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On some new extremal Type II $\mathbb{Z}_4$-codes of length 40*

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Abstract. Using the building-up method and a modification of the doubling method we construct new extremal Type II $\mathbb{Z}_4$-codes of length 40. The constructed codes of type $4^{k_1}2^{k_2}$, for $k_1 \in \{8, 9, 10, 11, 12, 14, 15\}$, are the first examples of extremal Type II $\mathbb{Z}_4$-codes of given type and length 40 whose residue codes have minimum weight greater than or equal to 8. Further, we use minimum weight codewords for a construction of 1-designs, some of which are self-orthogonal.

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1. Introduction

Extremal Type II $\mathbb{Z}_4$-codes are a class of self-dual $\mathbb{Z}_4$-codes with Euclidean weights divisible by 8 and the largest possible minimum Euclidean weight for a given length. A construction of such codes is of special interest for both theoretical and practical reasons.

The first extremal Type II $\mathbb{Z}_4$-codes of length 40 were constructed in the late 1990s. Pless, Solé and Qian constructed an extremal Type II $\mathbb{Z}_4$-code with the residue code of dimension 13 and minimum weight 12 by extension and augmentation of a nontrivial cyclic self-dual $\mathbb{Z}_4$-code of length 39 (see [20]). Further, an extremal Type II $\mathbb{Z}_4$-code of length 40 with the self-dual residue code was constructed by Calderbank and Sloane in [6] and also by Harada in [12]. Twenty inequivalent extremal Type II $\mathbb{Z}_4$-codes of length 40 with self-dual residue codes were constructed by Gaborit and Harada in [8]. In 2011, Harada proved that there is an extremal Type II $\mathbb{Z}_4$-code of length 40 whose residue code has dimension k if and only if $k \in \{7, 8, ..., 20\}$, and constructed codes $C_{40,i}$ of type $4^i 2^{40-2i}$, for $i = 8, 9, ..., 19$, with minimum Hamming weight 4 and minimum Lee weight 8 (see [14]). Moreover, the existence of two inequivalent extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^7 2^{26}$ ($C_{40,7}$ and $C'_{40,7}$) is given in [14]. In 2012, Chan constructed 133 new inequivalent extremal Type II $\mathbb{Z}_4$-codes of type $4^{17}2^6$, 501 new inequivalent extremal Type II $\mathbb{Z}_4$-codes of type $4^{18}2^4$ and 431 new inequivalent extremal Type II $\mathbb{Z}_4$-codes of type $4^{19}2^2$ (see [7]). Recently, Harada established the existence of 94343 inequivalent

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extremal Type II $\mathbb{Z}_4$-codes of length 40 with self-dual residue codes (see [16]). So, a vast majority of known extremal Type II $\mathbb{Z}_4$-codes is of type 4^{20}.

In this paper, we construct new extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^{k_1}2^{k_2}$ for $k_1 \in \{7, 8, 9, 10, 11, 12, 13, 14, 15, 18\}$. The constructed codes of type $4^{k_1}2^{k_2}$, $k_1 \in \{8, 9, 10, 11, 12, 14, 15\}$ are the first examples of extremal Type II $\mathbb{Z}_4$-codes of given type and length 40 whose residue codes have minimum weight greater than or equal to 8.

In [14], Harada determined the dimensions of the residue codes of extremal Type II $\mathbb{Z}_4$-codes of lengths 32 and 40, and in [15], a study of residue codes for lengths $24k$ and $24k+8$ yielded a construction of new extremal $\mathbb{Z}_4$-codes. In this paper, we relate the minimum weights of residue codes of Type II $\mathbb{Z}_4$-codes and the residue codes of corresponding codes obtained by the doubling method.

It is shown in [10] that the sets of supports of codewords with Hamming weight 10 in certain extremal Type II $\mathbb{Z}_4$-codes of length 24 form 5-$(24, 10, 36)$ designs. Different families of 3-designs were constructed in [18] from different types of codewords with supports of size 6 in Preparata code $\mathcal{P}_m$, where m is odd. These are the reasons why our interest is also in observing types of the minimum weight codewords in constructed extremal Type II $\mathbb{Z}_4$-codes.

The paper is organized as follows. In the next section, we provide the relevant background information. In Section 3, we present a construction of new extremal Type II $\mathbb{Z}_4$-codes of length 40. In Section 4, we investigate the residue codes of Type II $\mathbb{Z}_4$-codes obtained by the doubling method and show that the minimum weight of the corresponding residue code can not be decreased when applying the doubling method. Finally, in the last section, we observe different types of minimum weight codewords in extremal Type II $\mathbb{Z}_4$-codes of length $24, 32$ or 40 whose residue codes have minimum weight not less than 8. For the constructed extremal Type II $\mathbb{Z}_4$-codes of length 40 whose residue codes have minimum weight equal to 16 we obtained 1-designs. Some of those 1-designs are self-orthogonal.

In this paper, we have used computer algebra systems GAP [21] and Magma [5].

2. Preliminaries

We assume that the reader is familiar with the basic facts of coding theory. We refer the reader to [19] in relation to terms not defined in this paper.

Let $\mathbb{F}_q$ be the field of order q, where q is a prime power. A k-dimensional subspace of $\mathbb{F}_q^n$ is called an $[n, k]$ *q-ary linear code*, and we say that n is the *length* of the code. An element of a code is called a *codeword*. A *generator matrix* for an $[n, k]$ code C is any $k \times n$ matrix whose rows form a basis for C.

If $q = 2$, then the code is called *binary*. Two binary linear codes are *equivalent* if one can be obtained from the other by permuting the coordinates. The *weight* of a codeword $x \in \mathbb{F}_2^n$ is the number of non-zero coordinates in x. If the minimum weight d of an $[n, k]$ binary linear code is known, then we refer to the code as an $[n, k, d]$ binary linear code. Binary linear codes for which all codewords have even weight are called *even* and those among them for which all codewords have weight divisible by four are called *doubly even*. An $[n, k]$ binary linear code C is said to be an *optimal*

$[n,k]$ *binary linear* code if the minimum weight of C achieves the theoretical upper bound on the minimum weight of $[n,k]$ binary linear codes.

Let $\mathbb{Z}_4$ denote the ring of integers modulo 4. A $\mathbb{Z}_4$-*code* C of length n is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. Two $\mathbb{Z}_4$-codes are *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates.

Let $x \in \mathbb{Z}_4^n$. The *support* of a codeword $x \in \mathbb{Z}_4^n$ is the set of non-zero positions in x. Denote the number of coordinates i (where $i = 0,1,2,3$) in the codeword x by $n_i(x)$. A codeword $x \in \mathbb{Z}_4^n$ is of *type* $1^{n_1}2^{n_2}3^{n_3}0^{n_0}$ if $i = 0,1,2,3$ appears n_i times among the coordinates of x. The codeword $x \in \mathbb{Z}_4^n$ is *even* if $n_1(x) = n_3(x) = 0$. The *Hamming weight* of a codeword x is $wt_H(x) = n_1(x) + n_2(x) + n_3(x)$, the *Lee weight* of x is $wt_L(x) = n_1(x) + 2n_2(x) + n_3(x)$, and the *Euclidean weight* of x is $wt_E(x) = n_1(x) + 4n_2(x) + n_3(x)$. We will denote by $d_H(C)$, $d_L(C)$ and $d_E(C)$, the minimum Hamming weight, the minimum Lee weight and the minimum Euclidean weight of the code C, respectively.

Let C be a $\mathbb{Z}_4$-code of length n. The *dual code* $C^\perp$ of the code C is defined as

$$C^\perp = \{x \in \mathbb{Z}_4^n \mid \langle x,y\rangle = 0 \text{ for all } y \in C\},$$

where $\langle x,y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \pmod 4$ for $x = (x_1,x_2,\ldots,x_n)$ and $y = (y_1,y_2,\ldots,y_n)$. The code C is *self-orthogonal* when $C \subseteq C^\perp$ and *self-dual* if $C = C^\perp$. If C is a self-orthogonal $\mathbb{Z}_4$-code, then $wt_L(c)$ is even for all $c \in C$.

Type II $\mathbb{Z}_4$*-codes* are self-dual $\mathbb{Z}_4$-codes which have the property that all Euclidean weights are divisible by eight. For such codes the following theorem holds (see [4, Corollary 13]).

Theorem 1. *Let C be a Type II $\mathbb{Z}_4$-code of length n. Then the minimum Euclidean weight $d_E(C)$ of C is at most* $8\left\lfloor \frac{n}{24}\right\rfloor + 8$.

Type II $\mathbb{Z}_4$-codes meeting the bound in Theorem 1 with equality are called *extremal*.

Every $\mathbb{Z}_4$-code C contains a set of $k_1 + k_2$ codewords $\{c_1, c_2, \ldots, c_{k_1}, c_{k_1+1}, \ldots, c_{k_1+k_2}\}$ such that every codeword in C is uniquely expressible in the form

$$\sum_{i=1}^{k_1} a_i c_i + \sum_{i=k_1+1}^{k_1+k_2} a_i c_i,$$

where $a_i \in \mathbb{Z}_4$ and c_i has at least one coordinate equal to 1 or 3, for $1 \le i \le k_1$, $a_i \in \mathbb{Z}_2$ and c_i has all coordinates equal to 0 or 2, for $k_1 + 1 \le i \le k_1 + k_2$. We say that C is of *type* $4^{k_1}2^{k_2}$. The matrix whose rows are c_i, $1 \le i \le k_1 + k_2$, is called a *generator matrix* for C. A generator matrix G of a $\mathbb{Z}_4$-code C is in *standard form* if

$$G = \begin{bmatrix} I_{k_1} & A & B_1 + 2B_2 \\ O & 2I_{k_2} & 2D \end{bmatrix},$$

where A, B_1, B_2 and D are matrices with entries from $\mathbb{Z}_2$, O is the $k_2 \times k_1$ null matrix, and I_m denotes the identity matrix of order m.

Let C be a $\mathbb{Z}_4$-code of length n. There are two binary linear codes of length n associated with C: $C^{(1)} = \{c \pmod 2) \,|\, c \in C\}$ and $C^{(2)} = \{c \in \mathbb{Z}_2^n \,|\, 2c \in C\}$. The code $C^{(1)}$ is called the *residue code* of C and the code $C^{(2)}$ is called the *torsion code* of C. If C is a $\mathbb{Z}_4$-code of type $4^{k_1}2^{k_2}$, then $C^{(1)}$ is a binary code of dimension k_1. The residue code of a self-orthogonal $\mathbb{Z}_4$-code is a doubly even self-orthogonal binary code.

According to [17], the following theorem holds.

Theorem 2. *Let C be a Type II $\mathbb{Z}_4$-code of length n. Then $C^{(2)}$ is an even binary code, $C^{(1)}$ contains the all-one binary vector, C contains a codeword with all entries ± 1, and $n \equiv 0 \ (mod\ 8)$.*

For an extremal Type II $\mathbb{Z}_4$-code C the lower bound for the minimum weight of $C^{(2)}$ is given in [13, Lemma 2] as follows.

Proposition 1. *Let C be an extremal Type II $\mathbb{Z}_4$-code of length n. Then the torsion code $C^{(2)}$ has minimum weight at least $2\left\lfloor \frac{n}{24} \right\rfloor + 2$.*

3. New extremal Type II $\mathbb{Z}_4$-codes of length 40

The known results on extremal Type II $\mathbb{Z}_4$-codes of length 40 are summarized in Table 1, where the second and the fifth column contains the number of known inequivalent codes of given type.

type	number	reference	type	number	reference
$4^7 2^{26}$	2	[14]	$4^{14} 2^{12}$	1	[14]
$4^8 2^{24}$	1	[14]	$4^{15} 2^{10}$	1	[14]
$4^9 2^{22}$	1	[14]	$4^{16} 2^{8}$	1	[14]
$4^{10} 2^{20}$	1	[14]	$4^{17} 2^{6}$	134	[7], [14]
$4^{11} 2^{18}$	1	[14]	$4^{18} 2^{4}$	502	[7], [14]
$4^{12} 2^{16}$	1	[14]	$4^{19} 2^{2}$	432	[7], [14]
$4^{13} 2^{14}$	2	[14], [20]	4^{20}	94343	[6], [8], [16], [12]

Table 1: *Extremal Type II $\mathbb{Z}_4$-codes of length* 40 *- known results*

In this section, we construct new extremal Type II $\mathbb{Z}_4$-codes of length 40. For a construction of new extremal Type II $\mathbb{Z}_4$-codes of length 40 we use extremal Type II $\mathbb{Z}_4$-codes of length 32. The method that we use is a *building-up method*, i.e., a version of that method for Type II $\mathbb{Z}_4$-codes developed in [7]. Furthermore, we will use the *doubling method* given in [7] together with *Algorithm A* given in [2], which guarantees extremality of the codes constructed by the doubling method when applied to extremal codes (see [2, Theorem 3.10]).

The extremal Type II $\mathbb{Z}_4$-codes of length 32 that we use are described in the sequel.

1. Six extremal Type II $\mathbb{Z}_4$-codes of length 32 were constructed in [2] from Hadamard matrices. In [2], those codes are denoted by $\mathbf{C}_1$, $\mathbf{C}_2$, $\widetilde{\mathbf{C}_2}$, $\mathbf{C}_7$, $\mathbf{C}_{10}$ and $\mathbf{C}_{14}$, and are of type $4^6 2^{20}, 4^9 2^{14}, 4^8 2^{16}, 4^9 2^{14}, 4^7 2^{18}$ and $4^{10} 2^{12}$, respectively. We will use notations $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_5$ and $\mathbf{C}_6$, respectively.

2. In [20], three new extremal Type II $\mathbb{Z}_4$-codes of length 32 were constructed. We will use codes $\tilde{C}_{31,2}$ and $\tilde{C}_{31,3}$, which are of type $4^{11}2^{10}$, and denote them by $\mathbf{C}_7$ and $\mathbf{C}_8$, respectively. We will also use 45 extremal Type II $\mathbb{Z}_4$-codes of length 32 obtained in [1] from codes $\mathbf{C}_7$ and $\mathbf{C}_8$ by using Algorithm A and the doubling method. The types of those 45 codes are presented in Table 2 and their generator matrices are available at:

   `http://www.math.uniri.hr/∼sanjar/structures/`.

type	$4^{10}2^{12}$	$4^{9}2^{14}$	$4^{8}2^{16}$	$4^{7}2^{18}$
number of codes obtained from $\mathbf{C}_7$	2	8	17	4
number of codes obtained from $\mathbf{C}_8$	1	4	8	1

Table 2: *Types of extremal Type II $\mathbb{Z}_4$-codes of length 32 obtained from $\mathbf{C}_7$ and $\mathbf{C}_8$*

3. Finally, we use 5147 extremal Type II $\mathbb{Z}_4$-codes of length 32 and type $4^{14}2^4$ from [2, Remark 3.12].

3.1. New extremal Type II $\mathbb{Z}_4$-codes of length 40 by the building-up method

A method for constructing a binary self-dual code of length $n+2$ from a binary self-dual code of length n was introduced by Harada in [11]. We will use a similar *building-up method* for constructing a Type II $\mathbb{Z}_4$-code of length $n+8$ from a Type II $\mathbb{Z}_4$-code of length n which is developed by Chan in [7]. That method is given in the following theorem (see [7, Theorem 12]).

Theorem 3. *Let C be a Type II $\mathbb{Z}_4$-code of length n. Let G be a generator matrix of C. Suppose X is a 4×8 matrix and Y is a $4 \times n$ matrix, where the rows of the matrix $[X \;\; Y]$ are all mutually orthogonal and have Euclidean weights divisible by 8.*

Suppose the rows of

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 3 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 1 & 3 & 2 \end{bmatrix}$$

and X span $\mathbb{Z}_4^8$. Then the matrix

$$G' = \begin{bmatrix} X & Y \\ -GY^T(HX^T)^{-1}H & G \end{bmatrix}$$

is a generator matrix of a Type II $\mathbb{Z}_4$-code C' of length $n+8$.

Many possibilities for $[X \; Y]$ were calculated in [7]. We will use the matrix

$$M_1 = [X \; Y] = \begin{bmatrix} 2100200111011022011111020100101001010020 \\ 0011101010111001000031130021210111101020 \\ 0000010202001102010011031002320112012000 \\ 0101000000011020112220031000100013000201 \end{bmatrix}.$$

From the codes described in the introduction of this section we obtain the following results.

1. Applying the building-up method with the matrix M_1 to codes $\mathbf{C}_1, ..., \mathbf{C}_6$ we obtained five extremal Type II $\mathbb{Z}_4$-codes $\mathbf{C}_2'$, $\mathbf{C}_3'$, $\mathbf{C}_4'$, $\mathbf{C}_5'$ and $\mathbf{C}_6'$, of length 40. The results are presented in Table 3, where d denotes the minimum weight of the corresponding residue code.

	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$	$\mathbf{C}_6$
type of C'	$4^{13}2^{14}$	$4^{12}2^{16}$	$4^{13}2^{14}$	$4^{11}2^{18}$	$4^{14}2^{12}$
d	8	8	8	12	8

Table 3: *Extremal Type II $\mathbb{Z}_4$-codes of length 40 constructed from $\mathbf{C}_2, ..., \mathbf{C}_6$*

$\mathbb{Z}_4$-codes $C_{40,i}$ of type $4^i 2^{40-2i}$, for $i \in \{10, 11, 12, 13, 14\}$, constructed in [14] have residue codes with minimum weight 4. The extremal Type II $\mathbb{Z}_4$-code of type $4^{13}2^{14}$ and length 40 from [20] has a residue code with minimum weight 12. Furthermore, the residue code of $\mathbf{C}_2'$ and the residue code of $\mathbf{C}_4'$ are mutually inequivalent. So, $\mathbf{C}_2', ..., \mathbf{C}_6'$, are new extremal Type II $\mathbb{Z}_4$-codes of length 40 and the first examples with the specified minimum weight of the residue code.

Remark 1. *Using M_1 for building-up on $\mathbf{C}_1$, we get a Type II $\mathbb{Z}_4$-code of length 40 and type $4^{10}2^{20}$ which is not extremal, but with the matrix*

$$M_2 = [X\ Y] = \begin{bmatrix} 0001110211111022020011120100111000010020 \\ 1011000000011101011131131011200211100020 \\ 0100101002201002000211231022300010011000 \\ 0000111020011020101020331000120000000201 \end{bmatrix}$$

an extremal Type II $\mathbb{Z}_4$-code $\mathbf{C}_1'$ of length 40 and type $4^{10}2^{20}$ whose residue code has minimum weight equal to 12 is obtained. An extremal code of the same type constructed in [14] has a residue code with minimum weight equal to 4, so, this is the first example of an extremal Type II $\mathbb{Z}_4$-code of length 40 and type $4^{10}2^{20}$ whose residue code has minimum weight equal to 12.

2. The building-up method with the matrix M_1 applied to the codes $\mathbf{C}_7, \mathbf{C}_8$ and the 45 codes presented in Table 2 yields 35 extremal Type II $\mathbb{Z}_4$-codes of length 40. Table 4 contains information about obtained codes, where d denotes the minimum weight of the corresponding residue code.

type of C'	$4^{15}2^{10}$	$4^{14}2^{12}$	$4^{13}2^{14}$	$4^{12}2^{16}$	$4^{11}2^{18}$
no. of obtained extremal codes	2	3	11	18	1
d	8	8	8	8 or 12	12

Table 4: *Extremal Type II $\mathbb{Z}_4$-codes of length 40 constructed from $\mathbf{C}_7, \mathbf{C}_8$ and the 45 codes from Table 2*

All constructed codes of the same type have inequivalent residue codes. Therefore, they are inequivalent. Moreover, they are not equivalent to the previously constructed extremal Type II $\mathbb{Z}_4$-codes of length 40. The $\mathbb{Z}_4$-code $C_{40,15}$ of type $4^{15}2^{10}$ constructed in [14] has the residue code with minimum weight 4. Therefore, all obtained codes are new extremal Type II $\mathbb{Z}_4$-codes of length 40.

3. Finally, by using M_1, we apply the building-up method to the 5147 extremal Type II $\mathbb{Z}_4$-codes of length 32 and type $4^{14}2^4$ from [2, Remark 3.12]. In that way, we construct 400 inequivalent extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^{18}2^4$. All of them have a residue code with minimum weight equal to 8. So, they are not equivalent to the extremal code of the same type constructed in [14] since it has a residue code with minimum weight 4. To explore if the constructed codes are equivalent to those constructed in [7] we need a more detailed analysis of the corresponding residue codes.

   By W_i we denote the number of codewords of weight i in binary code. From Theorem 2 it follows that for the residue code of a Type II $\mathbb{Z}_4$-code of length n the following holds: $W_i = W_{n-i}$, for $i = 0, \ldots, n$, and $W_0 = W_n = 1$. Furthermore, the residue code is doubly even, and $W_i = 0$ for i not divisible by 4. So, the weight enumerator of the residue code of a Type II $\mathbb{Z}_4$-code of length 40 can be represented by $(W_4, W_8, W_{12}, W_{16}, W_{20})$.

   The residue codes of the 501 extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^{18}2^4$ constructed in [7] have $W_4 = 0$ and one of the following 21 weight enumerators:

$$\begin{aligned} P_{i,j} &= (W_8, W_{12}, W_{16}, W_{20}) \\ &= (57 + 4i, 5432 - 24i + 64j, 59654 + 60i - 256j, 131856 - 80i + 384j), \end{aligned}$$

   for $(i, j) \in \{(1,0), (2,0), (3,0), (4,0), (5,0), (6,0), (7,0), (3,1), (4,1), (5,1), (6,1), (7,1), (8,1), (5,2), (6,2), (7,2), (8,2), (9,2), (12,2), (9,3), (12,4)\}$.

   All 400 constructed extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^{18}2^4$ have mutually inequivalent residue codes with $W_4 = 0$ and weight enumerators different from $P_{i,j}$ obtained in [7]. Therefore, we obtained 400 new extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^{18}2^4$.

   There are 46 different weight enumerators among weight enumerators of the 400 corresponding residue codes. Table 5 contains those weight enumerators and the number of constructed extremal $\mathbb{Z}_4$-codes whose residue codes have a certain weight enumerator.

$(W_8, W_{12}, W_{16}, W_{20})$, no. of codes	$(W_8, W_{12}, W_{16}, W_{20})$, no. of codes
(87,5252,60104,131256), 6	(87,5380,59592,132024), 1
(75,5324,59924,131496), 17	(83,5404,59532,132104), 1
(81,5352,59758,131760), 9	(89,5240,60134,131216), 2
(67,5372,59804,131656), 10	(91,5292,59908,131560), 1
(81,5288,60014,131376), 14	(97,5384,59486,132208), 1
(89,5368,59622,131984), 3	(65,5384,59774,131696), 3
(77,5312,59954,131456), 16	(99,5372,59516,132168), 1
(73,5336,59894,131536), 16	(93,5280,59938,131520), 1
(77,5376,59698,131840), 10	(77,5248,60210,131072), 16
(69,5360,59834,131616), 13	(73,5272,60150,131152), 17
(71,5348,59864,131576), 16	(61,5344,59970,131392), 16
(79,5300,59984,131416), 17	(69,5296,60090,131232), 17
(83,5276,60044,131336), 12	(65,5320,60030,131312), 17
(83,5340,59788,131720), 8	(71,5284,60120,131192), 17
(85,5328,59818,131680), 6	(59,5356,59940,131432), 7
(85,5264,60074,131296), 6	(75,5260,60180,131112), 16
(87,5316,59848,131640), 4	(67,5308,60060,131272), 17
(75,5388,59668,131880), 4	(79,5236,60240,131032), 13
(73,5400,59638,131920), 1	(63,5332,60000,131352), 17
(79,5364,59728,131800), 10	(83,5212,60300,130952), 7
(85,5392,59562,132064), 1	(81,5224,60270,130992), 7
(95,5332,59712,131864), 1	(85,5200,60330,130912), 2
(89,5304,59878,131600), 2	(87,5188,60360,130872), 1

Table 5: *Extremal Type II* $\mathbb{Z}_4$*-codes of length* 40 *and type* $4^{18}2^4$

3.2. New extremal Type II $\mathbb{Z}_4$-codes of length 40 by the doubling method

In 2012, Chan introduced a method for a construction of Type II $\mathbb{Z}_4$-codes of type 4^a2^b from a given Type II $\mathbb{Z}_4$-code of type $4^{a+1}2^{b-2}$ called *the doubling method* (see [7]). That method is given in the following theorem.

Theorem 4. *Let C be a Type II $\mathbb{Z}_4$-code of length n. Let $2u \in \mathbb{Z}_4^n$ be an even codeword. Suppose $2u \notin C$ and $2u$ has an even number of $2'$s in its coordinates. Let $C_0 = \{v \in C \mid \langle 2u, v\rangle = 0\}$. Then $\widetilde{C} = C_0 \oplus \langle 2u\rangle$ is a Type II $\mathbb{Z}_4$-code.*

The following theorem is a generalization of Theorem 5 from [7], which covers the case $k_2 = 0$.

Theorem 5. *Let C be a Type II $\mathbb{Z}_4$-code of length n and type $4^{k_1}2^{k_2}$. Let G be a generator matrix of C in standard form and G_i the i^{th} row of G. Let $2u \in \mathbb{Z}_4^n \setminus C$ be a codeword with zeroes on the first $k_1 + k_2$ coordinate positions such that $wt_E(2u)$ is divisible by 8. Let $B = \{G_1, \ldots, G_{k_1+k_2}\}$. The following process will yield a generator matrix $\widetilde{G}$ for the $\mathbb{Z}_4$-code $\widetilde{C}$ obtained from C and $2u$ by the doubling method.*

Step 1: Let $B_E = \{G_i \in B \mid \langle G_i, 2u\rangle = 0\}$ and $B_O = B \setminus B_E$.

Step 2: Pick $G_i \in B_O$ arbitrarily. Define $B'_O = \{G_i + G_j \mid G_j \in B_O\}$.

Step 3: Let $\widetilde{G}$ be a matrix whose rows are the elements of the set $B'_O \cup B_E \cup \{2u\}$.

The resultant code $\widetilde{C}$ is independent of the choice of G_i in Step 2.

Proof. The set B_O is not empty because C is self-dual and $2u \notin C$. Further, for all $k = k_1+1, \ldots, k_1+k_2$ it follows $G_k \in B_E$. For $G_i, G_j \in B_O$ we have $\langle G_i + G_j, 2u\rangle = \langle G_i, 2u\rangle + \langle G_j, 2u\rangle = 2+2 = 0$, and $G_i + G_j$ is an even codeword if and only if $i = j$. It follows that there are $k_1 - 1$ non-even codewords and $k_2 + 1$ even codewords in $B'_O \cup B_E$. (Note that $\langle B'_O \cup B_E\rangle = \{v \in C \mid \langle 2u, v\rangle = 0\}$).

The independence follows from the fact that $G_k + G_j = (G_i + G_k) + (G_i + G_j) + (G_i + G_i)$. $\square$

An algorithm, which will be called Algorithm A, for a construction of all candidates $2u$ for which the obtained Type II $\mathbb{Z}_4$-code is extremal in the case when the starting code C in Theorem 4 is an extremal Type II $\mathbb{Z}_4$-code of length 24, 32 or 40 is given in [2]. The necessary condition for the application of Algorithm A is that code C does not contain codewords with exactly four odd coordinates. Examples of the application of Algorithm A to extremal Type II $\mathbb{Z}_4$-codes of length 32 are given in [2]. In the sequal, we demonstrate the application of Algorithm A to extremal Type II $\mathbb{Z}_4$-codes of length 40.

We apply the doubling method to the code $\mathbf{C}'_1$. This code is extremal Type II $\mathbb{Z}_4$-code of length 40 and type $4^{10}2^{20}$ with a residue code whose minimum weight is 12, which allows us to use Algorithm A. By applying Algorithm A, 99 codewords $2u$ are obtained for $\mathbf{C}'_1$. We list the 10 rightmost coordinates of those codewords, and the other coordinates are equal to zero.
2222222002, 2222000022, 2220202222, 2220200202, 2220000002, 2202222202, 2202202222, 2202200202, 2202002000, 2202002022, 2202000200, 2200222200, 2200220022, 2200202220, 2200022000, 2200020200, 2200020002, 2022222000, 2022222220, 2022222202, 2022222022, 2022220222, 2022022200, 2022020022, 2022000200, 2022000020, 2022000002, 2020222222, 2020220202, 2020202000, 2020202220, 2020200200, 2020200020, 2020200002, 2020022000, 2020022220, 2020020200, 2020000220, 2002220000, 2002220202, 2002202202, 2002202022, 2002200002, 2002022202, 2002002020, 2002000202, 2000220222, 2000202200, 2000202002, 2000200022, 2000022002, 2000020220, 2000020022, 0222220222, 0222220020, 0222220002, 0222202002, 0222022222, 0222020220, 0220222222, 0220200020, 0220000022, 0202220000, 0202222020, 0202202000, 0202200222, 0202200002, 0202020200, 0202002200, 0202002222, 0202002020, 0202000220, 0200220020, 0200202200, 0200202002, 0200200022, 0200022200, 0200022020, 0200020220, 0200020202, 0200000222, 0022220000, 0022222200, 0022022000, 0022022220, 0022020200, 0022002002, 0022000220, 0020220002, 0020202200, 0020022222, 0020022020, 0020020202, 0020002202, 0002222220, 0002202002, 0002022222, 0002000222, 0000020222.

$\mathbb{Z}_4$-codes $(\mathbf{C}'_1)_0 \oplus \langle 2u\rangle$ are extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^9 2^{22}$. The residue codes of the obtained $\mathbb{Z}_4$-codes $(\mathbf{C}'_1)_0 \oplus \langle 2u\rangle$ are all mutually inequivalent with $W_4 = W_8 = 0$ and have one of the six weight enumerators presented in Table 6. The second column in Table 6 contains the number of codes $(\mathbf{C}'_1)_0 \oplus \langle 2u\rangle$ whose residue code has a given weight enumerator.

(W_{12}, W_{16}, W_{20})	no. of codes
$(5, 115, 270)$	13
$(8, 103, 288)$	23
$(6, 111, 276)$	21
$(4, 119, 264)$	5
$(7, 107, 282)$	30
$(9, 99, 294)$	7

Table 6: *Weight enumerators of the codes obtained by the doubling method on* $\mathbf{C}'_1$

Since the code $C_{40,9}$ given in [14] has a residue code with minimum weight 4, we constructed 99 new extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^9 2^{22}$.

We used the doubling method on the obtained 99 $\mathbb{Z}_4$-codes $(\mathbf{C}'_1)_0 \oplus \langle 2u \rangle$. Those codes have residue codes with minimum weight 12, which makes them suitable for the application of Algorithm A to construct $2u$ in the doubling method. Altogether, 681 codewords $2u$ are obtained for these codes, which yields 681 extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^8 2^{24}$. All residue codes of the obtained 681 $\mathbb{Z}_4$-codes have $W_4 = W_8 = 0$ and one of the five weight enumerators presented in Table 7. The second column contains the number of codes whose residue code has a given weight enumerator, and the third column gives the number of inequivalent residue codes among them.

(W_{12}, W_{16}, W_{20})	no. of codes	no. of inequivalent residue codes	no. of $2u$
$(0, 55, 144)$	42	14*	0
$(1, 51, 150)$	267	89	2
$(2, 47, 156)$	276	92	5
$(4, 39, 168)$	15	5	3
$(3, 43, 162)$	81	27	4

Table 7: *Extremal codes of type* $4^8 2^{24}$ *obtained by the doubling method on* $(\mathbf{C}'_1)_0 \oplus \langle 2u \rangle$

Since the code $C_{40,8}$ from [14] has a residue code with minimum weight 4, we constructed at least 227 new extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^8 2^{24}$. Those codes have residue codes with minimum weight 12 or 16. Residue codes given in the first row and marked with a star are $[40, 8, 16]$ binary codes and those 14 codes are optimal.

Codes given in Table 7 are suitable for the application of Algorithm A to construct $2u$ in the doubling method. Altogether, 14 suitable codewords $2u$ are obtained as presented in the fourth column of Table 7. The construction yields 14 extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^7 2^{26}$. Their residue codes have $W_4 = W_8 = W_{12} = 0$ and $(W_{16}, W_{20}) = (15, 96)$, and they split into two equivalence classes. Codes from one of those classes are equivalent to the residue code of the code $C_{40,7}$ from [14]. Since the code $C'_{40,7}$ from [14] has a residue code with minimum weight 12, we obtained at least one new extremal Type II $\mathbb{Z}_4$-code of length 40 and type $4^7 2^{26}$, as given in Table 8. The new code is obtained by the doubling method from one of the codes of type $4^8 2^{24}$ with $(W_{12}, W_{16}, W_{20}) = (1, 51, 150)$.

The results from this section on the constructed new extremal Type II $\mathbb{Z}_4$-codes of length 40 are summarized in Table 8, where d presents the minimum weight of the corresponding residue codes. The constructed codes of type $4^{k_1}2^{k_2}$ for $k_1 \in \{8, 9, 10, 11, 12, 13, 14, 15\}$ are the first examples of extremal Type II $\mathbb{Z}_4$-codes of given type and length 40 with given d.

type	(no. of new codes, d)	type	(no. of new codes, d)
$4^7 2^{26}$	$(1, 16)$	$4^{12} 2^{16}$	$(13, 8)$, $(6, 12)$
$4^8 2^{24}$	$(213, 12)$, $(14, 16)$	$4^{13} 2^{14}$	$(13, 8)$
$4^9 2^{22}$	$(99, 12)$	$4^{14} 2^{12}$	$(4, 8)$
$4^{10} 2^{20}$	$(1, 12)$	$4^{15} 2^{10}$	$(2, 8)$
$4^{11} 2^{18}$	$(2, 12)$	$4^{18} 2^4$	$(400, 8)$

Table 8: *New extremal Type II $\mathbb{Z}_4$-codes of length* 40

Generator matrices of the constructed extremal Type II $\mathbb{Z}_4$-codes are available at `http://www.math.uniri.hr/~sanjar/structures/`.

4. On the minimum weight of the residue code of a Type II $\mathbb{Z}_4$-code obtained by the doubling method

In the previous section, using the doubling method on extremal Type II $\mathbb{Z}_4$-codes of type $4^9 2^{22}$ whose residue codes have minimum weight 12 we obtained 14 inequivalent optimal $[40, 8, 16]$ binary codes as residue codes of $\mathbb{Z}_4$-codes of type $4^8 2^{24}$ (Table 8). All those codes have weight enumerator $(W_{16}, W_{20}) = (55, 144)$. In all other cases from Section 3 the minimum weights of residue codes of Type II $\mathbb{Z}_4$-codes are equal to or less than the minimum weights of the residue codes of corresponding $\mathbb{Z}_4$-codes obtained by the doubling method. In this section, we prove that the minimum weight of the corresponding residue code can not be decreased when applying the doubling method.

Let $x \in \mathbb{Z}_4^n$. Denote the set of positions with element $i \in \mathbb{Z}_4$ in the codeword x by $S_i(x)$.

Theorem 6. *Let C be a self-dual $\mathbb{Z}_4$-code of length n and type $4^{k_1}2^{k_2}$. Let $2u \in \mathbb{Z}_4^n \backslash C$ be an even codeword. Let G be the generator matrix of C in standard form and G_i the i^{th} row of G. We define the following sets:*

$$\begin{aligned} B &= \{G_1, \ldots, G_{k_1}, G_{k_1+1}, \ldots, G_{k_1+k_2}\}, \\ B_E &= \{G_i \in B \mid \langle G_i, 2u \rangle = 0\}, \\ B_O &= B \setminus B_E. \end{aligned}$$

For $G_i \in B_O$ we define

$$B_O^i = \{G_i + G_j \mid G_j \in B_O\}.$$

Let $\widetilde{C}$ be a $\mathbb{Z}_4$-code with a generator matrix whose rows are the elements of the set $\widetilde{B} = B_O^i \cup B_E \cup \{2u\}$. Then:

(*i*) $\widetilde{C}^{(1)} \oplus \langle G_i \ (mod\ 2) \rangle = C^{(1)}$, (*ii*) $\widetilde{C}^{(2)} = C^{(2)} \oplus \langle u \rangle$, *where* $S_1(u) = S_2(2u)$.

Proof. The set

$$\{G_1 \pmod 2, \ldots, G_{k_1} \pmod 2\}$$

is a base for the binary code $C^{(1)}$. Let $G'_k \in \mathbb{Z}_2^n$ be a codeword with $S_1(G'_k) = S_2(G_k)$, for all $k = k_1 + 1, \ldots, k_1 + k_2$. Then

$$\{G_1 \pmod 2, \ldots, G_{k_1} \pmod 2, G'_{k_1+1}, \ldots, G'_{k_1+k_2}\}$$

is a base for the binary code $C^{(2)}$.

The set B_O is nonempty. Suppose to the contrary that $\langle G_i, 2u \rangle = 0$, for all $i = 1, \ldots, k_1 + k_2$. In that case, since C is self-dual, it follows that $2u \in C$, which is not possible.

The following equation holds for all $u, v \in \mathbb{Z}_4^n$:

$$u + v \pmod 2 = u \pmod 2 +_2 v \pmod 2.$$

So, the set

$$\{G_i \pmod 2 +_2 G_j \pmod 2 \,|\, G_j \in B_O, j \neq i\} \cup \{G_l \pmod 2 \,|\, G_l \in B_E, l = 1, \ldots, k_1\}$$

is a base for $\widetilde{C}^{(1)}$.

The set

$$\{G_i \pmod 2 +_2 G_j \pmod 2 \,|\, G_j \in B_O, j \neq i\} \cup \{G_i \pmod 2\} \cup \{G_l \pmod 2 \,| \\ G_l \in B_E, l = 1, \ldots, k_1\} \cup \{G'_{k_1+1}, \ldots, G'_{k_1+k_2}\} \cup \{u\}$$

is a base for $\widetilde{C}^{(2)}$.

Hence, $\widetilde{C}^{(1)} \oplus \langle G_i \pmod 2 \rangle = C^{(1)}$ and $\widetilde{C}^{(2)} = C^{(2)} \oplus \langle u \rangle$. □

As a consequence of Theorem 6 and Theorem 5 we have the following statement.

Corollary 1. *Let C be a Type II $\mathbb{Z}_4$-code. Let $\widetilde{C}$ be a $\mathbb{Z}_4$-code obtained from C by the doubling method. Then:*

(*i*) *the minimum weight of $\widetilde{C}^{(1)}$ is greater than or equal to the minimum weight of $C^{(1)}$,*

(*ii*) *the minimum weight of $\widetilde{C}^{(2)}$ is less than or equal to the minimum weight of $C^{(2)}$.*

Remark 2. *According to [9], the upper bound on the minimum weight of a binary $[40, 17]$ code is 12 and such a code has not been constructed yet. The residue codes of new extremal Type II $\mathbb{Z}_4$-codes of length 40 and type $4^{18}2^4$ constructed in previous section are $[40, 18]$ binary codes with the minimum weight 8. By Corollary 1 this means that they are candidates for a construction of an optimal $[40, 17]$ binary linear code with minimum weight 12 by the doubling method. We took one representative for each weight enumerator given in Table 5 and using Algorithm A for those 46 codes we obtained 5 518 532 codewords $2u$ for the application of the doubling method. We constructed residue codes for the corresponding $4^{17}2^6$ extremal Type II $\mathbb{Z}_4$-codes and checked their minimum weights, which are all equal to 8.*

5. Minimum weight codewords

All extremal Type II $\mathbb{Z}_4$-codes of length 40 constructed in this paper have a residue code with minimum weight at least 8. In Theorem 7, we observe types of codewords with the Euclidean weight equal to 16 in extremal Type II $\mathbb{Z}_4$-codes of length $24, 32$ or 40 whose residue codes have minimum weight at least 8. There are no codewords of type $1^a 2^3 3^b 0^{n-a-b-3}, a+b=4$, in such a code because there are no codewords of weight 4 in its residue code.

Lemma 1. *Let C be an extremal Type II $\mathbb{Z}_4$-code of length $n = 24, 32, 40$ such that the minimum weight of $C^{(1)}$ is at least 8. Then C does not contain an even codeword c with $n_2(c) \in \{1,2,3\}$.*

Proof. It follows from Theorem 2 that $C^{(2)}$ is even. Suppose $c \in C$ is an even codeword with $n_2(c) = 1$ or $n_2(c) = 3$. That implies the existence of a codeword $c_1 \in C^{(2)}$ of weight 1 or 3, where $2c_1 = c$, which is not possible.

According to Proposition 1, $C^{(2)}$ has minimum weight at least 4. Suppose $c \in C$ is an even codeword with $n_2(c) = 2$. That implies the existence of a codeword $c_1 \in C^{(2)}$ of weight 2, where $2c_1 = c$, which is not possible. □

Theorem 7. *Let C be an extremal Type II $\mathbb{Z}_4$-code of length $n = 24, 32, 40$ such that the minimum weight of $C^{(1)}$ is at least 8. Let S_1 be the set of all codewords of type $1^0 2^4 3^0 0^{n-4}$, S_2 the set of all codewords of type $1^a 2^0 3^b 0^{n-a-b}, a+b=16$, S_3 the set of all codewords of type $1^a 2^1 3^b 0^{n-a-b-1}, a+b=12$, and S_4 the set of all codewords of type $1^a 2^2 3^b 0^{n-a-b-2}, a+b=8$, in C.*

The following statements hold:

(*i*) $d_H(C) \geq 4, d_L(C) \geq 8, d_E(C) = 16$,

(*ii*) *S_1 is the set of all codewords in C of Hamming weight* 4,

(*iii*) *S_1 is the set of all codewords in C of Lee weight* 8,

(*iv*) *$S_1 \cup S_2 \cup S_3 \cup S_4$ is the set of all codewords in C of Euclidean weight* 16,

(*v*) *the set of supports of all codewords in S_1 is the set of supports of codewords of weight 4 in $C^{(2)}$.*

Proof. (i): $C^{(1)}$ is doubly even. Therefore, for $c \in C$ with $w_H(c) = i$, $i \in \{1,2,3\}$, we have $n_2(c) \neq j$, where $0 \leq j \leq i-1$. Furthermore, $n_2(c) \neq i$ by Lemma 1. It follows that $d_H(C) \geq 4$.

C is self-orthogonal, and therefore all codewords in C have even Lee weight. If $c \in C$ and $w_L(c) = 2i$, $i \in \{1,2,3\}$, then $n_2(c) \neq i-1$ because $C^{(1)}$ is doubly even. The case $n_2(c) = i$ is not possible by Lemma 1 and from the fact that the minimum weight of $C^{(1)}$ is at least 8 we have $n_2(c) \neq i-2$, for $i \in \{2,3\}$. Furthermore, since $C^{(1)}$ is doubly even, it follows that $n_2(c) \neq 0$ when $w_L(c) = 6$. Finally, we can conclude that $d_L(C) \geq 8$.

Since C is extremal, $d_E(C) = 16$ follows from $n \in \{24, 32, 40\}$.

(ii): If $c \in C$ has Hamming weight 4, then $n_1(c) + n_2(c) + n_3(c) = 4$. The case $n_2(c) = 0$ is not possible because the minimum weight of $C^{(1)}$ is at least

8. Furthermore, since $C^{(1)}$ is doubly even, we have $n_2(c) \notin \{1, 2, 3\}$. Therefore, $n_2(c) = 4$ and c is an even codeword of type $1^0 2^4 3^0 0^{n-4}$.

(iii): If $c \in C$ has Lee weight 8, then $n_1(c) + 2n_2(c) + n_3(c) = 8$. From $n_2(c) = 0$, it follows that $w_E(c) = 8$, which is not possible because C is extremal of length $n = 24, 32, 40$. $C^{(1)}$ is doubly even, so $n_2(c) \notin \{1, 3\}$. Since the minimum weight of $C^{(1)}$ is at least 8, the case $n_2(c) = 2$ is not possible. Therefore, $n_2(c) = 4$ and c is an even codeword of type $1^0 2^4 3^0 0^{n-4}$.

(iv): If $c \in C$ has Euclidean weight 16, then $n_1(c) + 4n_2(c) + n_3(c) = 16$. The case $n_2(c) = 3$ is not possible because the minimum weight of $C^{(1)}$ is at least 8.

(v): The support of a codeword $c \in S_1$ is equal to the support of a codeword $c_1 \in C^{(2)}$ of weight 4, where $2c_1 = c$. $\square$

Remark 3. *Let d be the minimum weight of $C^{(1)}$ in Theorem 7. If $d = 12$, then S_4 is empty. If $d = 16$, then sets S_3 and S_4 are empty and the set of supports of all codewords in S_2 is the set of supports of the codewords of minimum weight in $C^{(1)}$.*

5.1. Construction of 1-designs

We assume that the reader is familiar with the basic facts of design theory (see, e.g., [3]). An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. Sometimes, an investigation of different types of codewords yields a construction of combinatorial designs (see, e.g., [10, 18]). We use Theorem 7 to examine whether the sets of the supports of codewords in S_1, S_2, S_3 and S_4 in constructed extremal Type II $\mathbb{Z}_4$-codes of length 40 form the set of blocks of some design. Designs were obtained only from codes constructed in Subsection 3.2 whose residue codes have minimum weight 16. All of the obtained designs are 1-designs and results are summarized in Table 9, where the last column represents the number of non-isomorphic designs. As already mentioned in Theorem 7 and Remark 3, if the minimum weight of the residue code is equal to 16, then the blocks of design obtained from the set S_1 correspond to the codewords of minimum weight in the corresponding torsion code and the blocks of design obtained from the set S_2 correspond to the codewords of minimum weight in that residue code. We checked the conditions of the well-known Assmus-Mattson theorem (see, e.g., [19], p. 303.) for those binary codes, and none of them satisfies the conditions, which means that the existence of the obtained designs does not follow from that theorem.

type	set	parameters, no. of blocks	block intersection numbers	#
$4^7 2^{26}$	S_1	1-$(40, 4, 151)$, 1510 blocks	$0, 1, 2$	2
	S_2	1-$(40, 16, 6)$, 15 blocks	$4, 6, 8$	2
$4^8 2^{24}$	S_1	1-$(40, 4, 71)$, 710 blocks	$0, 1, 2$	14
	S_2	1-$(40, 16, 22)$, 55 blocks	$4, 6, 8$	14

Table 9: *Constructed designs*

Notice in Table 9 that the blocks of designs obtained from sets S_2, i.e. from the minimum weight codewords in the corresponding residue codes, hold 1-designs with even block intersection numbers. That is to say, the condition

$$|B_i \cap B_j| \equiv |B_k| \equiv 0 \ (mod\ 2)$$

is satisfied. So, those codes hold self-orthogonal 1-designs. Moreover, codewords of weight 20 in those residue codes also correspond to the sets of blocks of self-orthogonal 1-designs. In that way, one can obtain two mutually non-isomorphic self-orthogonal 1-$(40, 20, 48)$ designs with 96 blocks and 14 mutually non-isomorphic self-orthogonal 1-$(40, 20, 72)$ designs with 144 blocks. The intersection numbers of those designs are $0, 8, 10$ and 12.

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