q-FRACTIONAL DIRAC TYPE SYSTEMS

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ABSTRACT. This paper is devoted to study a regular q-fractional Dirac type system. We investigate the properties of the eigenvalues and the eigenfunctions of this system. By using a fixed point theorem we give a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions.

1. INTRODUCTION

q-calculus deals with the investigation and applications of quantum derivatives and quantum integrals. It is an interesting topic having interconnections with various problems of mathematical physics and quantum mechanics ([8–10, 12, 14, 16, 17, 23, 24, 30]). For the q-calculus, we refer the reader to the books [7, 13, 18].

The fractional q-calculus is the generalization of the q-calculus. In the recent years, some results have been derived in q-fractional equations [5–7, 15, 20–22, 25, 26]. Mansour [25] introduced q-fractional Sturm-Liouville problems containing the left-sided Caputo q-fractional derivative and the right-sided Riemann-Liouville q-fractional derivative. The author used a fixed point theorem to introduce a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions of q-fractional Sturm-Liouville problems. Al-Towailb studied the regular q-fractional Sturm-Liouville problems. The author proved properties of the eigenvalues and the eigenfunctions in [5]. In [26], the author introduced the essential q-fractional variational analysis needed in proving the existence of a countable set of real eigenvalues and associated orthogonal eigenfunctions for the regular q-fractional Sturm-Liouville problems. Allahverdiev and Tuna [3] proved a theorem on the completeness of the system of eigenvectors and associated vectors of the dissipative q-fractional Sturm-Liouville operators.

It is well known that the Dirac systems defined by

(1.1)
$$\begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$
$$= \lambda \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

where $x \in [a, b]$, play an important role in relativistic quantum mechanics. These systems describe spin $1/2$ particles, including electrons, neutrinos, muons, protons, neutrons, quarks, and their corresponding anti-particles. For the history and details of the Dirac systems, see $[11, 19, 29, 31]$ and their references. In this paper, we are interest in a $q-$fractional version of the system (1.1) defined by

$$\begin{pmatrix} 0 & -\mathcal{D}_{q,a^-}^{\alpha} \\ {}^c\mathcal{D}_{q,0^+}^{\alpha} & 0 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$
$$= \lambda \begin{pmatrix} \omega_1(x) & 0 \\ 0 & \omega_2(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

where $x \in (0, a)$. To the best of the authors' knowledge there are no results available in the literature considering this system. These results are a generalization of the regular q-Dirac system introduced in [2].

2. PRELIMINARIES

First of all, we recall the notations and some basic properties for q-fractional calculus theory, which are useful in the following discussion (see $[1, 4, 7, 13, 18, 25, 27, 28]$). Throughout this paper, we assume that $0 < q < 1$ and A is a q-geometric set, i.e., $qx \in A$ whenever $x \in A$. For every $t > 0$, we define the sets $A_{t,q}, A_{t,q}^*$ and $\mathcal{A}_{t,q}$, respectively, by

$$A_{t,q} := \{tq^n : n \in \mathbb{N}\}, \ A_{t,q}^* := A_{t,q} \cup \{0\},$$

and

$$\mathcal{A}_{t,q} := \{\pm tq^n : n \in \mathbb{N}\}.$$

Let $y(.)$ be a complex-valued function on A. The q-difference operator $\mathcal{D}_q$ is defined by

$$\mathcal{D}_q y(x) = \frac{y(qx) - y(x)}{(q-1)x} \text{ for all } x \in A \backslash \{0\}.$$

The q-derivative at zero is defined by

$$\mathcal{D}_q y(0) = \lim_{n \to \infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on x. A *right-inverse* to D_q, the *Jackson q-integration* is given by

$$\int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t \quad (a, b \in A).$$

Let $L_q^2(0, a)$ be the space of all complex-valued functions defined on $[0, a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|^2 \, d_q x \right)^{1/2} < \infty.$$

$L_q^2(0, a)$ is a separable Hilbert space with the inner product

$$(f, g) := \int_0^a f(x)\, \overline{g(x)} d_q x, \quad f, g \in L_q^2(0, a),$$

and the orthonormal basis

$$\phi_n(x) = \begin{cases} \frac{1}{\sqrt{x(1-q)}}, & x = aq^n, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 0, 1, 2, ...$(see [7]).

DEFINITION 2.1. *A function f which is defined on A, $0 \in A$, is said to be q-regular at zero if*

$$\lim_{n \to \infty} f(xq^n) = f(0)$$

for every $x \in A$ (see [7]).

Let $C(A)$ denote the space of all q-regular at zero functions on A. This space is a normed space with the norm function

$$\|f\| = \sup\{|f(xq^n)|,\ x \in A,\ n \in \mathbb{N}\}.$$

(see [7]).

DEFINITION 2.2. *A q-regular at zero function f which is defined on $A_{t,q}^*$ is said to be q-absolutely continuous if*

$$\sum_{j=0}^{\infty} \left| f(uq^j) - f(uq^{j+1}) \right| \leq K, \quad \forall u \in A_{t,q}^*,$$

for K is a constant depending on the function f (see [7]).

The space of all q-absolutely continuous functions on $A_{t,q}^*$ is denoted by $AC_q\left(A_{t,q}^*\right)$. Note that $\mathcal{AC}_q(A_{q,t}^*) \subseteq C(A_{q,t}^*)$.

For $n \in \mathbb{N}$ and $\alpha, a_1, .., a_n \in \mathbb{C}$; the q-*shifted factorial*, the *multiple q-shifted factorial* and the q-*binomial coefficients* are defined by

$$(a;q)_0 = 1, \ (a;q)_n = \prod_{k=0}^{n-1}\left(1 - aq^k\right), \ (a;q)_\infty = \prod_{k=0}^{\infty}\left(1 - aq^k\right),$$

$$(a_1, a_2, ..., a_k : q) = \prod_{j=1}^{k}(a_j;q)_n$$

and

$$\begin{bmatrix}\alpha\\0\end{bmatrix}_q = 1, \ \begin{bmatrix}\alpha\\n\end{bmatrix}_q = \frac{\left(1 - q^\alpha\right)\left(1 - q^{\alpha-1}\right)...\left(1 - q^{\alpha-n+1}\right)}{(q;q)_n},$$

respectively (see [7]). The *generalized q-shifted factorial* is defined by

$$(a;q)_\nu = \frac{(a;q)_\infty}{(aq^\nu;q)_\infty} \ (\nu \in \mathbb{R})$$

(see [7]). The q-*Gamma function* is defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1 - q)^{1-z}, \ z \in \mathbb{C}, \ |q| < 1$$

(see [7]).

DEFINITION 2.3. *Let $0 < \alpha \leq 1$. The left-sided and right-sided Riemann-Liouville q-fractional operator are given by the formulas*

$$(2.1) \qquad \mathcal{I}_{q,a^+}^\alpha f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x\left(\frac{qt}{x};q\right)_{\alpha-1}f(t)\,d_q t,$$

$$(2.2) \qquad \mathcal{I}_{q,b^-}^\alpha f(x) = \frac{1}{\Gamma_q(\alpha)}\int_{qx}^b t^{\alpha-1}\left(\frac{qx}{t};q\right)_{\alpha-1}f(t)\,d_q t,$$

respectively (see [25]).

DEFINITION 2.4. *Let $\alpha > 0$ and $\lceil\alpha\rceil = m$. The left-sided and right-sided Riemann-Liouville fractional q-derivatives of the order α are defined, respectively, as follows:*

$$(2.3) \qquad \mathcal{D}_{q,a^+}^\alpha f(x) = \mathcal{D}_q^m \mathcal{I}_{q,a^+}^{m-\alpha}f(x),$$

$$(2.4) \qquad \mathcal{D}_{q,b^-}^\alpha f(x) = \left(\frac{-1}{q}\right)^m \mathcal{D}_{q^{-1}}^m \mathcal{I}_{q,b^-}^{m-\alpha}f(x).$$

Similar formulas give the left-sided and right-sided Caputo fractional q-derivatives of order α, respectively as follows:

$$^{c}\mathcal{D}_{q,a+}^{\alpha}f(x) = \mathcal{I}_{q,a+}^{m-\alpha}\mathcal{D}_{q}^{m}f(x),$$

$$^{c}\mathcal{D}_{q,b-}^{\alpha}f(x) = \left(\frac{-1}{q}\right)^{m}\mathcal{I}_{q,b-}^{m-\alpha}\mathcal{D}_{q^{-1}}^{m}f(x)$$

(see [25]).

In order to prove the main results, we also need the following lemmas. One can find them in [25].

LEMMA 2.5.

i) *The left-sided Riemann-Liouville q-fractional operator satisfies the semi-group property*

(2.5) $$\mathcal{I}_{q,a+}^{\alpha}\mathcal{I}_{q,a+}^{\beta} = \mathcal{I}_{q,a+}^{\alpha+\beta}f(x), \ \ x \in A_{q,a}^{*},$$

for any function defined on $A_{q,a}$ and for any values of α and β.

ii) *The right-sided Riemann-Liouville q-fractional operator satisfies the semi-group property*

$$\mathcal{I}_{q,b-}^{\alpha}\mathcal{I}_{q,b-}^{\beta}f(x) = \mathcal{I}_{q,b-}^{\alpha+\beta}f(x), \ \ x \in A_{q,b}^{*},$$

for any function defined on $A_{q,b}$ and for any values of α and β.

LEMMA 2.6. *Let $\alpha \in (0,1)$.*

i) *If $f \in L_{q}^{1}\left(A_{q,a}^{*}\right)$ such that $\mathcal{I}_{q,0+}^{\alpha}f \in AC_{q}\left(A_{t,q}^{*}\right)$ then*

$$^{c}\mathcal{D}_{q,0+}^{\alpha}\mathcal{I}_{q,0+}^{\alpha}f(x) = f(x) - \frac{\mathcal{I}_{q,0+}^{\alpha}f(0)}{\Gamma_{q}(1-\alpha)}x^{-\alpha}.$$

Moreover, if f is bounded on $A_{t,q}^{*}$ then

$$^{c}\mathcal{D}_{q,0+}^{\alpha}\mathcal{I}_{q,0+}^{\alpha}f(x) = f(x).$$

ii) *If $f \in L_{q}^{1}\left(A_{q,a}\right)$ then*

$$\mathcal{D}_{q,0+}^{\alpha}\mathcal{I}_{q,0+}^{\alpha}f(x) = f(x).$$

iii) *If f is a function defined on $A_{t,q}^{*}$ then*

$$^{c}\mathcal{D}_{q,a-}^{\alpha}\mathcal{I}_{q,a-}^{\alpha}f(x) = f(x) - \frac{a^{-\alpha}}{\Gamma_{q}(1-\alpha)}\left(\frac{qx}{a};q\right)_{-\alpha}\left(\mathcal{I}_{q,a-}^{\alpha}f\right)\left(\frac{a}{q}\right),$$

$$\mathcal{D}_{q,a-}^{\alpha}\mathcal{I}_{q,a-}^{\alpha}f(x) = f(x),$$

$$\mathcal{I}_{q,a-}^{\alpha}\mathcal{D}_{q,a-}^{\alpha}f(x) = f(x) - \frac{a^{\alpha-1}}{\Gamma_{q}(\alpha)}\left(\frac{qx}{a};q\right)_{\alpha-1}\left(\mathcal{I}_{q,a-}^{1-\alpha}f\right)\left(\frac{a}{q}\right).$$

iv) *If $f \in AC_{q}\left(A_{t,q}^{*}\right)$ then*

$$\mathcal{I}_{q,0+}^{\alpha}{}^{c}\mathcal{D}_{q,0+}^{\alpha}f(x) = f(x) - f(0).$$

We denote by $L^2_{q,\omega}\left(A^*_{t,\alpha};E\right)$ $(E := \mathbb{R}^2)$ the Hilbert space which consists of vector-valued functions with inner product

$$(2.6) \qquad (f,g) := \int_0^a f_1(x)\overline{g_1(x)}\omega_1(x)d_q x$$

$$+ \int_0^a f_2(x)\overline{g_2(x)}\omega_2(x)d_q x,$$

where $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$, $f_i(x)$, $g_i(x)$, $\omega_{i\alpha}(x)$ $(i=1,2)$ are real-valued functions on $A^*_{t,\alpha}$ and $\omega_i(x) > 0$, $\forall x \in A^*_{t,\alpha}$, $(i=1,2)$.

3. q-FRACTIONAL DIRAC SYSTEMS

In the present section, our goal is to study the q-fractional Dirac system which includes the right-sided Caputo and the left-sided Riemann-Liouville fractional derivatives of same order α. Throughout this section, we assume $\alpha \in (0,1)$.

Let

$$\tau_{q,\alpha}y := \begin{pmatrix} 0 & -\mathcal{D}^\alpha_{q,a^-} \\ {}^c\mathcal{D}^\alpha_{q,0^+} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} -\mathcal{D}^\alpha_{q,a^-}y_2 + p(x)y_1 \\ {}^c\mathcal{D}^\alpha_{q,0^+}y_1 + r(x)y_2 \end{pmatrix},$$

where $y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. With this notation, we consider the q-fractional Dirac type system:

$$(3.1) \qquad \tau_{q,\alpha}f_\lambda = \lambda\omega f_\lambda, \ a \le x \le b < \infty,$$

where $f_\lambda = \begin{pmatrix} f_{\lambda 1} \\ f_{\lambda 2} \end{pmatrix}$, $p(.)$, $r(.)$ are real-valued functions defined in $A^*_{t,\alpha}$, $\omega(x) = \begin{pmatrix} \omega_1(x) & 0 \\ 0 & \omega_2(x) \end{pmatrix}$, $\omega_i(.)$ are real-valued functions defined in $A^*_{t,\alpha}$ and $\omega_{i\alpha}(x) > 0$, $\forall x \in A^*_{t,\alpha}$, $(i=1,2)$, λ is a complex eigenvalue parameter and boundary conditions

$$(3.2) \qquad c_{11}f_{\lambda 1}(0) + c_{12}\mathcal{I}^{1-\alpha}_{q,a^-}f_{\lambda 2}(0) = 0,$$

$$(3.3) \qquad c_{21}f_{\lambda 1}(a) + c_{22}\mathcal{I}^{1-\alpha}_{q,a^-}f_{\lambda 2}\left(\frac{a}{q}\right) = 0,$$

with $c_{11}^2 + c_{12}^2 \ne 0$ and $c_{21}^2 + c_{22}^2 \ne 0$.

To pass from the differential expression $T_{q,\alpha} := \omega^{-1}\tau_{q,\alpha}$ to operators, we introduce the space $H \subseteq L^2_{q,\omega}\left(A^*_{t,\alpha};E\right) \cap C\left(A^*_{t,\alpha};E\right)$ which consists of all q-regular at zero functions satisfying the conditions (3.2) and (3.3) with inner product (2.6).

THEOREM 3.1. *The operator $T_{q,\alpha}$ generated by q-fractional Dirac type system (FD) defined by (3.1)-(3.3) is formally self-adjoint on H.*

PROOF. Let $u\left(.\right), z\left(.\right) \in H$. Then, we have

$$(T_{q,\alpha}u, z) - (u, T_{q,\alpha}z) = \int_0^a \left({}^c\mathcal{D}_{q,0^+}^\alpha u_1 + r\left(x\right)u_2\right)\overline{z_2}d_q x$$

$$+ \int_0^a \left(-\mathcal{D}_{q,a^-}^\alpha u_2 + p\left(x\right)u_1\right)\overline{z_1}d_q x$$

$$- \int_0^a u_2\overline{\left({}^c\mathcal{D}_{q,0^+}^\alpha z_1 + r\left(x\right)z_2\right)}d_q x$$

$$- \int_0^a u_1\overline{\left(-\mathcal{D}_{q,a^-}^\alpha z_2 + p\left(x\right)z_1\right)}d_q x$$

$$= \int_0^a \left({}^c\mathcal{D}_{q,0^+}^\alpha u_1\right)\overline{z_2}d_q x - \int_0^a \left(\mathcal{D}_{q,a^-}^\alpha u_2\right)\overline{z_1}d_q x$$

$$- \int_0^a u_2\overline{\left({}^c\mathcal{D}_{q,0^+}^\alpha z_1\right)}d_q x + \int_0^a u_1\overline{\left(\mathcal{D}_{q,a^-}^\alpha z_2\right)}d_q x.$$

Since

$$\int_0^a \left({}^c\mathcal{D}_{q,0^+}^\alpha u_1\right)\overline{z_2}d_q x = \int_0^a u_1\overline{\left(-\mathcal{D}_{q,a^-}^\alpha z_1\right)}d_q x$$

$$- \left[u_1\left(a\right)\overline{\mathcal{I}_{q,a^-}^{1-\alpha}z_2\left(\frac{a}{q}\right)} - u_1\left(0\right)\overline{\mathcal{I}_{q,a^-}^{1-\alpha}z_2\left(0\right)}\right]$$

and

$$\int_0^a u_2\overline{\left({}^c\mathcal{D}_{q,0^+}^\alpha z_1\right)}d_q x = \int_0^a \left(-\mathcal{D}_{q,a^-}^\alpha u_2\right)\overline{z_1}d_q x$$

$$- \left[z_1\left(a\right)\overline{\mathcal{I}_{q,a^-}^{1-\alpha}u_2\left(\frac{a}{q}\right)} - z_1\left(0\right)\overline{\mathcal{I}_{q,a^-}^{1-\alpha}u_2\left(0\right)}\right],$$

we get

(3.4) $$(T_{q,\alpha}u, z) - (u, T_{q,\alpha}z) = [u, z]\left(a\right) - [u, z]\left(0\right),$$

where $[y, z]\left(x\right) := y_1\left(x\right)\overline{\mathcal{I}_{q,a^-}^{1-\alpha}z_2\left(x\right)} - \overline{z_1\left(x\right)}\mathcal{I}_{q,a^-}^{1-\alpha}y_2\left(x\right)$. We proceed to show that the equality $(T_{q,\alpha}u, z) = (u, T_{q,\alpha}z)$ for any $u\left(.\right), z\left(.\right) \in H$. From the boundary conditions (3.2) and (3.3), we get $[u, z]_a = 0$ and $[u, z]_0 = 0$. Consequently,

(3.5) $$(T_{q,\alpha}u, z) = (u, T_{q,\alpha}z).$$

This completes the proof.   □

LEMMA 3.2. *All eigenvalues of the operator $T_{q,\alpha}$ generated by q-FD system defined by (3.1)-(3.3) are real.*

PROOF. Let μ be an eigenvalue with an eigenfunction $z(x)$. From the equality (3.5), we get

(3.6) $(T_{q,\alpha}z, z) = (z, T_{q,\alpha}z) = (z, \mu z) = \overline{\mu}(z, z).$

On the other hand,

(3.7) $(T_{q,\alpha}z, z) = (\mu z, z) = \mu(z, z).$

It follows from (3.6) and (3.7) that

$$\mu(z, z) = \overline{\mu}(z, z), \quad (\mu - \overline{\mu})(z, z) = 0.$$

Since $z \neq 0$, we get $\mu = \overline{\mu}$.   □

LEMMA 3.3. *If μ_1 and μ_2 are two different eigenvalues of the operator $T_{q,\alpha}$ generated by q-FD system defined by (3.1)-(3.3), then the corresponding eigenfunctions θ and η are orthogonal.*

PROOF. Let μ_1 and μ_2 be two different real eigenvalues with corresponding eigenfunctions θ and η, respectively. From (3.5), we obtain

$$(T_{q,\alpha}\theta, \eta) = (\theta, T_{q,\alpha}\eta), (\mu_1\theta, \eta) = (\theta, \mu_2\eta)$$

$$(\mu_1 - \mu_2)(\theta, \eta) = 0.$$

Since $\mu_1 \neq \mu_2$, we obtain that $\theta(x)$ and $\eta(x)$ are orthogonal.   □

Now let $u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$, $z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in H$. Then, we define the Wronskian of $u(x)$ and $z(x)$ by

$$W(u, z)(x) = u_1(x)\mathcal{I}_{q,a^-}^{1-\alpha}z_2(x) - z_1(x)\mathcal{I}_{q,a^-}^{1-\alpha}u_2(x).$$

THEOREM 3.4. *The Wronskian of any solution of Eq. (3.1) is independent of x.*

PROOF. Let $u(x)$ and $z(x)$ be two solutions of Eq. (3.1). By Green's formula (3.4), we have

$$(T_{q,\alpha}u, z) - (u, T_{q,\alpha}z) = [u, z](a) - [u, z](0).$$

Since $T_{q,\alpha}u = \lambda u$ and $T_{q,\alpha}z = \lambda z$, we have

$$(\lambda u, z) - (u, \lambda z) = [u, z](a) - [u, z](0),$$

$$(\lambda - \overline{\lambda})(u, z) = [u, z](a) - [u, z](0).$$

Since $\lambda \in \mathbb{R}$, we have $[u, z](a) = [u, z](0) = W(u, \overline{z})(0)$, i.e., the Wronskian is independent of x.   □

COROLLARY 3.5. *If $u(x)$ and $z(x)$ are both solutions of Equation (3.1), then either $W(u, z)(x) = 0$ or $W(u, z)(x) \neq 0$ for all $x \in [0, a]$.*

THEOREM 3.6. *Any two solutions of Equation (3.1) are linearly dependent if and only if their Wronskian is zero.*

PROOF. Let $u(x)$ and $z(x)$ be two linearly dependent solutions of equation (3.1). Then, there exists a constant $k > 0$ such that $u(x) = k\ z(x)$. Hence

$$W(u, z) = \begin{vmatrix} u_1(x) & \mathcal{I}_{q,a^-}^{1-\alpha} u_2(x) \\ z_1(x) & \mathcal{I}_{q,a^-}^{1-\alpha} z_2(x) \end{vmatrix} = \begin{vmatrix} kz_1(x) & k\mathcal{I}_{q,a^-}^{1-\alpha} z_2(x) \\ z_1(x) & \mathcal{I}_{q,a^-}^{1-\alpha} z_2(x) \end{vmatrix} = 0.$$

Conversely, the Wronskian $W(u, z) = 0$ and therefore, $u(x) = kz(x)$, i.e., $u(x)$ and $z(x)$ are linearly dependent. □

Before proceeding further, we need the following auxiliary functions.

We introduce the function $\phi(x) := \begin{pmatrix} (\mathcal{I}_{q,a^-}^{\alpha} 1)(x) \\ (\mathcal{I}_{q,0^+}^{\alpha} 1)(x) \end{pmatrix}$. Further, the general solution of the equation $\tau_{q,\alpha}\psi = 0$, i.e.,

$$\begin{pmatrix} 0 & \mathcal{D}_{q,a^-}^{\alpha} \\ {}^c\mathcal{D}_{q,0^+}^{\alpha} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

is given by

$$\psi = \begin{pmatrix} \xi_1 \\ \xi_2 \varphi(\alpha, a, x) \end{pmatrix},$$

where

(3.8) $$\varphi(\alpha, a, x) = \frac{a^{\alpha-1} \left(\frac{qx}{a} : q\right)_{\alpha-1}}{\Gamma_q(\alpha)}.$$

LEMMA 3.7. *Let*

$$\Delta := c_{11}c_{12} - c_{11}c_{21}$$

and

(3.9) $$F_\lambda(f) := \{V - \lambda\omega\} f_\lambda,$$

where $V(x) := \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}$. *Assume* $\Delta \neq 0$. *Then on the space* $C(A_{t,\alpha}^*)$, *the q-FD system defined by (3.1)-(3.3) is equivalent to the integral equation*

$$f_\lambda(x) = -MF_\lambda(f) + A(x)T + B(x)Z,$$

where the coefficients M, A, T, B and Z are

$$M := \begin{pmatrix} 0 & \mathcal{I}^{\alpha}_{q,0^+} \\ \mathcal{I}^{\alpha}_{q,a^-} & 0 \end{pmatrix},$$

$$A(x) := \begin{pmatrix} \frac{c_{12}c_{22}}{\Delta} \\ -\frac{c_{21}c_{12}}{\Delta}\varphi(\alpha, a, x) \end{pmatrix},$$

$$T := -\mathcal{I}^{\alpha}_{q,a^-} F_{\lambda 1}(y)\mid_{x=0},$$

$$B(x) := \begin{pmatrix} \frac{c_{12}c_{21}}{\Delta} \\ -\frac{c_{21}c_{11}}{\Delta}\varphi(\alpha, a, x) \end{pmatrix},$$

$$Z := -\mathcal{I}^{1}_{q,0^+} F_{\lambda 2}(y)\mid_{x=a},$$

and the function $\varphi(\alpha, a, x)$ is defined in (3.8).

PROOF. Using fractional composition rules and (3.9), we can rewrite the equation (3.1) as follows:

$$\tau_{q,\alpha}[f_{\lambda}(x) + MF_{\lambda}(f)] = 0.$$

Thus, we get

$$f_{\lambda}(x) + MF_{\lambda}(f) = \begin{pmatrix} \xi_1 \\ \xi_2\varphi(\alpha, a, x) \end{pmatrix},$$

i.e.,

$$(3.10) \qquad f_{\lambda}(x) = -MF_{\lambda}(f) + \begin{pmatrix} \xi_1 \\ \xi_2\varphi(\alpha, a, x) \end{pmatrix}.$$

Now, we shall connect the coefficients ξ_i $(i = 1, 2)$ to the values c_{ij} $(i, j = 1, 2)$ in the boundary conditions (3.2)-(3.3). From the equation (3.10), we obtain

$$Kf_{\lambda}(x) = -KMF_{\lambda}(f) + K\begin{pmatrix} \xi_1 \\ \xi_2\varphi(\alpha, a, x) \end{pmatrix},$$

where $K := \begin{pmatrix} 0 & \mathcal{I}^{1-\alpha}_{q,a^-} \\ 1 & 0 \end{pmatrix}$. Then we have

$$\begin{pmatrix} \mathcal{I}^{1-\alpha}_{q,a^-}f_{\lambda 2} \\ f_{\lambda 1} \end{pmatrix} = -\begin{pmatrix} \mathcal{I}^{1}_{q,a^-} & 0 \\ 0 & \mathcal{I}^{\alpha}_{q,0^+} \end{pmatrix} F_{\lambda}(f) + \begin{pmatrix} \mathcal{I}^{1-\alpha}_{q,a^-}[\xi_2\varphi(\alpha, a, x)] \\ \xi_1 \end{pmatrix},$$

i.e,

$$\begin{pmatrix} \mathcal{I}^{1-\alpha}_{q,a^-}f_{\lambda 2} \\ f_{\lambda 1} \end{pmatrix} = \begin{pmatrix} -\mathcal{I}^{1}_{q,a^-}F_{\lambda 1}(f) \\ -\mathcal{I}^{\alpha}_{q,0^+}F_{\lambda 2}(f) \end{pmatrix} + \begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix}.$$

By virtue of (3.2) and (3.3), we conclude that

$$f_{\lambda 1}(0) = \xi_1,$$
$$f_{\lambda 1}(a) = -\mathcal{I}_{q,0^+}^{\alpha} F_{\lambda 2}(y)\,|_{x=a} + \xi_1,$$
$$\mathcal{I}_{q,a^-}^{1-\alpha} f_{\lambda 2}(0) = -\mathcal{I}_{q,a^-}^{1} F_{\lambda 1}(y)\,|_{x=0} + \xi_2,$$
$$\mathcal{I}_{q,a^-}^{1-\alpha} f_{\lambda 2}\left(\frac{a}{q}\right) = \xi_2.$$

This leads to the system of equations

$$c_{11}\xi_1 + c_{12}\xi_2 = -c_{12}, \ \ Tc_{21}\xi_1 + c_{22}\xi_2 = -c_{21}Z.$$

Since $\Delta \neq 0$, the solutions for coefficients $\xi_j, j = 1, 2$ is unique:

$$\xi_1 = \frac{c_{12}(c_{21}Z - c_{22}T)}{\Delta},$$
$$\xi_2 = \frac{c_{21}(c_{12}T - c_{11}Z)}{\Delta}.$$

We have finished the proof of the lemma. $\qquad\qquad\qquad\qquad$ □

Now, we prove the existence and uniqueness of eigenfunction of the regular q-FD system defined by (3.1)-(3.3). In the next result, we use the following notations:

$$A := \|A(x)\|_C, \ \ B := \|B(x)\|_C, \ \ S_\phi := \|\phi(x)\|_C,$$

where $\|.\|_C$ denotes the supremum norm on the space $C\left(A_{t,\alpha}^*, E\right).$

THEOREM 3.8. *Let $\alpha \in (0, 1)$ and assume $\Delta \neq 0$. Then unique continuous function y_λ for the regular q-FD system defined by (3.1)-(3.3) corresponding to each eigenvalue obeying*

$$(3.11) \qquad\qquad \|V - \lambda\omega\|_C \leq \frac{1}{S_\phi + A\|\phi(a)\|_C + Ba}$$

exists and such eigenvalue is simple.

PROOF. Let us define the mapping $L : C\left(A_{t,\alpha}^*, E\right) \to C\left(A_{t,\alpha}^*, E\right)$ by

$$Lf := -MF_\lambda(f) + A(x)T + B(x)Z.$$

Now, we show that the equation (3.1) can be interpreted as a fixed point condition on the space $C\left(A_{t,\alpha}^*, E\right).$ Using the following estimate

$$\|F_\lambda(g) - F_\lambda(h)\|_C \leq \|g - h\|_C \|V - \lambda\omega\|_C,$$

we conclude that

$$\begin{aligned}
\|Lg - Lh\|_C &\leq \|g - h\|_C \|V - \lambda\omega\|_C S_\phi + A \|g - h\|_C \|\phi(a)\|_C \\
&\quad + Ba \|g - h\|_C \|V - \lambda\omega\|_C \\
&= \|V - \lambda\omega\|_C \|g - h\|_C (S_\phi + A \|\phi(a)\|_C + Ba) \\
&= \Pi \|g - h\|_C,
\end{aligned}$$

where $\Pi = \|V - \lambda\omega\|_C (S_\phi + A \|\phi(a)\|_C + Ba)$. By the condition (3.11), the mapping L is a contraction on the space $C\left(A_{t,\alpha}^*, E\right)$ so it has a unique fixed point. Therefore, such eigenvalue is simple. □

CONCLUSION 3.9. In this paper, we study regular q-fractional Dirac systems. In this context, we investigate the properties of the eigenvalues and the eigenfunctions of this system. Finally, we give a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions.

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q-frakcijski sustavi Diracovog tipa

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SAŽETAK. Ovaj članak je posvećen proučavanju regularnih q-frakcijskih sustava Diracovog tipa. Proučavaju se svojstva svojstvenih vrijednosti i svojstvenih funkcija tih sustava. Korištenjem teorema o fiksnoj točki, daje se dovoljan uvjet na svojstvene vrijednosti za postojanje i jedinstvenost pridruženih svojstvenih funkcija.

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