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A note on the curve complex of the 3-holed projective plane*

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Abstract. Let S be a projective plane with 3 holes. We prove that there is an exhaustion of the curve complex $\mathcal{C}(S)$ by a sequence of finite rigid sets. As a corollary, we obtain that the group of simplicial automorphisms of $\mathcal{C}(S)$ is isomorphic to the mapping class group $\mathrm{Mod}(S)$. We also prove that $\mathcal{C}(S)$ is quasi-isometric to a simplicial tree.

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1. Introduction

The complex of curves $\mathcal{C}(S)$ of a surface S, first introduced by Harvey [7], is the simplicial complex with k-simplices representing collections of homotopy classes of $k+1$ non-isotopic disjoint simple closed curves in S. In this paper, we let $S = N_{1,3}$ be a 3-holed projective plane. Then $\mathcal{C}(S)$ is one-dimensional and its combinatorial structure was described by Scharlemann [22]. The first purpose of this note is to prove some rigidity results about $\mathcal{C}(N_{1,3})$, which are known for most surfaces, but have not been proved in the literature in this particular case. The second purpose is to show that $\mathcal{C}(N_{1,3})$ is quasi-isometric to a simplicial tree.

By the celebrated theorem of Ivanov [12], Korkmaz [14] and Luo [17], the group $\mathrm{Aut}(\mathcal{C}(S))$ of simplicial automorphisms of $\mathcal{C}(S)$ for orientable surface S is, with a few well understood exceptions, isomorphic to the extended mapping class group $\mathrm{Mod}^{\pm}(S)$. A stronger version of this result, due to Shackleton [24], says that every locally injective simplicial map from $\mathcal{C}(S)$ to itself is induced by some element of $\mathrm{Mod}^{\pm}(S)$ (simplicial map is locally injective if its restriction to the star of every vertex is injective). Analogous results for nonorientable surfaces were proved by Atalan-Korkmaz [3] and Irmak [10], omitting the case of $N_{1,3}$.

In [1], Aramayona and Laininger introduced the notion of a *rigid set*. It is a subcomplex $X \subset \mathcal{C}(S)$, with the property that every locally injective simplicial map $X \to \mathcal{C}(S)$ is induced by some homeomorphism of S. In [1], they constructed a finite rigid set in $\mathcal{C}(S)$, for every orientable surface S, and in [2] they proved that there is an exhaustion of $\mathcal{C}(S)$ by a sequence of finite rigid sets.

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Let $S = N_{g,n}$ be a nonorientable surface of genus g with n holes. Ilbira and Korkmaz [9] constructed a finite rigid set in $\mathcal{C}(S)$ for $g+n \neq 4$. Irmak [11] proved that $\mathcal{C}(S)$ can be exhausted by a sequence of finite rigid sets for $g+n \geq 5$ or $(g,n) = (3,0)$. The methods used in [9, 11] fail for $g+n < 5$ due to the exceptional combinatorial structure of $\mathcal{C}(S)$. While this structure is rather simple for $g+n < 4$ (see [22]), it is quite complicated for $g+n = 4$. In this paper, we show that the main results of Ilbira-Korkmaz [9] and Irmak [11] are true for $N_{1,3}$.

Theorem 1. *There exists a sequence* $\mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots \subset \mathcal{C}(N_{1,3})$ *such that:*

(1) $\mathcal{X}_i$ *is a finite rigid set for all* $i \geq 1$*;*

(2) $\mathcal{X}_i$ *has a trivial pointwise stabilizer in* $\mathrm{Mod}(N_{1,3})$ *for all* $i \geq 1$*;*

(3) $\bigcup_{i\geq 1} \mathcal{X}_i = \mathcal{C}(N_{1,3})$.

Our proof is independent of [9, 11]. The following corollary is an extension of the main results of Atalan and Korkmaz [3] and Irmak [10]. It follows easily from Theorem 1 (see the proof of the analogous corollary in [2]).

Corollary 1. *If* $\phi\colon \mathcal{C}(N_{1,3}) \to \mathcal{C}(N_{1,3})$ *is a locally injective simplicial map, then there exists a unique* $f \in \mathrm{Mod}(N_{1,3})$ *such that* $\phi = f$.

In particular, the group of simplicial automorphisms of $\mathcal{C}(N_{1,3})$ is isomorphic to $\mathrm{Mod}(N_{1,3})$.

Masur and Minsky [19] proved that $\mathcal{C}(S)$ is δ-hyperbolic for orientable S. Their result was extended to nonorientable surfaces by Bestvina and Fujiwara [4] and Masur and Schleimer [20]. The coarse structure of $\mathcal{C}(S)$ is central in low-dimensional topology, providing a key to a better understanding of the mapping class group, the Teichmüller space, and geometry of 3-manifolds. Our next result determines the coarse structure of $\mathcal{C}(N_{1,3})$.

Theorem 2. *The curve graph* $\mathcal{C}(N_{1,3})$ *is quasi-isometric to a simplicial tree.*

It follows that the Gromov boundary $\partial_\infty\mathcal{C}(N_{1,3})$ of $\mathcal{C}(N_{1,3})$ is totally disconnected. We expect that $\partial_\infty\mathcal{C}(N_{g,n})$ is connected for large enough g and n, similarly to orientable surfaces [6, 16]. Recall that for orientable S, $\partial_\infty\mathcal{C}(S)$ is homeomorphic to the space of ending laminations of S [13].

2. Preliminaries

Let S be a surface of finite type. By a *hole* in a surface we mean a boundary component. A *curve* on S is an embedded simple closed curve. A curve is one-sided (resp. two-sided) if its regular neighbourhood is a Möbius band (resp. an annulus). If α is a curve on S, then $S\backslash\alpha$ is the subsurface obtained by removing from S an open regular neighbourhood of α. A curve α is *essential* if no boundary component of $S\backslash\alpha$ is a disc or an annulus or a Möbius band.

The *curve complex* $\mathcal{C}(S)$ is a simplicial complex whose k-simplices correspond to sets of $k+1$ isotopy classes of essential curves on S with pairwise disjoint representatives. To simplify the notation, we will confuse a curve with its isotopy class and

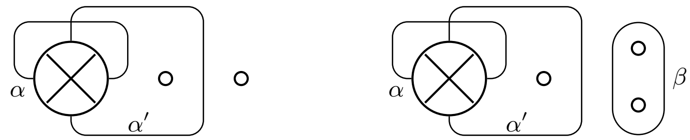

Figure 1: *Vertices of* $\mathcal{C}(N_{1,2})$ *(left) and* $\mathcal{C}(N_{1,3})$ *(right)*

the corresponding vertex of $\mathcal{C}(S)$. Simplices of dimension 1, 2 and 3 will be called *edges*, *triangles* and *tetrahedra*, respectively. For $\alpha, \beta \in \mathcal{C}^0(S)$, by $i(\alpha, \beta)$ we denote their geometric intersection number.

The *mapping class group* $\mathrm{Mod}(S)$ of a nonorientable surface S (resp. the *extendend mapping class group* $\mathrm{Mod}^{\pm}(S)$ of an orientable surface S) is the group of isotopy classes of all self-homeomorphisms of S. If S is orientable, then the mapping class group $\mathrm{Mod}(S)$ is defined to be the group of isotopy classes of orientation preserving homeomorphisms. Note that $\mathrm{Mod}(S)$ and $\mathrm{Mod}^{\pm}(S)$ act on $\mathcal{C}(S)$ by simplicial automorphisms.

If S is a four-holed sphere (or a torus with at most one hole), then $\mathcal{C}(S)$ is a countable set of vertices. In order to obtain a connected complex, the definition of $\mathcal{C}(S)$ is modified by declaring $\alpha, \beta \in \mathcal{C}^0(S)$ to be adjacent in $\mathcal{C}(S)$ whenever $i(\alpha, \beta) = 2$ (or $i(\alpha, \beta) = 1$). Furthermore, triangles are added to make $\mathcal{C}(S)$ into a flag complex. The complex $\mathcal{C}(S)$ obtained in such way is isomorphic to the well-known *Farey complex* [21, 23]. Two adjacent vertices of $\mathcal{C}(S)$ will be called *Farey neighbours*, and 2-simplices of $\mathcal{C}(S)$ will be called *Farey triangles*.

We represent the surface $N_{1,n}$ as a sphere with one crosscap and n holes. The following two lemmas are easy to prove, and otherwise, they can be found in [22].

Lemma 1. *$\mathcal{C}(N_{1,2})$ consists of two one-sided vertices α, α' such that $i(\alpha, \alpha') = 1$ (Figure 1).*

Lemma 2. *In $\mathcal{C}(N_{1,3})$ every two-sided vertex β is connected by an edge with exactly two vertices α, α', which are one-sided and $i(\alpha, \alpha') = 1$. Conversely, for every pair of one-sided vertices α, α' such that $i(\alpha, \alpha') = 1$, there exists exactly one two-sided vertex β connected by an edge with α and α' (Figure 1).*

3. Finite rigid sets

In this section, S denotes a three-holed projective plane. The complex $\mathcal{C}(S)$ was studied by Scharlemann [22]. It is a bipartite graph: its vertex set can be partitioned as $\mathcal{C}^0(S) = V_1 \sqcup V_2$, where V_1 and V_2 denote the sets of one-sided and two-sided vertices, respectively, and every edge of $\mathcal{C}(S)$ connects a one-sided vertex with a two-sided one. Furthermore, by Lemma 2, every $\beta \in V_2$ is connected by an edge with exactly two $\alpha, \alpha' \in V_1$ such that $i(\alpha, \alpha') = 1$. We say that β is *determined* by α and α'.

We define an auxiliary simplicial complex $\mathcal{D}$ whose vertex set is V_1, and a set of vertices $\{\alpha_0, \dots, \alpha_k\}$ is a simplex if $i(a_i, a_j) = 1$ for $0 \le i < j \le k$. It follows from the above discussion that $\mathcal{C}(S)$ is isomorphic to the graph obtained by subdividing every edge of $\mathcal{D}^1$ – the 1-skeleton of $\mathcal{D}$. Indeed, the subdivision of an edge of $\mathcal{D}^1$ corresponds to adding the two-sided vertex determined by this edge.

Proposition 1. *(a) The link of each vertex of $\mathcal{D}$ is isomorphic to the Farey complex.*

(b) $\dim \mathcal{D} = 3$.

(c) Every triangle of $\mathcal{D}$ is contained in exactly two different tetrahedra.

Proof. Fix a vertex $\alpha \in \mathcal{D}$ and consider the four-holed sphere $S\backslash\alpha$. Recall that $\mathcal{C}(S\backslash\alpha)$ is the Farey complex. We define a map $\theta_\alpha \colon \mathrm{Lk}(\alpha) \to \mathcal{C}(S\backslash\alpha)$, where $\mathrm{Lk}(\alpha)$ is the link of α in $\mathcal{D}$. For a vertex $\alpha' \in \mathrm{Lk}(\alpha)$, $\theta_\alpha(\alpha')$ is a two-sided curve determined by α and α'. It follows from Lemma 2 that θ_α is a bijection on vertices, and we claim that it is a simplicial isomorphism. Indeed, note that for $\alpha', \alpha'' \in \mathrm{Lk}(\alpha)$ we have $i(\alpha', \alpha'') = 1 \iff i(\theta_\alpha(\alpha'), \theta_\alpha(\alpha'')) = 2$. This proves (a). The other assertions are consequences of (a) and well-known properties of the Farey complex; namely $\dim \mathcal{C}(S\backslash\alpha) = 2$ and every edge of $\mathcal{C}(S\backslash\alpha)$ is contained in exactly two different triangles. □

Given a one-sided curve, α_0 we can construct infinitely many tetrahedra of $\mathcal{D}$ containing α_0 as a vertex. Let $\{\beta_i\}_1^3$ be any Farey triangle of $\mathcal{C}(S\backslash\alpha_0)$ and, for $1 \le i \le 3$, let α_i be a one-sided curve such that β_i is determined by α_i and α_0. Then $\{\alpha_i\}_0^3$ is a tetrahedron of $\mathcal{D}$.

Lemma 3. *Suppose that Σ is a 4-holed sphere, and C is one of its boundary components. For any two Farey triangles $\{\beta_0, \beta_1, \beta_2\}$ and $\{\beta'_0, \beta'_1, \beta'_2\}$ in $\mathcal{C}(\Sigma)$ there exists a unique $f \in \mathrm{Mod}^{\pm}(\Sigma)$ such that $f(C) = C$ and $f(\beta_i) = \beta'_i$ for $i = 0, 1, 2$.*

Proof. We denote by $\mathrm{Mod}(\Sigma, C)$ (resp. $\mathrm{Mod}^{\pm}(\Sigma, C)$) the subgroup of $\mathrm{Mod}(\Sigma)$ (resp. $\mathrm{Mod}^{\pm}(\Sigma)$) consisting of elements fixing C. By cutting Σ along Farey neighbours we obtain four annuli, each containing one boundary component of S. Therefore, there exists an orientation preserving $f' \in \mathrm{Mod}(\Sigma, C)$ such that $f'(\beta_i) = \beta'_i$ for $i = 1, 2$. Furthermore, since $f'(C) = C$, such f' is easily shown to be unique by the Alexander method [5, Prop. 2.8]. The pointwise stabilizer of $\{\beta'_1, \beta'_2\}$ in $\mathrm{Mod}^{\pm}(\Sigma, C)$ is a cyclic group of order 2 generated by an orientation reversing involution τ fixing every hole and such that β'_0 and $\tau(\beta'_0)$ are the unique common Farey neighbours of both β'_1 and β'_2. By composing f' with τ if necessary we obtain the desired f. □

Lemma 4. *For any two tetrahedra $\{\alpha_i\}_{i=0}^3$ and $\{\alpha'_i\}_{i=0}^3$ of $\mathcal{D}$ there exists a unique $f \in \mathrm{Mod}(S)$ such that $f(\alpha_i) = \alpha'_i$ for $0 \le i \le 3$.*

Proof. For $1 \le i \le 3$, let β_i (respectively β'_i) be a two-sided curve determined by α_0 and α_i (respectively α'_0 and α'_i). Note that $\{\beta_1, \beta_2, \beta_3\}$ and $\{\beta'_1, \beta'_2, \beta'_3\}$ are Farey triangles in $\mathcal{C}(S\backslash\alpha_0)$ and $\mathcal{C}(S\backslash\alpha'_0)$, respectively. By Lemma 3 there exists a unique $f \in \mathrm{Mod}(S)$ such that $f(\alpha_0) = \alpha'_0$ and $f(\beta_i) = \beta'_i$ for $1 \le i \le 3$. Since α'_i is the unique vertex of $\mathcal{C}(S)$ different from α'_0 and adjacent to β'_i, we have $f(\alpha_i) = \alpha'_i$ for $1 \le i \le 3$. □

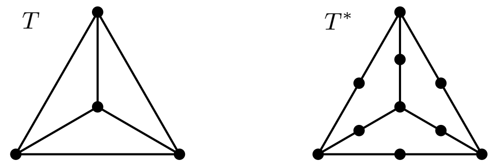

Figure 2: *A tetrahedron of $\mathcal{D}$ and the corresponding subgraph of $\mathcal{C}(S)$*

We define a "dual" graph $\mathcal{T}$ whose vertices are tetrahedra of $\mathcal{D}$. Two tetrahedra are connected by an edge in $\mathcal{T}$ if their intersection is a triangle. Sharlemann proved in [22, Theorem 3.1] that $\mathcal{D}^1$ is the 1-skeleton of the complex obtained from a tetrahedron by repeated stellar subdivision of the faces, but not the edges. This result can be rephrased in terms of the graph $\mathcal{T}$ as follows.

Theorem 3 (Sharlemann)**.** *$\mathcal{T}$ is a 4-regular tree.*

Recall that a k-regular tree is the infinite tree whose every vertex has degree k.

Let T be a tetrahedron of $\mathcal{D}$. We denote by T^* the full subcomplex of $\mathcal{C}(S)$ spanned by the four vertices of T and the six two-sided vertices determined by the edges of T (Figure 2). The following proposition says that T^* is rigid. It is thus an extension of the main result of [9].

Proposition 2. *Suppose that T is a tetrahedron $\mathcal{D}$ and $\phi\colon T^* \to \mathcal{C}(S)$ is a locally injective simplicial map. Then there exists a unique $f \in \mathrm{Mod}(S)$ such that $\phi = f$ on T^*.*

Proof. First note that ϕ is injective because it is locally injective and T^* has diameter 2. Let $T = \{\alpha_i\}_{i=0}^{3}$. We claim that $\{\phi(\alpha_i)\}_{i=0}^{3}$ is a tetrahedron of $\mathcal{D}$. Indeed, for $1 \leq i \leq 3$, $\phi(\alpha_i)$ is adjacent in $\mathcal{C}(S)$ to three different vertices, and hence it is one-sided as two-sided vertices of $\mathcal{C}(S)$ have degree 2. For $i \neq j$, the distance in $\mathcal{C}(S)$ between $\phi(\alpha_i)$ and $\phi(\alpha_j)$ is 2, and hence $\phi(\alpha_i)$ and $\phi(\alpha_j)$ are adjacent in $\mathcal{D}$.

By Lemma 4, there exists a unique $f \in \mathrm{Mod}(S)$ such that $f(\alpha_i) = \phi(\alpha_i)$ for $0 \leq i \leq 3$. Let β be a two-sided vertex of T^* determined by α_i and α_j. Then $\phi(\beta)$ is adjacent to $\phi(\alpha_i)$ and $\phi(\alpha_j)$, and since such a curve is unique, $\phi(\beta) = f(\beta)$. □

We denote by $\mathcal{T}^0$ the vertex set of $\mathcal{T}$, that is the set of tetrahedra of $\mathcal{D}$. Let $d_{\mathcal{T}}$ denote the path metric on $\mathcal{T}$. We fix a reference tetrahedron T_0 and define

$$\mathcal{T}_n^0 = \{T \in \mathcal{T}^0 \mid d_{\mathcal{T}}(T, T_0) \leq n\}.$$

In other words, $\mathcal{T}_n^0$ is the set of tetrahedra within distance at most n from T_0 in the path metric on $\mathcal{T}$.

Proof of Theorem 1. Let $\mathcal{X}_1 = T_0^*$ and for $n \geq 1$:

$$\mathcal{X}_{n+1} = \bigcup_{T \in \mathcal{T}_n^0} T^*.$$

We prove by induction that $\mathcal{X}_n$ is rigid for all $n \geq 1$. By Proposition 2, $\mathcal{X}_1$ is rigid. Assume that $\mathcal{X}_n$ is rigid and let $\phi\colon \mathcal{X}_{n+1} \to \mathcal{C}(S)$ be a locally injective simplicial map. Since $\mathcal{X}_n$ is rigid, there exists a unique $f \in \mathrm{Mod}(S)$ such that $f = \phi$ on $\mathcal{X}_n$. Let $\phi' = f^{-1} \circ \phi$.

Let $T \in \mathcal{T}_{n+1} \backslash \mathcal{T}_n$. We need to show that ϕ' fixes every vertex of T^*. It suffices to show that ϕ' fixes every vertex of T because then it also has to fix the two-sided vertices of T^* determined by edges of T. The tetrahedron T has a common face with some (unique) tetrahedron $T' \in \mathcal{T}_n$. Let $T = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ and $T' = \{\alpha_0', \alpha_1, \alpha_2, \alpha_3\}$. Let β (resp. β') be the two-sided vertex of T^* (resp. $(T')^*$) determined by α_0 and α_1 (resp. α_0' and α_1). By local injectivity of ϕ', $\phi'(\beta) \neq \phi'(\beta') = \beta'$, and hence also $\phi'(\alpha_0) \neq \phi'(\alpha_0') = \alpha_0'$. By Proposition 2, $\phi'(T)$ is a tetrahedron different from T' and having a common face with T'. Since such a tetrahedron is unique by (c) of Proposition 1, $\phi'(T) = T$ and $\phi'(\alpha_0) = \alpha_0$. We have shown that ϕ' pointwise fixes T^*, and it follows that it pointwise fixes $\mathcal{X}_{n+1}$. Hence $\phi = f$ on $\mathcal{X}_{n+1}$.

Since $\mathcal{X}_n$ contains T_0^* for all $n \geq 1$, it has a trivial pointwise stabilizer in $\mathrm{Mod}(S)$. Finally, it follows from the connectedness of $\mathcal{T}$ that $\bigcup_{n\geq 1} \mathcal{X}_n = \mathcal{C}(S)$. □

4. Coarse geometry

In this section, we consider $\mathcal{C}(S)$ and $\mathcal{D}^1$ as metric graphs with all edges of length 1. We denote the metrics on these graphs by $d_{\mathcal{C}}$ and $d_{\mathcal{D}}$, respectively.

There is a natural piecewise-linear homeomorphism $\phi\colon \mathcal{C}(S) \to \mathcal{D}^1$ equal to the identity on one-sided vertices which forgets the two-sided vertices. That is, if β is the two-sided vertex of $\mathcal{C}(S)$ determined by α and α', then $\phi(\beta)$ is defined to be the midpoint of the edge of $\mathcal{D}$ connecting α and α'. We have

$$d_{\mathcal{C}}(x,y) = 2d_{\mathcal{D}}(\phi(x), \phi(y))$$

for all $x, y \in \mathcal{C}(S)$. In particular, ϕ is a quasi-isometry.

Since $\mathcal{T}$ is a tree, every triangle of $\mathcal{D}$ is separating, i.e. the space obtained by removing a triangle from $\mathcal{D}$ has two connected components. If Δ is a triangle of $\mathcal{D}$, and x and y are points lying in different connected components of $\mathcal{D}\backslash\Delta$, then we say that Δ separates x from y.

Lemma 5. *Let p be a vertex on a geodesic in $\mathcal{D}^1$ from x to y, such that $d_{\mathcal{D}}(p,x) \geq 1$ and $d_{\mathcal{D}}(p,y) \geq 1$. There exists a triangle Δ of $\mathcal{D}$ such that $p \in \Delta$ and Δ separates x from y.*

Proof. Let $[x,y]$ be a geodesic in $\mathcal{D}^1$ from x to y containing p, and let q be the vertex preceding p on $[x,y]$. Let $(T_i)_0^n$ be any sequence of tetrahedra forming a geodesic in $\mathcal{T}$ such that $q \in T_0$ and $y \in T_n$. Note that $q \notin T_n$ since $d_{\mathcal{D}}(q,y) = 1 + d_{\mathcal{D}}(p,y) \geq 2$. Let T_i be the first tetrahedron in this sequence such that $q \notin T_i$. Then $\Delta = T_i \cap T_{i-1}$ is a triangle separating q from y. The segment $[q,y]$ must pass through a vertex of Δ, and since $q \in T_{i-1} \backslash T_i$, the distance from q to Δ is 1, hence $p \in \Delta$. Finally, notice that Δ separates x from y, for otherwise $[x,y]$ could not contain q (there would be a shorter path from x to y avoiding q). □

Proof of Theorem 2. Since $\mathcal{C}(S)$ is quasi-isometric to $\mathcal{D}^1$, it suffices to show that $\mathcal{D}^1$ is quasi-isometric to a simplicial tree. By [18, Theorem 4.6], this is equivalent to $\mathcal{D}^1$ satisfying the following bottleneck property: There is some $L > 0$ so that for all $x, y \in \mathcal{D}^1$ there is a midpoint $m = m(x, y)$ with $d(x, m) = d(y, m) = \frac{1}{2}d(x, y)$ and the property that any path from x to y must pass within less than L of the point m.

Let $L > \frac{3}{2}$ and define $m = m(x, y)$ to be the midpoint of any geodesic from x to y. Clearly we can assume $d_{\mathcal{D}}(x, m) \geq L$. Let p be a vertex on a geodesic from x to y such that $d_{\mathcal{D}}(m, p) \leq \frac{1}{2}$. By Lemma 5, there exists a triangle Δ separating x from y such that $p \in \Delta$. Any path from x to y must pass through Δ, and hence within at most $\frac{3}{2}$ of the point m. ☐

Recall that a geodesic metric space (X, d) is δ-hyperbolic if, for any geodesic triangle $[x, y] \cup [x, z] \cup [y, z]$ and any $p \in [x, y]$ there exists some $q \in [x, z] \cup [y, z]$ with $d(p, q) \leq \delta$. A triangle satisfying the condition above is called δ-thin. The curve complex is known to be 17-hyperbolic for every surface for which it is connected [8, 15]. Inspired by Minsky's proof of the hyperbolicity of the Farey graph [21], we give a better bound for the hyperbolicity constant of $\mathcal{C}(N_{1,3})$.

Proposition 3. *The graph $\mathcal{C}(N_{1,3})$ is 3-hyperbolic.*

Proof. First we prove that $\mathcal{D}^1$ is $\frac{3}{2}$-hyperbolic. Let $[x, y] \cup [x, z] \cup [y, z]$ be a geodesic triangle in $\mathcal{D}^1$ and $p \in [x, y]$. Clearly we can assume $d_{\mathcal{D}}(x, p) \geq \frac{3}{2}$. Let p' be a vertex on $[x, y]$ such that $d_{\mathcal{D}}(p, p') \leq \frac{1}{2}$. By Lemma 5, there exists a triangle Δ separating x from y such that $p' \in \Delta$. It follows that $[x, z] \cup [y, z]$ has a non-empty intersection with Δ, and for any point q in this intersection $d_{\mathcal{D}}(q, p) \leq \frac{3}{2}$.

To finish the proof we use the homeomorphism $\phi\colon \mathcal{C}(S) \to \mathcal{D}^1$. Observe that ϕ maps geodesics triangles to geodesic triangles and $d_{\mathcal{C}}(x, y) = 2d_{\mathcal{D}}(\phi(x), \phi(y))$ for all $x, y \in \mathcal{C}(S)$. Since geodesic triangles in $\mathcal{D}^1$ are $\frac{3}{2}$-thin, geodesic triangles in $\mathcal{C}(S)$ are 3-thin. ☐

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