. Since $\phi(z)$ satisfy b), $g(w)$ is meromorphic in $\mathbb{C}$. We have $\tilde{\phi}(z) = g(w)$ such that $w = \frac{1}{z}$; then $\tilde{\phi}'(z) = \frac{-1}{z^2} g'(w)$ and then

$$\frac{\tilde{\phi}'(z)}{\tilde{\phi}(z)} = \frac{-1}{z^2}\frac{g'(w)}{g(w)}. \tag{3.4}$$

By Lemma 2.1, there exists a set $E_1 \subset (1,\infty)$ that has a finite logarithmic measure such that for all $|w| = \frac{1}{|z|} = \frac{1}{r}$ satisfying $\frac{1}{r} \notin [0,1) \cup E_1$, we have

$$\left|\frac{g'(w)}{g(w)}\right| \leq C\left[T\left(\frac{\alpha}{r}, g\right) r \log^{\alpha}\left(\frac{1}{r}\right) \log T\left(\frac{\alpha}{r}, g\right)\right],\ \frac{1}{r} \notin E_1,$$

and by Lemma 2.2 and (3.4), we get

$$\left|\frac{\tilde{\phi}'(z)}{\tilde{\phi}(z)}\right| \leq C\left[\frac{1}{r} T_0\left(\frac{r}{\alpha}, \tilde{\phi}\right) \log^{\alpha}\left(\frac{1}{r}\right) \log T_0\left(\frac{r}{\alpha}, \tilde{\phi}\right)\right],\ r \notin E_1^*; \tag{3.5}$$

where $\frac{1}{r} = R \notin E_1 \Leftrightarrow r \notin E_1^*$ and $\int\limits_0^{r_0} \frac{\chi_{E_1^*}}{t} dt = \int\limits_{1/r_0}^{\infty} \frac{\chi_{E_1}}{T} dT < \infty$, (the constant $C > 0$ is not the same at each occurrence). Combining (3.2)-(3.3) with (3.5) and by taking into account Remark 1.5, we get

$$\left|\frac{f'(z)}{f(z)}\right| \leq C\left[\frac{1}{r}T_0\left(\frac{r}{\alpha},f\right)\log^{\alpha}\left(\frac{1}{r}\right)\log T_0\left(\frac{r}{\alpha},f\right)\right],\ r\notin E_1^*.$$

We have $\tilde{\phi}''(z) = \frac{1}{z^4}g''(w) + \frac{2}{z^3}g'(w)$; and so

$$\frac{\tilde{\phi}''(z)}{\tilde{\phi}(z)} = \frac{1}{z^4}\frac{g''(w)}{g(w)} + \frac{2}{z^3}\frac{g'(w)}{g(w)}.$$

By Lemma 2.1 and Lemma 2.2, we obtain

$$\left|\frac{\tilde{\phi}''(z)}{\tilde{\phi}(z)}\right| \leq C\left[\frac{1}{r}T_0\left(\frac{r}{\alpha},\tilde{\phi}\right)\log^{\alpha}\left(\frac{1}{r}\right)\log T_0\left(\frac{r}{\alpha},\tilde{\phi}\right)\right]^2\ r\notin E_1^*. \tag{3.6}$$

We have

$$\frac{f''(z)}{f(z)} = \frac{\tilde{\phi}''(z)}{\tilde{\phi}(z)} + \frac{\mu''(z)}{\mu(z)} + 2\frac{\tilde{\phi}'(z)}{\tilde{\phi}(z)}\frac{\mu'(z)}{\mu(z)}. \tag{3.7}$$

Combining (3.6)-(3.7) with (3.3) and by Remark 1.5, we get

$$\left|\frac{f''(z)}{f(z)}\right| \leq C\left[\frac{1}{r}T_0\left(\frac{r}{\alpha},f\right)\log^{\alpha}\left(\frac{1}{r}\right)\log T_0\left(\frac{r}{\alpha},f\right)\right]^2,\ r\notin E_1^*.$$

In general, we can find that

$$\tilde{\phi}^{(k)}(z) = \frac{1}{z^{2k}}g^{(k)}(w) + \frac{a_{k-1}}{z^{2k-1}}g^{(k-1)}(w) + ... + \frac{a_1}{z^{k+1}}g'(w);$$

where $a_1, ..., a_{k-1}$ are integers; thus

$$\frac{\tilde{\phi}^{(k)}(z)}{\tilde{\phi}(z)} = \frac{1}{z^{2k}}\frac{g^{(k)}(w)}{g(w)} + \frac{a_{k-1}}{z^{2k-1}}\frac{g^{(k-1)}(w)}{g(w)} + ... + \frac{a_1}{z^{k+1}}\frac{g'(w)}{g(w)}. \tag{3.8}$$

Also by making use of Lemma 2.1 and Lemma 2.2 with (3.8), for $r = |z| < r_0$, we get,

$$\left|\frac{\tilde{\phi}^{(k)}(z)}{\tilde{\phi}(z)}\right| \leq C\left[\frac{1}{r}T_0\left(\frac{r}{\alpha},\tilde{\phi}\right)\log^{\alpha}\left(\frac{1}{r}\right)\log T_0\left(\frac{r}{\alpha},\tilde{\phi}\right)\right]^k\ r\notin E_1^*. \tag{3.9}$$

We can generalize the equality of $\frac{f^{(k)}(z)}{f(z)}$ as follows

$$\frac{f^{(k)}(z)}{f(z)} = \sum_{j=0}^{j=k}\binom{k}{j}\frac{\tilde{\phi}^{(k-j)}(z)}{\tilde{\phi}(z)}\frac{\mu^{(j)}(z)}{\mu(z)}, \tag{3.10}$$

where $\binom{k}{j} = \frac{k!}{j!(k-j)!}$ is the binomial coefficient. Combining (3.9)-(3.10), with (3.3) and Remark 1.5, we obtain

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq C\left[\frac{1}{r}T_0\left(\frac{r}{\alpha},f\right)\log^{\alpha}\left(\frac{1}{r}\right)\log T_0\left(\frac{r}{\alpha},f\right)\right]^k\ (k\in\mathbb{N}),$$

The same reasoning applies to the case (ii); noting that $\theta \in E_2 \Leftrightarrow 2\pi - \theta \in E_2^*$; so, if $E_2 \subset [0, 2\pi)$ has linear measure zero, then $E_2^* \subset [0, 2\pi)$ has also linear measure zero. $\square$

Proof of Theorem 1.9. We divide the proof into three parts:

1) If $\sigma_n(A_j, 0) \leq \alpha$ for all $j = 0, 1, \ldots, k-1$, then by Lemma 2.7 all solutions f of (1.16) satisfy $\sigma_{n+1}(f, 0) \leq \alpha$.

2) Suppose that $\sigma_n(A_j, 0) = \alpha_j$, and let $q \in \{0, 1, ..., k-1\}$ be the largest index such that $\alpha_q = \max\limits_{0 \leq j \leq k-1} \{\alpha_j\}$. By Part 1) all solutions f of (1.16) satisfy $\sigma_{n+1}(f, 0) \leq \alpha_q$. Assume that there are $q+1$ linearly independent solutions $f_{0,1}, f_{0,2}, ..., f_{0,q+1}$ of (1.16) satisfy $\sigma_{n+1}(f_{0,j}, 0) < \alpha_q$ for all $j = 1, \ldots, q+1$. By Lemma 2.9 with $m = q+1$, there exists a solution $f_{q,1} \not\equiv 0$ of (2.17) such that $\sigma_{n+1}(f_{q,1}) < \alpha_q$ and for any $\varepsilon > 0$

$$|A_{q,j}(z)| \leq \exp_n \left\{ \frac{1}{r^{\max\{\tau_{q+j}, \beta\} + \varepsilon}} \right\}, \; r \notin E. \tag{3.11}$$

where $\tau_{q+j} = \max\limits_{q+j \leq l \leq k-1} \{\alpha_l\}$ and $j = 1, \ldots, k-q-1$. We have $\max\{\tau_{q+j}, \beta\} < \alpha_q$, and then

$$|A_{q,j}(z)| \leq \exp_n \left\{ \frac{1}{r^{\alpha_q - 2\varepsilon}} \right\}, \; r \notin E, \tag{3.12}$$

for all $j = 1, \ldots, k-q-1$ and for $\varepsilon > 0$ small enough. Now, by Lemma 2.10, $\sigma_n(A_{q,0}, 0) = \sigma_n(A_{0,q}, 0) = \alpha_q$ and by Lemma 2.4, there exists a set $F \subset (0, R')$ of infinite logarithmic measure such that for all $r \in F$ we have

$$|A_{q,0}(z)| \geq \exp_n \left\{ \frac{1}{r^{\alpha_q - \varepsilon}} \right\}, \tag{3.13}$$

where $|A_{q,j}(z)| = M_0(r, A_{q,j})$. On the other hand, by (2.17)

$$|A_{q,0}(z)| \leq |\frac{f_{q,1}^{(k-q)}}{f_{q,1}}| + |A_{q,k-q-1}(z)||\frac{f_{q,1}^{(k-q-1)}}{f_{q,1}}| + \cdots + |A_{q,1}(z)||\frac{f'_{q,1}}{f_{q,1}}|,$$

and so by (3.12) and Corollary 1.8 with $\sigma_{n+1}(f_{q,1}) < \alpha_q$, we get

$$|A_{q,0}(z)| \leq \exp_n \left\{ \frac{1}{r^{\alpha_q - 2\varepsilon}} \right\}, \; r \notin E. \tag{3.14}$$

By taking $r \in F \backslash E$, (3.14) contradicts (3.13). Hence, there are at most q linearly independent solutions f of (1.16) such that $\sigma_{n+1}(f) < \alpha_q$. Since $\sigma_{n+1}(f) \leq \alpha_q$ for all solutions f of (1.16), there are at least $k-q$ linearly independent solutions f of (1.16) such that $\sigma_{n+1}(f, 0) = \alpha_q$.

3) Suppose that all solutions f of (1.16) satisfy $\sigma_{n+1}(f, 0) \leq \alpha$, and assume that there is a coefficient $A_j(z)$ of (1.16) such that $\sigma_n(A_j) > \alpha$. If $q \in \{0, 1, ..., k-1\}$ is the largest index such that $\alpha_q = \max\limits_{0 \leq j \leq k-1} \{\alpha_j\}$,

then by part 2), (1.16) has at least $k-q$ linearly independent solutions f such that $\sigma_{n+1}(f,0)=\alpha_q>\alpha$. A contradiction. So, $\sigma_n(A_j)\leq\alpha$ for all $j=0,1,\ldots,k-1$. □

Proof of Theorem 1.13. From (1.16), we can write

$$|A_0(z)|\leq\left|\frac{f^{(k)}}{f}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_1(z)|\left|\frac{f'}{f}\right|. \tag{3.15}$$

Case (i): $\sigma(A_j,0)<\sigma(A_0,0)<\infty$ $(j=1,...,k-1)$. Set $\max\{\sigma(A_j,0): j\neq 0\}<\beta<\alpha<\sigma(A_0,0)$. By (1.9), there exists $r_0>0$ such that for all r satisfying $r_0\geq r>0$, we have

$$|A_j(z)|\leq\exp\{\frac{1}{r^\beta}\},\quad j=1,2,\ldots,k-1. \tag{3.16}$$

By Lemma 2.3, there exists a set $F\subset(0,R')$ of infinite logarithmic measure such that for all $r\in F$, we have

$$|A_0(z)|>\exp\{\frac{1}{r^\alpha}\}, \tag{3.17}$$

where $|A_0(z)|=M_0(r,A_0)$. From Theorem 1.6, there exists a set $E_1^*\subset(0,R')$ that has finite logarithmic measure and a constant $C>0$ such that for all $r=|z|$ satisfying $r\in(0,R')\setminus E_1^*$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\frac{C}{r^{2k}}\left[T_0\left(\frac{r}{\alpha},f\right)\right]^{2k}\ (j=1,...,k-1). \tag{3.18}$$

Using (3.16)–(3.18) in (3.15), for $r\in F\setminus E_1^*$, we obtain

$$\exp\{\frac{1}{r^\alpha}\}\leq\frac{C}{r^{2k}}\left[T_0\left(\frac{r}{\alpha},f\right)\right]^{2k}\exp\{\frac{1}{r^{\beta+\varepsilon}}\}. \tag{3.19}$$

From (3.19), we obtain that $\sigma_2(f,0)\geq\alpha$.

On the other hand, applying Lemma 2.7 with (1.16), we obtain that $\sigma_2(f,0)\leq\sigma(A_0,0)$. Since $\alpha\leq\sigma_2(f,0)\leq\sigma(A_0,0)$ holds for all $\alpha<\sigma(A_0,0)$, then $\sigma_2(f,0)=\sigma(A_0,0)$.

Case (ii): $0<\sigma(A_j,0)\leq\sigma(A_0,0)<\infty$ and

$$\max\{\tau_M(A_j,0):\sigma(A_j,0)=\sigma(A_0,0)\}<\tau_M(A_0,0)\ (j=1,...,k-1).$$

Set $\max\{\tau_M(A_j,0):\sigma(A_j,0)=\sigma(A_0,0)\}<\mu<\nu<\tau_M(A_0,0)$ and $\sigma(A_0,0)=\sigma$. By (1.10), there exists $r_0>0$ such that for all r satisfying $r_0\geq r>0$, we have

$$|A_j(z)|\leq\exp\{\frac{\mu}{r^\sigma}\},\quad j=1,2,\ldots,k-1. \tag{3.20}$$

By Lemma 2.3, there exists a set $F\subset(0,R')$ of infinite logarithmic measure such that for all $r\in F$ and $|A_0(z)|=M_0(r,A_0)$, we have

$$|A_0(z)|>\exp\{\frac{\nu}{r^\sigma}\}. \tag{3.21}$$

Combining (3.20)-(3.21) with (3.18) and (3.15), we get for $r \in F \backslash E_1^*$,

$$\exp\{\frac{\nu}{r^{\sigma}}\} \leq \frac{C}{r^{2k}} \left[T_0\left(\frac{r}{\alpha}, f\right)\right]^{2k} \exp\{\frac{\mu}{r^{\sigma}}\}. \tag{3.22}$$

From (3.22), we get $\sigma_2(f,0) \geq \sigma$, and combining this with Lemma 2.7, we obtain that $\sigma_2(f,0) = \sigma(A_0, 0)$. □

Proof of Theorem 1.14. We begin with the case $a = cb$ $(0 < c < 1)$. It is easy to see that $\tau_M\left(A(z)\exp\left\{\frac{a}{z^n}\right\}, 0\right) = |a|$ and $\tau_M\left(B(z)\exp\left\{\frac{b}{z^n}\right\}, 0\right) = |b|$. By Theorem 1.13 case (ii), we get $\sigma_2(f,0) = n$. Now, suppose that $\arg a \neq \arg b$. Then, there exist $(\varphi_1, \varphi_2) \subset [0, 2\pi)$ such that for $\arg(z) = \varphi \in (\varphi_1, \varphi_2)$, we have $\delta_b(\varphi) > 0$ and $\delta_a(\varphi) < 0$. From (1.19), we can write

$$|B(z)\exp\left\{\frac{b}{z^n}\right\}| \leq \left|\frac{f''}{f}\right| + |A(z)\exp\left\{\frac{a}{z^n}\right\}|\left|\frac{f'}{f}\right|. \tag{3.23}$$

Since $\max\{\sigma(A,0), \sigma(B,0)\} < n$, then by Lemma 2.8, (1.14) and (3.23), we obtain

$$\exp\left\{(1-\varepsilon)\,\delta_b(\varphi)\frac{1}{r^n}\right\} \leq \frac{C}{r^4}\left[T_0\left(\frac{r}{\alpha}, f\right)\right]^4 \exp\left\{(1-\varepsilon)\,\delta_a(\varphi)\frac{1}{r^n}\right\}. \tag{3.24}$$

From (3.24) we get $\sigma_2(f,0) \geq n$ and combining this with Lemma 2.7, we obtain that $\sigma_2(f,0) = n$. □

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Ocjene za logaritamsku derivaciju u okolini singularne točke i primjene

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Sažetak. U ovom članku dane su ocjene u okolini od $z = 0$ za logaritamsku derivaciju $\left|\frac{f^{(k)}(z)}{f(z)}\right|$, gdje je f meromofna funkcija u području oblika $D(0, R) = \{z \in \mathbb{C} : 0 < |z| < R\}$. Dane su neke primjene na rast rješenja linearnih diferencijalnih jednadžbi u okolini singularne točke.

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