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On the smallest integer vector at which a multivariable polynomial does not vanish

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Abstract. We prove that for any polynomial P of degree d in $\mathbb{C}[x_1,\dots,x_n]$ there exists a vector $(u_1,\dots,u_n)\in\mathbb{Z}^n$ such that $P(u_1,\dots,u_n)\neq 0$ and $\sum_{i=1}^n |u_i| \le \min\{d, \lfloor (d+n)/2\rfloor\}$. We also show that this bound is best possible. Similarly, for any $P\in\mathbb{C}[x_1,\dots,x_n]$ of degree d and any real number $p\ge \log 3/\log 2$ there is a vector $(u_1,\dots,u_n)\in\mathbb{Z}^n$ such that $P(u_1,\dots,u_n)\neq 0$ and $\sum_{i=1}^n |u_i|^p \le \max\{1+\lfloor d/2\rfloor^p, \lfloor (d+1)/2\rfloor^p\}$. The latter bound is also best possible for every $n\ge 2$.
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1. Introduction

Let d and n be positive integers, and let $P\in\mathbb{C}[x_1,\dots,x_n]$ be a polynomial in n variables of degree d. Then, by [6, Lemma 2.4], there is a vector $(v_1,\dots,v_n)\in\mathbb{Z}^n$ such that $P(v_1,\dots,v_n)\neq 0$ and

$$\max_{1\le i\le n} |v_i| \le \lfloor (d+1)/2\rfloor. \tag{1}$$

The proof of [6, Lemma 2.4] is straightforward by induction on n. Inequality (1) is then used in getting an upper bound for the number of fields of given degree and bounded discriminant in [6] and [12].

In fact, the following more general statement is also true (see [1]):

Theorem 1. *Let d and n be positive integers, and let $V_1,\dots,V_n$ be any sets containing at least $d+1$ complex numbers each. Then, for any $P\in\mathbb{C}[x_1,\dots,x_n]$ of degree d there is a vector $(v_1,\dots,v_n)\in V_1\times\dots\times V_n$ such that $P(v_1,\dots,v_n)\neq 0$.*

Theorem 1 implies upper bound (1) by choosing $V_1=\dots=V_d=\mathcal{S}_d$, where

$$\mathcal{S}_d=\{-\lfloor (d+1)/2\rfloor, -\lfloor (d+1)/2\rfloor+1,\dots,\lfloor (d+1)/2\rfloor\}\subset\mathbb{Z}, \tag{2}$$

since $|V_i|=|\mathcal{S}_d|=2\lfloor (d+1)/2\rfloor+1\ge d+1$ for $i=1,\dots,n$. Theorem 1, whose short proof is included here for the sake of completeness, is a version of the so-called combinatorial Nullstellensatz which has many applications. See, for instance,

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[2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15] for stronger versions of Theorem 1 and for its applications to graph theory, sumsets, finite fields, etc.

We emphasize that bound (1) does not depend on n. In the next section, we will give an example showing that bound (1) is sharp.

In this note, we first look at the same problem for the quantity $\sum_{i=1}^{n}|u_i|$ instead of $\max_{1\le i\le n}|u_i|$ and prove the following:

Theorem 2. *Let d and n be positive integers. Then, for any polynomial $P \in \mathbb{C}[x_1,\dots,x_n]$ of degree d there is a vector $(u_1,\dots,u_n)\in\mathbb{Z}^n$ such that $P(u_1,\dots,u_n)\neq 0$ and*

$$\sum_{i=1}^{n}|u_i| \le \lfloor (d+n)/2 \rfloor \tag{3}$$

if $d \ge n$ or

$$\sum_{i=1}^{n}|u_i| \le d \tag{4}$$

if $d < n$.

In Section 2, we will give two examples illustrating that for any positive integers d, n bounds (3) and (4) are sharp.

Inequalities (1), (3) and (4) give optimal bounds for the norms L^{∞} and L^1 of a vector in $\mathbb{Z}^n$ at which a complex polynomial of degree d in n variables does not vanish. In the next theorem, we consider the same problem for the norm L^p:

Theorem 3. *Let d and n be positive integers, and let p be a real number satisfying*

$$p \ge \frac{\log 3}{\log 2} = 1.584962\dots. \tag{5}$$

Then, for any polynomial $P\in\mathbb{C}[x_1,\dots,x_n]$ of degree d there is a vector $(u_1,\dots,u_n)$ $\in\mathbb{Z}^n$ such that $P(u_1,\dots,u_n)\neq 0$ and

$$\sum_{i=1}^{n}|u_i|^p \le \max\{1+\lfloor d/2\rfloor^p, \lfloor (d+1)/2\rfloor^p\}. \tag{6}$$

Note that the right-hand side of (6) equals $1+(d/2)^p$ for d even, and $((d+1)/2)^p$ for d odd, by the inequality

$$a^p + b^p \le (a+b)^p, \tag{7}$$

where $a, b \ge 0$ and $p \ge 1$. (Select $a = (d-1)/2$, $b = 1$ in (7) and d odd on the right-hand side of (6).) Unlike (3), bound (6) is independent of n.

In Section 2, we will give some examples showing that upper bound (6) for the L^p-norm is sharp for any pair $(d,n)\in\mathbb{N}^2$ (except for the pair $(d,n) = (2k,1)$, where $k\in\mathbb{N}$, when the bound $|u_1|^p \le (d/2)^p$ is tight by (3)), and that lower bound (5) on p cannot be relaxed.

Finally, in Section 3, we will prove theorems 1, 2 and 3.

2. Examples showing that bounds are sharp

We first show bound (1) is sharp for every pair $(d, n) \in \mathbb{N}^2$. To see this, for a positive integer d we define

$$\psi_d(x) = \prod_{\alpha \in S_d} (x - \alpha), \tag{8}$$

where

$$S_d = \{-k+1, -k+2, \dots, k-2, k-1\} \tag{9}$$

if $d = 2k-1$ with $k \in \mathbb{N}$, and

$$S_d = \{-k+1, \dots, k-1, k\} \tag{10}$$

if $d = 2k$ with $k \in \mathbb{N}$. Then, by (8), (9) and (10), we have

$$\psi_1(x) = x,\ \psi_2(x) = x(x-1),\ \psi_3(x) = x(x-1)(x+1),$$

etc. Note that $\deg f_d = d$ for each $d \in \mathbb{N}$ and

$$\min_{\alpha \in \mathbb{Z} \setminus S_d} |\alpha| = \lfloor (d+1)/2 \rfloor. \tag{11}$$

With this notation, the polynomial

$$P(x_1, \dots, x_n) = \psi_d(x_1) + \dots + \psi_d(x_n) \in \mathbb{Z}[x_1, \dots, x_n] \tag{12}$$

of degree d satisfies $P(v_1, \dots, v_n) = 0$ if $v_1, \dots, v_n \in S_d$. Consequently, $P(v_1, \dots, v_n) \neq 0$ for some $(v_1, \dots, v_n) \in \mathbb{Z}^n$ only if at least one v_i, $i = 1, \dots, n$, does not belong to the set S_d. This yields

$$\max_{1 \le i \le n} |v_i| \ge \lfloor (d+1)/2 \rfloor$$

by (11). Thus, bound (1) is tight.

To show that bound (3) is best possible for $d \ge n$ we consider the polynomial

$$P(x_1, \dots, x_n) = x_1 \cdots x_{n-1} \psi_{d-n+1}(x_n) \in \mathbb{Z}[x_1, \dots, x_n]$$

of degree d. Notice that $P(u_1, \dots, u_n) = 0$ for $(u_1, \dots, u_n) \in \mathbb{Z}^n$ if $u_i = 0$ for at least one $i = 1, \dots, n-1$ or if $u_n \in S_{d-n+1}$. It follows that $P(u_1, \dots, u_n) \neq 0$ for $(u_1, \dots, u_n) \in \mathbb{Z}^n$ only if $|u_i| \ge 1$ for $i = 1, \dots, n-1$ and also $|u_n| \ge \lfloor (d-n+2)/2 \rfloor$ by (11). Then,

$$\sum_{i=1}^{n} |u_i| \ge n - 1 + \lfloor (d-n+2)/2 \rfloor = n + \lfloor (d-n)/2 \rfloor = \lfloor (d+n)/2 \rfloor,$$

which shows that bound (3) is sharp.

Similarly, in the case $d < n$, we can select, for instance,

$$P(x_1, \dots, x_n) = x_1 \dots x_{d-1}(x_d + \dots + x_n) \in \mathbb{Z}[x_1, \dots, x_n].$$

Then, $P(u_1, \dots, u_n) \neq 0$ for some $(u_1, \dots, u_n) \in \mathbb{Z}^n$ only if $u_i \neq 0$ for each $i = 1, \dots, d-1$ and $u_i \neq 0$ for at least one i in the range $d \le i \le n$. Thus, at least d

integers u_i $(1 \le i \le n)$ are nonzero, and so $\sum_{i=1}^n |u_i| \ge d$. This example shows that bound (4) is sharp for any pair of positive integers satisfying $d < n$.

Next, to show that bound (6) is sharp we first observe that for the polynomial P defined in (12) and any vector $(u_1, \dots, u_n) \in \mathbb{Z}^n$ satisfying $P(u_1, \dots, u_n) \ne 0$ we must have

$$\sum_{i=1}^n |u_i|^p \ge \max_{1 \le j \le n} |u_j|^p \ge \lfloor (d+1)/2 \rfloor^p.$$

Likewise, for the polynomial

$$P(x_1, \dots, x_n) = (x_1 + \dots + x_{n-1})\psi_{d-1}(x_n) \in \mathbb{Z}[x_1, \dots, x_n],$$

where $d, n \ge 2$, and for $(u_1, \dots, u_n) \in \mathbb{Z}^n$ such that $P(u_1, \dots, u_n) \ne 0$ at least one u_i, where $i = 1, \dots, n-1$, must be nonzero, and $|u_n| \ge \lfloor d/2 \rfloor$ by (11). This implies

$$\sum_{i=1}^n |u_i|^p \ge 1 + \lfloor d/2 \rfloor^p,$$

and hence bound (6) is tight for every pair $(d, n) \in \mathbb{N}^2$, where $n \ge 2$. For $n = 1$, bound $|u_1|^p \le \lfloor (d+1)/2 \rfloor^p$ is tight for d odd by (3), whereas the right-hand side of (6) is $1 + (d/2)^p$ for d even.

Finally, to show that condition (5) cannot be relaxed we consider

$$P(x_1, \dots, x_n) = x_1 x_2 x_3 (x_4 + \dots + x_n) \in \mathbb{Z}[x_1, \dots, x_n],$$

where $n \ge 4$. This polynomial P is of degree $d = 4$. For the smallest L^p-norm of the integer vector $(u_1, \dots, u_n) \in \mathbb{Z}^n$ satisfying $P(u_1, \dots, u_n) \ne 0$ we have $\sum_{i=1}^n |u_i|^p = 4$, since $u_1, u_2, u_3 \ne 0$ and $u_i \ne 0$ for at least one $i \in \{4, \dots, n\}$. Therefore, by (6) with $d = 4$, it follows that $4 \le 1 + 2^p$. This is equivalent to $p \ge \log 3 / \log 2$, and so condition (5) cannot be relaxed.

3. Proofs

We begin with two simple lemmas.

Lemma 1. *Let $P \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial of degree d and let S be the set of all $\alpha \in \mathbb{C}$ for which $P(x_1, \dots, x_{n-1}, \alpha)$ is zero identically. Then, $s = |S| \le d$ and*

$$P(x_1, \dots, x_n) = Q(x_1, \dots, x_n) \prod_{\alpha \in S} (x_n - \alpha) \tag{13}$$

for some $Q \in \mathbb{C}[x_1, \dots, x_n]$ of degree $d - s$.

Here, for $s = 0$, which happens if S is empty, we have $P = Q$ and the last product in (13) is omitted. Also, in principle, Q can be divisible by $x_n - \alpha$ for some $\alpha \in S$.

Proof. Observe that for any $\alpha \in \mathbb{C}$ the polynomial $P(x_1, \dots, x_{n-1}, \alpha)$ is either not zero identically or, if it is zero identically, P can be written in the form

$$P(x_1, \dots, x_n) = (x_n - \alpha) Q(x_1, \dots, x_n) \tag{14}$$

for some $Q \in \mathbb{C}[x_1, \dots, x_n]$ of degree $d-1$. Indeed, if $P(x_1, \dots, x_{n-1}, \alpha)$ is zero identically, then

$$P(x_1, \dots, x_{n-1}, x_n) = P(x_1, \dots, x_{n-1}, x_n) - P(x_1, \dots, x_{n-1}, \alpha)$$

is divisible by $x_n - \alpha$, which implies (14). Now, (14) implies (13) by the definition of S, and the inequality $|S| \le d$ holds by degree consideration. □

Lemma 2. *Let $a, b \ge 1$ be real numbers. Then, for any real number $p \ge 1$ we have*

$$\lfloor a \rfloor^p + \lfloor b \rfloor^p \le 1 + \lfloor a + b - 1 \rfloor^p. \tag{15}$$

Furthermore, for any real $a, b \ge 1$ and

$$p \ge \frac{\log 3}{\log 2}$$

we have

$$a^p + b^p + 1 \le (a+b)^p. \tag{16}$$

Proof. Set $u = \lfloor a \rfloor$ and $v = \lfloor b \rfloor$. Then, the left-hand side of (15) is $u^p + v^p$, whereas its right-hand side is greater than or equal to $1 + (u+v-1)^p$. The inequality $u^p + v^p \le 1 + (u+v-1)^p$ is the equality for $p = 1$. Assume that $p > 1$. Fix $t = u + v \ge 2$. Without loss of generality, we can assume that $1 \le u \le t/2$. Then, the function

$$g(u) = (t-1)^p + 1 - u^p - (t-u)^p$$

is increasing in u in the interval $u \in [1, t/2]$ (which can be a singleton), so $g(u) \ge g(1) = 0$ for each $u \in [1, t/2]$. This proves (15).

For the proof of (16) we fix $t = a + b \ge 2$. Without loss of generality, we can assume that $1 \le a \le t/2$. For any $p > 1$ the function

$$h(a) = t^p - 1 - a^p - (t-a)^p$$

is increasing in a in the interval $a \in [1, t/2]$, so

$$h(a) \ge h(1) = t^p - (t-1)^p - 2$$

for $a \in [1, t/2]$ (the interval can be a singleton). Thus, in order to complete the proof of (16) is remains to verify the inequality

$$t^p - (t-1)^p - 2 \ge 0$$

for $t \ge 2$ and $p \ge \log 3/\log 2$.

It is clear that for each fixed $p > 1$ the function $t^p - (t-1)^p - 2$ is increasing in $t \in [2, \infty)$, since its derivative in t is positive. Consequently, for $t \ge 2$ and $p \ge \log 3/\log 2$ we obtain

$$t^p - (t-1)^p - 2 \ge 2^p - (2-1)^p - 2 = 2^p - 3 \ge 0,$$

which is the desired conclusion. □

Proof of Theorem 1. The result is clear for $n = 1$. Let $P(x_1, \dots, x_n)$ be a polynomial of degree d in $n \geq 2$ variables. Suppose the assertion of the theorem is true for polynomials in at most $n-1$ variables. Write P as in (13). By $|V_n| \geq d+1$, there exists $v_n \in V_n$ such that $P(x_1, \dots, x_{n-1}, v_n)$ is not zero identically. As $P(x_1, \dots, x_{n-1}, v_n)$ is a polynomial of degree at most d in at most $n-1$ variables, by the induction hypothesis, for each $i = 1, \dots, n-1$ there are $v_i \in V_i$ such that $P(v_1, \dots, v_{n-1}, v_n) \neq 0$. (If $P(x_1, \dots, x_{n-1}, v_n)$ does not depend on the variable x_i, we can assign an arbitrary value of V_i to the corresponding v_i.) □

Proof of Theorem 2. We will prove the inequality

$$\sum_{i=1}^{n} |u_i| \leq \min\{d, \lfloor (d+n)/2 \rfloor\}, \tag{17}$$

which is a combination of (3) and (4), by induction on n. The result is clear for $n = 1$, since the set $\mathcal{S}_d$ defined in (2) has $2\lfloor (d+1)/2 \rfloor + 1 \geq d+1$ elements, and so the polynomial $P(x_1)$ of degree d does not vanish for some $x_1 = u_1 \in \mathcal{S}_d$. Furthermore, by the choice of $\mathcal{S}_d$, we have

$$|u_1| \leq \lfloor (d+1)/2 \rfloor = \min\{d, \lfloor (d+1)/2 \rfloor\}.$$

Let $P(x_1, \dots, x_n)$ be a polynomial of degree d in $n \geq 2$ variables. Let S be the set as in Lemma 1, i. e. the set of $\alpha \in \mathbb{C}$ for which $P(x_1, \dots, x_{n-1}, \alpha)$ is zero identically when $\alpha \in S$. Finally, let β be the integer with the smallest absolute value that is not in S. (If there are two such integers, then β is any of those two.) Then, it is clear that

$$|\beta| \leq \lfloor (s+1)/2 \rfloor, \tag{18}$$

where s is an integer in the range $0 \leq s \leq |S|$. Note that (18) holds with $\beta = 0$ if $0 \notin S$.

By the choice of β, the polynomial $P(x_1, \dots, x_{n-1}, \beta)$ is not zero identically. Thus, inserting $x_n = \beta$ into (13) we see that $Q(x_1, \dots, x_{n-1}, \beta)$ is a nonzero polynomial of degree $d_0 \leq d - |S| \leq d - s$ in $n_0 \leq n-1$ variables. By the induction hypothesis, we can choose $(u_1, \dots, u_{n-1}) \in \mathbb{Z}^{n-1}$ (where $u_i = 0$ if $Q(x_1, \dots, x_{n-1}, \beta)$ does not depend on the variable x_i) so that $Q(u_1, \dots, u_{n-1}, \beta) \neq 0$ and

$$\sum_{i=1}^{n-1} |u_i| \leq \min\{d_0, \lfloor (d_0+n_0)/2 \rfloor\} \leq \min\{d-s, \lfloor (d-s+n-1)/2 \rfloor\}. \tag{19}$$

(Here, it is possible that $d_0 = d - s = 0$, but (17) also holds for $d = 0$ if a zero degree polynomial is a nonzero constant.)

Note that $P(u_1, \dots, u_n, \beta) \neq 0$ by (13) and $(u_1, \dots, u_{n-1}, \beta) \in \mathbb{Z}^n$ by the choice of β. Now, combining (19) with (18) we deduce that

$$\begin{aligned}\sum_{i=1}^{n-1} |u_i| + \beta &\leq \min\{d-s, \lfloor (d-s+n-1)/2 \rfloor\} + \lfloor (s+1)/2 \rfloor \\ &\leq \min\{d, \lfloor (d+n)/2 \rfloor\},\end{aligned}$$

because $d-s+\lfloor (s+1)/2\rfloor \le d$ for each $s \ge 0$ and

$$\lfloor (d-s+n-1)/2\rfloor + \lfloor (s+1)/2\rfloor \le \lfloor (d+n)/2\rfloor.$$

This completes the proof of (17) for the vector

$$(u_1,\dots,u_{n-1},u_n) = (u_1,\dots,u_{n-1},\beta) \in \mathbb{Z}^n,$$

since $P(u_1,\dots,u_n) \ne 0$. □

Proof of Theorem 3. Observe that the result holds for $n=1$ by (1). Let $P(x_1,\dots,x_n)$ be a polynomial of degree d in $n \ge 2$ variables. If P is not divisible by x_n, we can select $u_n = 0$. Then, $P(x_1,\dots,x_{n-1},0)$ is a polynomial of degree d in at most $n-1$ variable, which is not zero identically. Then, the required inequality follows by induction on n.

Now, again let S be the set as in Lemma 1, i. e. the set of $\alpha \in \mathbb{C}$ for which $P(x_1,\dots,x_{n-1},\alpha)$ is zero identically when $\alpha \in S$, and let β be the integer with the smallest absolute value that is not in S. Then, as $\beta \ne 0$ by $0 \in S$, we must have

$$1 \le |\beta| \le \lfloor (s+1)/2\rfloor,$$

where $1 \le s \le |S|$. Moreover, in view of $n \ge 2$, we must have $|S| < d$, so that $1 \le s \le d-1$.

By the choice of β, the polynomial $P(x_1,\dots,x_{n-1},\beta)$ is not zero identically. Thus, inserting $x_n = \beta$ into (13) we find that $Q(x_1,\dots,x_{n-1},\beta)$ is a nonzero polynomial of degree $d_0 \le d-|S| \le d-s$ in $n_0 \le n-1$ variables. By the induction hypothesis, we can choose $(u_1,\dots,u_{n-1}) \in \mathbb{Z}^{n-1}$ (where $u_i = 0$ if $Q(x_1,\dots,x_{n-1},\beta)$ does not depend on the variable x_i) so that $Q(u_1,\dots,u_{n-1},\beta) \ne 0$ and

$$\sum_{i=1}^{n-1} |u_i|^p \le \max\{1+\lfloor (d-s)/2\rfloor^p, \lfloor (d-s+1)/2\rfloor^p\}.$$

So, in order to complete the proof of (6) it remains to verify that for each integer s in the range $1 \le s \le d-1$ the sum

$$\lfloor (s+1)/2\rfloor^p + \max\{1+\lfloor (d-s)/2\rfloor^p, \lfloor (d-s+1)/2\rfloor^p\}$$

does not exceed

$$\max\{1+\lfloor d/2\rfloor^p, \lfloor (d+1)/2\rfloor^p\}.$$

Firstly, we note that by (15) with $a=(s+1)/2$ and $b=(d-s+1)/2$, we have

$$\lfloor (s+1)/2\rfloor^p + \lfloor (d-s+1)/2\rfloor^p \le 1+\lfloor d/2\rfloor^p.$$

What is left is to show that

$$\lfloor (s+1)/2\rfloor^p + \lfloor (d-s)/2\rfloor^p + 1 \le \max\{1+\lfloor d/2\rfloor^p, \lfloor (d+1)/2\rfloor^p\}. \tag{20}$$

Suppose first that d is even. Then, the left-hand side of inequality (20) is either

$$\Big(\frac{s+1}{2}\Big)^p + \Big(\frac{d-s-1}{2}\Big)^p + 1$$

(if s is odd), or

$$\left(\frac{s}{2}\right)^p + \left(\frac{d-s}{2}\right)^p + 1$$

(if s is even), whereas the right-hand side of (20) is $1 + (d/2)^n$. Hence, inequality (20) holds by (7) with $a = (s+1)/2$, $b = (d-s-1)/2$ or with $a = s/2$, $b = (d-s)/2$.

We reduce to proving (20) for d odd. Then, the right-hand side of (20) is equal to $((d+1)/2)^p$. Recall that $1 \le s \le d-1$. Now, in the case when $d - s \ge 2$, we obtain

$$\lfloor (s+1)/2 \rfloor^p + \lfloor (d-s)/2 \rfloor^p + 1 \le \left(\frac{s+1}{2}\right)^p + \left(\frac{d-s}{2}\right)^p + 1 \le \left(\frac{d+1}{2}\right)^p$$

by (16) with $a = (s+1)/2$ and $b = (d-s)/2$. (We remark that this is the only place where we use condition (5) on p.)

Likewise, in the case when $d - s = 1$, we derive that

$$\lfloor (s+1)/2 \rfloor^p + \lfloor (d-s)/2 \rfloor^p + 1 = \left(\frac{s}{2}\right)^p + 1 = \left(\frac{d-1}{2}\right)^p + 1 \le \left(\frac{d+1}{2}\right)^p$$

by (7) with $a = (d-1)/2$, $b = 1$. This completes the proof of (20) and finishes the proof of Theorem 3. □

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