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Spectral expansion for singular conformable Sturm-Liouville problem

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Abstract. In this study, the spectral function for the singular conformable Sturm-Liouville problem is demonstrated. Further, we establish a Parseval equality and a spectral expansion formula by means of the spectral function.

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1. Introduction

Recently, a new definition of fractional derivative named conformable fractional derivative, which is based on the classical definition of derivative was introduced ([34]). Conformable derivative aims to extend the concept of derivative and provide new perspectives for differential equations via the concept of conformable differential equations ([21]). Examples of these perspectives can be found e.g. in ([2, 3, 26, 35, 1, 14, 15, 27, 28, 9, 10, 11, 24, 41]).

Conformable calculus started with the work [34] in which conformable derivative was presented. Next, the right and left conformable derivatives, the fractional chain rule and the fractional integrals of higher orders were put forward by Abdeljawad ([1]). Since then, this topic has attracted the attention of many researchers (see, e.g. [34, 16, 19, 17, 18, 40, 23, 22, 44, 33, 25, 45, 13]). In ([28]), Gulsen et al. studied the conformable Sturm-Liouville equation with separated boundary conditions on an arbitrary time scale $\mathbb{T}$ and extended some main spectral properties of the standard Sturm-Liouville equation to the conformable fractional case. In ([2]), the concept of Wronskian for conformable linear differential equations with variable coefficients was given, and an Abel formula for fractional equations with variable coefficients was obtained. In ([3]), the conformable heat equation was solved. In ([26]), the existence and uniqueness theorems of consecutive linear conformable equations were studied. In ([35]), some conformable equations are solved. The work in ([1]) deals with the concepts of the chain rule for conformable derivative, conformable partial integration, the conformable Gronwall inequality, a conformable exponential function and Laplace transformation. In ([14]), for the second-order conformable equations, the

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methods of order reduction and the change of constants are discussed and general solutions of differential equations with constant coefficients and of Cauchy-Euler type are given. Then, conformable problems and the properties of these problems are examined. In ([15]), fractional differential equations with proportional derivatives and their properties are examined. In ([45]), Wang et al. introduced fractional Sobolev spaces on time scales, characterized them, and defined weak conformable fractional derivatives. In ([49]), the authors provided a new version of the Gronwall inequality within the framework of the generalized proportional fractional derivative.

On the other hand, in ([27]), Dirac systems defined on the time scale were studied and some spectral properties of these problems were examined. Recently, Allahverdiev et al. have considered the conformable fractional Sturm-Liouville boundary-value problem ([10]). They proved the existence and uniqueness theorem for this equation, and constructed the associated Green function of this problem. Furthermore, the authors studied one-dimensional conformable fractional Dirac systems ([7, 9, 11]).

Today it is widely accepted that spectral expansion theorems are useful in the fields of science and engineering. If a partial differential equation is solved by separation of variables (i.e., by the Fourier method), then the expansion of an arbitrary function as a series of eigenfunctions and the completeness properties are obtained. The first study done for the spectral expansion problem is that of Weyl's in [46]. Later, this problem was investigated by many authors with different techniques. Thus there are a lot of studies on eigenfunction expanding problems in the literature (see [38, 29, 30, 4, 5, 6, 7, 8, 9, 10, 11, 12, 31, 32, 39, 47, 48, 37, 42, 43, 20]).

The primary aim of this study is to prove the existence of a spectral function for a singular conformable Sturm-Liouville equation of the form

$$-T_{\alpha}^{2}y(t)+v(t)y(t)=\lambda y(t),\ 0< t<\infty, \tag{1}$$

where λ is a complex parameter, and $v(.)$ is a real-valued conformable locally integrable function on $[0,\infty)$. Our work can be summarized as follows. In Section 2, some necessary concepts and properties are reviewed. In Section 3, we construct the resolvent in view of the Green function of the regular problem. We show that a regular conformable Sturm-Liouville operator has a compact resolvent, thus it has a purely discrete spectrum. Finally, in Section 4, the existence of a spectral function for the singular conformable Sturm-Liouville problem is proved. The Parseval equality and a spectral expansion formula by means of the spectral function of singular conformable Sturm-Liouville problem are constructed.

2. Preliminaries

In this section, we recall some basic definitions and properties related to conformable calculus and operator theory. For more details, the reader may refer to [34, 21, 2, 3, 26, 35, 1, 36, 39]. Throughout this paper, we will fix $\alpha\in(0,1)$.

Definition 1. *Let $0<\alpha<1$. For a function $f:(0,\infty)\to\mathbb{R}:=(-\infty,\infty)$, the conformable derivative of order α of f at $t>0$ is defined by*

$$T_{\alpha}f(t)=\lim_{\varepsilon\to0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \tag{2}$$

and the conformable derivative at 0 *is defined by*

$$(T_\alpha f)(0) = \lim_{t \to 0^+} (T_\alpha f(t)).$$

Definition 2. *The left conformable derivative of order* α *of a function* $f : [a, \infty) \to \mathbb{R}$ *is defined by*

$$(T_\alpha^a\ f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon\,(t-a)^{1-\alpha}) - f\,(t)}{\varepsilon},\ 0 < \alpha \leq 1. \tag{3}$$

When $a = 0$*, we write* T_α*. If* $(T_\alpha f)(t)$ *exists on* (a, b)*, then*

$$(T_\alpha^a f)(a) = \lim_{t \to a^+} (T_\alpha^a f)(t).$$

Definition 3. *The right conformable derivative of order* α *of* $f : (-\infty, b]$ *is defined by*

$$({}_{\alpha}^{b}Tf)(t) = -\lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(b-t)^{1-\alpha}) - f(t)}{\varepsilon}, \tag{4}$$

where $0 < \alpha \leq 1$ *and* $({}^bT_\alpha f)(t) = \lim_{t \to b^-} ({}^bT_\alpha f)(t)$.

In the next lemma, we consider some properties of the conformable derivative.

Lemma 1. *Let* f, g *be conformable differentiable of order* α $(0 < \alpha \leq 1)$ *at a point* t*. Then*

(*i*) $T_\alpha\,(\lambda f + \delta g) = \lambda T_\alpha\,(f) + \delta T_\alpha\,(g)$*, where* $\lambda, \delta \in \mathbb{R}$.

(*ii*) $T_\alpha\,(fg) = fT_\alpha\,(g) + gT_\alpha\,(f)$.

(*iii*) $T_\alpha \left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(*iv*) *If* f *is differentiable, then* $T_\alpha^a\,(f)\,(t) = (t-a)^{1-\alpha}\,f'\,(t)$.

(*v*) $T_\alpha\,(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$.

Next, we recall the concept of conformable integral.

Definition 4. *The conformable integral of order* α $(0 < \alpha \leq 1)$ *of a function* f : $[a, \infty)$ *is defined by*

$$(I_\alpha^a f)\,(t) = \int\limits_a^t f(x) d\alpha\,(x, a) = \int\limits_a^t (x-a)^{\alpha - 1} f(x) dx.$$

Similarly, in the right case, we have

$$({}^bI_\alpha f)\,(t) = \int\limits_t^b f(x) d\alpha\,(b, x) = \int\limits_t^b (b-x)^{\alpha - 1} f(x) dx.$$

Lemma 2. *Assume that f is a continuous function on (a,∞) and $0<\alpha<1$. Then, we have*

$$T_{\alpha}^{a} I_{\alpha}^{a} f(t) = f(t),$$

for all $t>a$.

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ be two functions such that f and g are conformable differentiable. Thus we have*

$$\int_a^b f(t) T_{\alpha}^{a}(g)(t)\, d\alpha(t,a) + \int_a^b g(t) T_{\alpha}^{a}(f)(t)\, d\alpha(t,a) = f(b)g(b) - f(a)g(a).$$

Let $L_{\alpha}^{2}(0,b)$ $(b\le+\infty)$ be the space of all complex-valued functions f defined on $[0,b)$, where

$$\|f\| := \left(\int_0^b |f(t)|^2 d_{\alpha}(t)\right)^{1/2} = \left(\int_0^b |f(t)|^2 t^{\alpha-1} dt\right)^{1/2} < +\infty,$$

and $d_{\alpha}(t) := d\alpha(t,0) = t^{\alpha-1}dt$.

The space $L_{\alpha}^{2}(0,b)$ is a Hilbert space (see [34]) with the inner product

$$(f,g) := \int_a^b f(t)\overline{g(t)} d_{\alpha}(t), \;\; f,g\in L_{\alpha}^{2}(0,b).$$

The conformable α-Wronskian of x and y is defined by

$$W_{\alpha}(x,y)(t) = x(t)T_{\alpha}y(t) - y(t)T_{\alpha}x(t),\;\; t\in[0,b). \tag{5}$$

Definition 5. *A function $M(t,x)$ of two variables with $0<t,\ x<b$ is called the $\alpha-$Hilbert-Schmidt kernel if*

$$\int_0^b\int_0^b |M(t,x)|^2 d_{\alpha}(t) d_{\alpha}(x) < +\infty.$$

We denote by l^2 the aggregate of all sequences $x=(x_1,x_2,...)$ of complex numbers where

$$\sum_{n=1}^{\infty} |x_n|^2 < +\infty.$$

The sequence space l^2 is a Hilbert space (see [39]) with the inner product

$$< x,y > := \sum_{n=1}^{\infty} x_n\overline{y_n}, \text{ where } x,y\in l^2.$$

Now we have a

Theorem 2 (see [39]). *If*

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < +\infty, \tag{6}$$

then the operator A defined by the formula

$$A\{x_i\} = \{y_i\}, \ i \in \mathbb{N} := \{1, 2, 3, ...\},$$

is compact on the sequence space l^2, where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k, \ i \in \mathbb{N}. \tag{7}$$

Definition 6 (see [36])**.** *A function f defined on an interval $[a, b]$ is said to be of bounded variation if there is a continual $C > 0$ as*

$$\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \leq C$$

for every partition

$$a = t_0 < t_1 < ... < t_n = b \tag{8}$$

of $[a, b]$ by points of subdivision $x_0, x_1, ..., x_n$.

Definition 7 (see [36])**.** *Let f be a function of bounded variation. Assume that f is a function of bounded variation. By the total variation of f on $[a, b]$, denoted by $\overset{b}{\underset{a}{V}}(f)$, we mean the quantity*

$$\overset{b}{\underset{a}{V}}(f) := \sup \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|,$$

in which the least upper bound is taken over all (finite) partitions (2) of the interval $[a, b]$.

Now we recall the following renowned theorems of Helly.

Theorem 3 (see [36])**.** *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real, nondecreasing functions defined on a finite interval $a \leq \lambda \leq b$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda), \ a \leq \lambda \leq b.$$

Theorem 4 (see [36])**.** *Assume that $(w_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of real, nondecreasing functions defined on a finite interval $a \leq \lambda \leq b$, and suppose that*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda), \ a \leq \lambda \leq b.$$

If f is any continuous function on $a \leq \lambda \leq b$, then we have

$$\lim_{n\to\infty} \int_a^b f(\lambda)\, dw_n(\lambda) = \int_a^b f(\lambda)\, dw(\lambda).$$

3. Regular conformable Sturm-Liouville problem

In this section, we construct the Green function and prove that a regular conformable Sturm-Liouville operator has a compact resolvent, thus it has a purely discrete spectrum, and we get a Parseval equality for this operator.

We consider the regular conformable Sturm-Liouville equation defined by

$$-T_\alpha^2 y(t) + v(t)y(t) = \lambda y(t),\ 0< t <b < \infty, \tag{9}$$

where $T_\alpha^2 := T_\alpha \circ T_\alpha$. Let $y(t,\lambda)$ satisfy the boundary conditions

$$y(0,\lambda)\cos\beta + T_\alpha y(0,\lambda)\sin\beta = 0, \tag{10}$$

$$y(b,\lambda)\cos\gamma + T_\alpha y(b,\lambda)\sin\gamma = 0,\ \ \gamma,\beta \in \mathbb{R}, \tag{11}$$

in which λ is a complex eigenvalue parameter, $v(.)$ is a real-valued function defined on $[0,\infty)$ and $v \in L^1_{\alpha,loc}(0,\infty)$, where

$$L^1_{\alpha,loc}(0,\infty) := \left\{ f : [0,\infty) \to \mathbb{C} : \int_0^b |f(t)|\, d_\alpha(t) < \infty, \forall b \in (0,\infty) \right\}.$$

We denote by $\theta(t,\lambda)$ and $\psi(t,\lambda)$ the solutions of (9) subject to the initial conditions

$$\theta(0,\lambda) = \sin\beta, \qquad T_\alpha\theta(0,\lambda) = -\cos\beta, \tag{12}$$

$$\psi(b,\lambda) = \sin\gamma, \qquad T_\alpha\psi(b,\lambda) = -\cos\gamma. \tag{13}$$

In this way, the Green *function of* the problem defined by $(9)-(11)$ is given by

$$G(t,x,\lambda) = \begin{cases} \frac{\psi(t,\lambda)\theta(x,\lambda)}{W(\theta,\psi)},\ 0 \le x < t \\ \\ \frac{\theta(t,\lambda)\psi(x,\lambda)}{W(\theta,\psi)},\ t < x < b. \end{cases} \tag{14}$$

(see [38, 10]). In what follows, without loss of generality, we assume that $\lambda = 0$ is not an eigenvalue of problem $(9)-(11)$. Hence, by (14), we have

$$G(t,x) = G(t,x,0) = \begin{cases} \frac{\psi(t)\theta(x)}{W(\theta,\psi)},\ 0 \le x < t \\ \\ \frac{\theta(t)\psi(x)}{W(\theta,\psi)},\ t < x < b. \end{cases} \tag{15}$$

Theorem 5. *$G(t,x)$ defined by (15) is an $\alpha-$Hilbert-Schmidt kernel.*

Proof. By the upper half of formula (14), we get

$$\int_0^b d_\alpha(t)\int_0^t |G(t,x)|^2 d_\alpha(x) < +\infty;$$

and by the lower half of (14), we have

$$\int_0^b d_\alpha(t)\int_t^b |G(t,x)|^2 d_\alpha(x) < +\infty$$

because the inner integral exists and is a product $\theta(x)\psi(t)$, and these products belong to $L^2_\alpha(0,b)\times L^2_\alpha(0,b)$ since each of the factors belongs to $L^2_\alpha(0,b)$. Then, we obtain

$$\int_0^b\int_0^b |G(t,x)|^2 d_\alpha(t)d_\alpha(x) < +\infty. \tag{16}$$

□

Theorem 6. *The operator* $\mathbf{S}$ *defined by the formula*

$$(\mathbf{S}f)(t) = \int_0^b G(t,x)f(x)d_\alpha(x)$$

is compact and self-adjoint on $L^2_\alpha(0,b)$.

Proof. Let $\phi_i = \phi_i(x)$ $(i \in \mathbb{N})$ be an orthonormal basis of $L^2_\alpha(0,b)$. Since $G(t,x)$ is a $\alpha-$Hilbert-Schmidt kernel, it can be defined as follows:

$$t_i = (f,\phi_i) = \int_0^b f(x)\overline{\phi_i(x)}d_\alpha(x),$$

$$y_i = (g,\phi_i) = \int_0^b g(x)\overline{\phi_i(x)}d_\alpha(x),$$

$$a_{ik} = \int_0^b\int_0^b G(t,x)\phi_i(t)\overline{\phi_k(x)}d_\alpha(t)d_\alpha(x)\ (i,k\in\mathbb{N}).$$

Then, $L^2_\alpha(0,b)$ is mapped isometrically onto l^2. By this mapping, our integral operator transforms into the operator defined by formula (7) on the space l^2. Condition (16) is translated into condition (6). Thus the original operator is compact. Now let $f,g \in L^2_\alpha(0,b)$. Since $G(t,x) = G(x,t)$, we have

$$\begin{aligned}(\mathbf{S}f,g) &= \int_0^b \mathbf{S}f(t)\overline{g(t)}d_\alpha(t)\\ &= \int_0^b\int_0^b G(t,x)f(t)d_\alpha(t)\overline{g(t)}d_\alpha(x)\\ &= \int_0^b f(t)\Big(\overline{\int_0^b G(t,x)g(t)d_\alpha(x)}\Big)d_\alpha(t) = (f,\mathbf{S}g).\end{aligned}$$

Thus we have proved that the operator $\mathbf{S}$ is self-adjoint. □

4. Parseval equality and spectral expansion for the singular conformable Sturm-Liouville problem

In this section, the existence of a spectral function for the singular conformable Sturm-Liouville problem given by (9) – (10) will be proved. The Parseval equality and a spectral expansion formula by means of the spectral function will be set up.

Let $\lambda_{m,b}$ $(m \in \mathbb{N})$ denote the eigenvalues of regular problem (9) − (11) and $\theta_{m,b}(t) = \theta(t, \lambda_{m,b})$ the corresponding eigenfunctions which satisfy conditions (10) − (11). If f is a real-valued function defined on $[0, b]$,

$$\int_0^b f^2(t) d_\alpha(t) < \infty,$$

and

$$\gamma_{m,b}^2 = \int_0^b \theta_{m,b}^2(t) d_\alpha(t),$$

i.e., $f(.) \in L_\alpha^2(0, b)$, then it follows from Theorem 6 and the Hilbert-Schmidt theorem ([36]) that

$$\int_0^b f^2(t) d_\alpha(t) = \sum_{m=1}^{\infty} \frac{1}{\gamma_{m,b}^2} \left\{ \int_0^b f(t) \theta_{m,b}(t) d_\alpha(t) \right\}^2. \tag{17}$$

Now let us define the non-decreasing step function ϱ_b on $\mathbb{R}$ by

$$\varrho_b(\lambda) = \begin{cases} -\sum\limits_{\lambda < \lambda_{m,b} < 0} \frac{1}{\gamma_{m,b}^2}, & \text{for } \lambda \leq 0 \\ \sum\limits_{\lambda < \lambda_{m,b} < 0} \frac{1}{\gamma_{m,b}^2}, & \text{for } \lambda \geq 0. \end{cases}$$

Then equality (17) can be stated as

$$\int_0^b f^2(t) d_\alpha(t) = \int_{-\infty}^{\infty} F^2(\lambda) d_{\varrho_b}(\lambda), \tag{18}$$

which is called the *Parseval equality*, where

$$F(\lambda) = \int_0^b f(t) \theta(t, \lambda) d_\alpha(t).$$

Letting $b \to \infty$, we will demonstrate that the Parseval equality for problem (9) − (10) can be obtained from (18).

Now we present

Lemma 3. *For any $N > 0$, there exists a positive constant $M = M(N)$ not depending on b such that*

$$\overset{N}{\underset{-N}{V}} \{\varrho_b(\lambda)\} = \sum_{-N \leq \lambda_{m,b} < N} \frac{1}{\gamma_{m,b}^2} = \varrho_b(N) - \varrho_b(-N) < M. \tag{19}$$

Proof. Let $\sin\beta \neq 0$. Since $\theta(t, \lambda)$ is continuous at zero, by condition $\theta(0, \lambda) = \sin\beta$, there exists a positive number k nearby 0 such that

$$\left(\frac{1}{k} \int_0^k \theta(t, \lambda) d_\alpha t \right)^2 > \frac{1}{2} \sin^2 \beta. \tag{20}$$

Let us define $f_k(x)$ by

$$f_k(t) = \begin{cases} \frac{1}{k}, & 0 \le t \le k \\ 0, & t > k \end{cases}.$$

From (18), (19) and (20), we get

$$\int_0^k f_k^2(t) d_\alpha t = \frac{1}{\alpha k^{2-\alpha}} = \int_{-\infty}^{\infty} \left(\frac{1}{k} \int_0^k \theta(t,\lambda) d_\alpha t \right)^2 d_{\varrho_b}(\lambda)$$

$$\geq \int_{-N}^{N} \left(\frac{1}{k} \int_0^k \theta(t,\lambda) d_\alpha t \right)^2 d_{\varrho_b}(\lambda)$$

$$> \frac{1}{2} \sin^2 \beta \left\{ \varrho_b(N) - \varrho_b(-N) \right\},$$

which proves inequality (19).

If $\sin\beta = 0$, the function $f_k(t)$ is defined by

$$f_k(x) = \begin{cases} (\frac{1}{k})^2, & 0 \le t \le k \\ 0, & t > k. \end{cases}$$

Thus we obtain inequality (19) by applying the Parseval equality. □

Let ϱ be any nondecreasing function on $-\infty < \lambda < \infty$. Denote by $L^2_\varrho(\mathbb{R})$ the Hilbert space of all functions $f : \mathbb{R} \to \mathbb{R}$ measurable with respect to the Lebesgue-Stieltjes measure defined by ϱ, with the condition

$$\int_{-\infty}^{\infty} f^2(\lambda) d_\varrho(\lambda) < \infty$$

and the inner product

$$(f,g)_\varrho := \int_{-\infty}^{\infty} f(\lambda) g(\lambda) d_\varrho(\lambda).$$

The leading conclusion of this study is the following theorem.

Theorem 7. *For the Sturm-Liouville problem* (9)−(10), *there exists a nondecreasing function $\varrho(\lambda)$ on $-\infty < \lambda < \infty$ with the following properties.*

(*i*) *If f is a real-valued function and $f \in L^2_\alpha(0,\infty)$, then there is a function $F \in L^2_\varrho(\mathbb{R})$ such that*

$$\lim_{b\to\infty} \int_{-\infty}^{\infty} \left\{ F(\lambda) - \int_0^b f(t)\theta(t,\lambda) d_\alpha(t) \right\} d_\varrho(\lambda) = 0, \tag{21}$$

and the Parseval equality

$$\int_0^\infty f^2(t) d_\alpha(t) = \int_{-\infty}^{\infty} F^2(\lambda) d_\varrho(\lambda) \tag{22}$$

holds.

(*ii*) *The integral*

$$\int_{-\infty}^{\infty} F(\lambda)\theta(t,\lambda)d_{\varrho}(\lambda)$$

converges to f *in* $L_{\alpha}^{2}(0,\infty)$. *That is,*

$$\lim_{n\to\infty}\int_{0}^{\infty}\left\{f(t)-\int_{-n}^{n}F(\lambda)\theta(t,\lambda)d_{\varrho}(\lambda)\right\}^{2}d_{\alpha}(t)=0.$$

We note that the function ϱ is named the *spectral function* for the singular boundary value problem given by $(9)-(10)$.

Proof. Assume that:

1. $f_{\xi}(t)$ vanishes outside the interval $[0,\xi]$, where $\xi<b$.
2. $f_{\xi}(t)$ and $T_{\alpha}f_{\xi}(t)$ are continuous.
3. $f_{\xi}(t)$ satisfies boundary condition (10).

If we apply the Parseval equality (18) to $f_{\xi}(t)$, then we obtain

$$\int_{0}^{\xi} f_{\xi}^{2}(t)d_{\alpha}(t)=\int_{-\infty}^{\infty}F_{\xi}^{2}(\lambda)d_{\varrho}(\lambda), \tag{23}$$

where

$$F_{\xi}(\lambda)=\int_{0}^{\xi} f_{\xi}(t)\theta(t,\lambda)d_{\alpha}(t). \tag{24}$$

Since $\theta\left(x,\lambda\right)$ satisfies equation (9), we see that

$$\theta(t,\lambda)=\frac{1}{\lambda}\left[-T_{\alpha}^{2}\theta(t,\lambda)+v(t)\theta(t,\lambda)\right].$$

By (24), we get

$$F_{\xi}(\lambda)=\frac{1}{\lambda}\int_{0}^{\xi} f_{\xi}(t)\left[-T_{\alpha}^{2}\theta(t,\lambda)+v(t)\theta(t,\lambda)\right]d_{\alpha}(t).$$

Since $f_{\xi}(t)$ and $\theta(t,\lambda)$ satisfy boundary condition (12) and $f_{\xi}(t)$ vanishes in a neighborhood of the point ξ, we obtain

$$F_{\xi}(\lambda)=\frac{1}{\lambda}\int_{0}^{b}\theta(t,\lambda)\left[-T_{\alpha}^{2}f_{\xi}(t)+v(t)f_{\xi}(t)\right]d_{\alpha}(t),$$

via the integration by parts.

For any finite $N > 0$, by using (18), we get

$$\int_{|\lambda|>N} F_\xi^2(\lambda) d_{\varrho_b}(\lambda)$$

$$\leq \frac{1}{N^2} \int_{|\lambda|>N} \left\{ \int_0^b \left[\theta(t,\lambda) \left[-T_\alpha^2 f_\xi(t) + v(t) f_\xi(t)\right]\right] d_\alpha(t) \right\}^2 d_{\varrho_b}(\lambda)$$

$$\leq \frac{1}{N^2} \int_{-\infty}^{\infty} \left\{ \int_0^b \left[\theta(t,\lambda) \left[-T_\alpha^2 f_\xi(t) + v(t) f_\xi(t)\right]\right] d_\alpha(t) \right\}^2 d_{\varrho_b}(\lambda)$$

$$= \frac{1}{N^2} \int_0^\xi \left[-T_\alpha^2 f_\xi(t) + v(t) f_\xi(t)\right]^2 d_\alpha(t).$$

From (23), we see that

$$\left| \int_0^\xi f_\xi^2(t) d_\alpha(t) - \int_{-N}^{N} F_\xi^2(\lambda) d_{\varrho_b}(\lambda) \right| \leq \frac{1}{N^2} \int_0^\xi \left[-T_\alpha^2 f_\xi(t) + v(t) f_\xi(t)\right]^2 d_\alpha(t). \tag{25}$$

By Lemma 3, the set $\{\varrho_b(\lambda)\}$ is bounded. By using theorems 3 and 4 with $a = 0$, we can find a sequence $\{b_{n_k}\}$ such that the sequence $\varrho_{b_{n_k}}(\lambda)$ converges to a monotone function $\varrho(\lambda)$. Passing to the limit as $b_{n_k} \to \infty$ in (25), we get

$$\left| \int_0^\xi f_\xi^2(t) d_\alpha(t) - \int_{-N}^{N} F_\xi^2(\lambda) d_{\varrho}(\lambda) \right| \leq \frac{1}{N^2} \int_0^\xi \left[-T_\alpha^2 f_\xi(t) + \theta(t) f_\xi(t)\right]^2 d_\alpha(t).$$

Hence, letting $N \to \infty$, we obtain

$$\int_0^\xi f_\xi^2(t) d_\alpha(t) = \int_{-\infty}^{\infty} F_\xi^2(\lambda) d_{\varrho}(\lambda).$$

Now suppose that f is an arbitrary real-valued function on $L_\alpha^2(a,\infty)$. It is known that there exists a sequence $\{f_s(t)\}$ satisfying conditions 1–3 and such that

$$\lim_{s\to\infty} \int_0^\infty (f(t) - f_s(t))^2 d_\alpha(t) = 0.$$

Let

$$F_s(\lambda) = \int_0^\infty f_s(t) \theta(t,\lambda) d_\alpha(t).$$

Then, we have

$$\int_0^\infty f_s^2(t)d_\alpha(t) = \int_{-\infty}^\infty F_s^2(\lambda)d_\varrho(\lambda).$$

Since

$$\int_0^\infty (f_{s_1}(t) - f_{s_2}(t))^2\, d_\alpha(t) \to 0 \text{ as } s_1, s_2 \to \infty,$$

we have

$$\int_{-\infty}^\infty (F_{s_1}(\lambda) - F_{s_2}(\lambda))^2\, d\varrho(\lambda) = \int_0^\infty (f_{s_1}(x) - f_{s_2}(x))^2\, d_\alpha(t) \to 0$$

as $s_1, s_2 \to \infty$. Consequently, there is a limit function F satisfying

$$\int_0^\infty f^2(t)d_\alpha(t) = \int_{-\infty}^\infty F^2(\lambda)d_\varrho(\lambda),$$

by the completeness of the space $L_\varrho^2(\mathbb{R})$.

Our next goal is to demonstrate that the sequence (K_s) defined by

$$K_s(\lambda) = \int_0^s f(t)\theta(t,\lambda)d_\alpha(t)$$

converges to F as $s \to \infty$, in the metric of the space $L_\varrho^2(\mathbb{R})$. Suppose that h is another function in $L_\alpha^2(0,\infty)$. By similar arguments, $H(\lambda)$ can be defined by h. It is clear that

$$\int_0^\infty (f(t) - h(t))^2\, d_\alpha(t) = \int_{-\infty}^\infty \{F(\lambda) - H(\lambda)\}^2\, d_\varrho(\lambda).$$

Now set

$$h(t) = \begin{cases} f(t), & t \in [0,s] \\ 0, & t \in (s,\infty). \end{cases}$$

Then we have

$$\int_{-\infty}^\infty \{F(\lambda) - K_s(\lambda)\}^2\, d_\varrho(\lambda) = \int_s^\infty f^2(t)d_\alpha(t) \to 0 \ (s \to \infty),$$

which proves that (K_s) converges to F in $L_\varrho^2(\mathbb{R})$ as $s \to \infty$. This proves (i).

Now, we will prove (ii). Suppose that $f(.), h(.) \in L_\alpha^2(0,\infty)$ and $F(\lambda), H(\lambda)$ are their Fourier transforms, respectively. Then $F \mp H$ are the transforms of $f \mp h$. Consequently, by (22), we get

$$\int_0^\infty [f(t) + h(t)]^2\, d_\alpha(t) = \int_{-\infty}^\infty (F(\lambda) + H(\lambda))^2\, d_\varrho(\lambda),$$
$$\int_0^\infty [f(t) - h(t)]^2\, d_\alpha(t) = \int_{-\infty}^\infty (F(\lambda) - H(\lambda))^2\, d_\varrho(\lambda).$$

Subtracting the second relation from the first, we get

$$\int_0^\infty f(t)h(t)d_\alpha(t) = \int_{-\infty}^\infty F(\lambda)H(\lambda)d_\varrho(\lambda), \tag{26}$$

which is called the *generalized Parseval equality.*

Now set

$$f_\Lambda(t) = \int_{-\Lambda}^{\Lambda} F(\lambda)\theta(t,\lambda)d_\varrho(\lambda),$$

where F is the function defined in (21) and Λ is a positive number. Let $h(.)$ be a function which is equal to zero outside the finite interval $[0,s]$. Thus we obtain

$$\begin{aligned}\int_0^s f_\Lambda(t)h(t)d_\alpha(t) &= \int_0^s \left\{\int_{-\Lambda}^{\Lambda} F(\lambda)\theta(t,\lambda)d_\varrho(\lambda)\right\} h(t)d_\alpha(t) \\ &= \int_{-\Lambda}^{\Lambda} F(\lambda)\left\{\int_0^s \theta(t,\lambda)h(t)d_\alpha(t)\right\} d_\varrho(\lambda) \\ &= \int_{-\Lambda}^{\Lambda} F(\lambda)H(\lambda)d_\varrho(\lambda). \qquad (27)\end{aligned}$$

From (26), we get

$$\int_0^\infty f_\Lambda(t)h(t)d_\alpha(t) = \int_{-\infty}^{\infty} F(\lambda)H(\lambda)d_\varrho(\lambda). \qquad (28)$$

By (27) and (28), we have

$$\int_0^\infty \left(f(t) - f_\Lambda(t)\right) h(t)d_\alpha(t) = \int_{|\lambda|>\Lambda} F(\lambda)H(\lambda)d_\varrho(\lambda).$$

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\int_0^\infty \left(f(t) - f_\Lambda(t)\right) h(t)d_\alpha(t)\right|^2 &\leq \int_{|\lambda|>\Lambda} F^2(\lambda)d_\varrho(\lambda)\int_{|\lambda|>\Lambda} H^2(\lambda)d_\varrho(\lambda) \\ &\leq \int_{|\lambda|>\Lambda} F^2(\lambda)d_\varrho(\lambda)\int_{-\infty}^{\infty} H^2(\lambda)d_\varrho(\lambda).\end{aligned}$$

If we apply this inequality to the function

$$h(t) = \begin{cases} f(t) - f_\Lambda(t), & t \in [0,s] \\ 0, & t \in (s,\infty) \end{cases},$$

then we get

$$\int_0^\infty \left(f(t) - f_\Lambda(t)\right)^2 d_\alpha(t) \leq \int_{|\lambda|>\Lambda} F^2(\lambda)d_\varrho(\lambda).$$

Letting $\Lambda \to \infty$ yields the desired result since the right-hand side does not depend on s. $\square$

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